A Class of Weibull Mixed Distributions

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Abstract
We have derived a class of mixture distributions which we call weibull mixtures of distributions. Estimation of unknown parameters along with some properties of these distributions are also prescribed.

Keywords: Mixing distribution; Mixture distribution; Weibull distribution

Introduction
Mixture distribution was first coined in 1894. A number of authors like Pearson, Rider, Blichke, Chahine, Roy, et al., authors [1-13] defined mixtures of two distributions and studied various mixed distributions which they called poisson mixture, binomial mixture, negative binomial mixture, chi-square mixture, Erlang mixture, Laplace mixture, Rayleigh mixture, F, Dual mixture of distributions. Weibull distribution is widely being used in bio-statistics, but weibull mixture distribution has not yet been premeditated. In the present paper, we define first the weibull mixture of some well known distributions which we call weibull mixtures of distributions and then weibull mixture distribution is presented in form of definitions and theorems.

Preliminaries
A mixture distribution is a weighted average of probability distributions of positive weights that sum to one. The distributions thus mixed are called the components of the mixture. The weights themselves comprise a probability distribution called the mixing distribution. Because of this property of weights, a mixture is in particular again a probability distribution. Mixtures occur most commonly when the parameter of the distributions, given by the density by the density function f(x, θ), is itself subject to the change variation. The mixing distribution g(θ) is then a probability distribution on the parameter of the distributions. The general formula for the finite mixture is 
\[ f(x;θ) = \sum f(x;θ)g(θ) \]  
the infinite analogue, in which g is a density function, is
\[ f(x;θ)g(θ)dθ. \]

Main Results
Here in this paper, we define the weibull mixtures of some well known distributions such as normal, lognormal, gamma, exponential, beta, rectangular, erlang, chi-square, t and F distributions. Then some characteristics of these distributions such as characteristic functions, moments, and shape characteristics are also obtained. The main results of the paper are presented in form of definitions and theorems. Comparison of the probability density functions and the first two moments are prescribed in the tertiary section.

Definition 3.1
A random variable X is said to have a weibull mixed distribution if its probability density function is defined as
\[ f(x;α,β) = \int abxe^{-\alpha x^b}g(x;α)d\alpha \]  
where g(x;α) is a probability density function. The name of weibull mixture distribution comes from the fact that the name of (3.1) is the weighted average of g(x,α) with weights equal to the ordinates of weibull distribution.

Definition 3.2
If X follows a weibull mixture of Normal distribution with parameters a and b, then the density function is given by
\[ f(x;a,b) = \int abxe^{-\alpha x^b}dx; -\infty < x < \infty \]  
with parameters a and b such that
\[ \int_{-\infty}^{\infty} f(x;a,b)dx = 1 \]

The characteristic function and moments of the same distribution are presented in the theorem below.

Theorem 3.1
If X has a weibull mixture of Normal distributions with parameters a and b then its characteristic function is represented as
\[ \phi(t) = \int abxe^{-\alpha x^b}e^{itx}dx; -\infty < x < \infty \]
and the 2s moment about origin is
\[ \int_{-\infty}^{\infty} abxe^{-\alpha x^b}e^{itx}dx \]
and (2s+1)th moment about origin is zero. Mean = 0,
\[ \text{Varinace} = 1 + 2a \left[ \left( 1 + \frac{1}{s} \right) \beta_1 \right] \]
Remark: For a = b = 0, \( \phi(t), \mu_2, \mu_4, \beta_1, \beta_2, \beta_3, \beta_4 \) and \( \beta_3 \) and \( \beta_4 \) are same for Normal distribution with mean zero and variance unity.

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Received March 06, 2012; Accepted March 24, 2012; Published March 24, 2012


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Definition 3.3

If a random variable $X$ has the density function
\[ f(x;a,b) = \int_0^\infty abr^{-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr; x > 0 \] \hspace{1cm} (3.5)
then it is said to have a weibull mixture of Lognormal distribution with parameters $a,b$ since
\[ \int_0^\infty f(x;a,b)dx = 1 \] \hspace{1cm} (3.6)

Various moments of the distribution are given in the next theorem.

Theorem 3.2

If $X$ is a weibull mixture of lognormal variable with parameters $a,b$ then its characteristic function is given by
\[ \phi(t) = \int_0^\infty abr^{-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr; x > 0 \] \hspace{1cm} (3.7)
and the $r^{th}$ moment about origin is
\[ \int_0^\infty abr^{-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr; x > 0 \] \hspace{1cm} (3.8)

Definition 3.4

A random variable $X$ having the density function
\[ f(x;a,b,\alpha,\beta) = \int_0^\infty abr^{-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr; x > 0 \] \hspace{1cm} (3.9)
is defined a weibull mixture of Gamma distribution with parameters $a,b,\alpha,\beta$ whereas
\[ \int_0^\infty f(x;a,b,\alpha,\beta)dx = 1. \] \hspace{1cm} (3.10)
The characteristic function and moments are followed by the next theorem.

Theorem 3.3

If $X$ denotes a weibull mixture of gamma variate with parameters $a,b,\alpha,\beta$ then its characteristic function is obtained as
\[ \phi(t) = ab\left(1 - \frac{it}{\beta}\right)^{-a}\int_0^\infty r^{a-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr; x > 0 \] \hspace{1cm} (3.11)
Mean - $\mu = \frac{1}{\beta}\left[\alpha + \frac{1}{\beta}\right]$,
variance - $\sigma^2 = \frac{1}{\beta^2}\left[\alpha + \frac{1}{\beta}\right]^2 + \frac{1}{\beta}\left[\frac{b}{\beta^2}\right] + \frac{\alpha}{\beta}\left[1 + \frac{b}{\beta}\right] - \left[\frac{\alpha}{\beta}\right]^2\left[1 + \frac{b}{\beta}\right]^2 + \left[\frac{1}{\beta}\right]$, 
\[ \beta_2 = \frac{1}{\beta^2}\left[\alpha + \frac{1}{\beta}\right]^2 + \frac{1}{\beta}\left[\frac{b}{\beta^2}\right] + \frac{\alpha}{\beta}\left[1 + \frac{b}{\beta}\right] - \left[\frac{\alpha}{\beta}\right]^2\left[1 + \frac{b}{\beta}\right]^2 + \left[\frac{1}{\beta}\right]. \] \hspace{1cm} (3.12)

Remarks: $\phi(t)\mu$, $\mu_1$, $\mu_2$, $\mu_3$, and $\beta_2$ are true for Gamma distribution with parameters $a,b,\alpha,\beta$ when $a = b = 0$. For $a = 1$, weibull mixture of Gamma distribution should be equivalent to weibull mixture of Exponential distribution. As such we also derived the weibull mixture of Exponential distribution.

Estimates of parameters by the method of moments: Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from the distribution (3.8). We assume that parameters $a,b,\alpha,\beta$ are known. Then the distribution contains only one unknown parameter $a$. We have
\[ \mu = \frac{1}{\beta}\left[\alpha + \frac{1}{\beta}\right]\frac{1}{1 + \frac{b}{\beta}}, \] and $m'_2 = \sum m^2 - \bar{X}$. Hence by the method of moments, we get,
\[ \frac{1}{\beta}\left[\alpha + \frac{1}{\beta}\right]\frac{1}{1 + \frac{b}{\beta}} = \bar{X}. \] Therefore, $\hat{a} = \bar{X}\hat{\beta} - a\frac{1}{1 + \frac{1}{\beta}}$. \hspace{1cm} (3.13)

Definition 3.5

A random variable $X$ having the density function
\[ f(x;a,b,\alpha) = \int_0^\infty abr^{a-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr; x > 0 \] \hspace{1cm} (3.14)
is said to have a weibull mixture of Exponential distribution with parameters $a,b,\alpha$, and
\[ \int_0^\infty f(x;a,b,\alpha)dx = 1. \] \hspace{1cm} (3.15)

Theorem 3.4

If $X$ follows weibull mixture of exponential variate with parameters $a,b,\alpha$ and then its characteristic function is given by
\[ \phi(t) = ab\left(1 - \frac{it}{\beta}\right)^{-a}\int_0^\infty r^{a-1}e^{-\alpha x}e^{-\beta x}r^{a-1}dr, \] \hspace{1cm} (3.16)
Mean - $\mu = \frac{1}{\beta}\left[\alpha + \frac{1}{\beta}\right]\frac{1}{1 + \frac{b}{\beta}} + \frac{\alpha}{\beta}\frac{1}{1 + \frac{b}{\beta}} - \left[\frac{\alpha}{\beta}\right]^2\frac{1}{1 + \frac{b}{\beta}} + \left[\frac{1}{\beta}\right]$, 
variance - $\sigma^2 = \frac{1}{\beta^2}\left[\alpha + \frac{1}{\beta}\right]^2 + \frac{1}{\beta}\frac{b}{\beta^2} + \frac{\alpha}{\beta}\left[1 + \frac{b}{\beta}\right] - \left[\frac{\alpha}{\beta}\right]^2\left[1 + \frac{b}{\beta}\right]^2 + \left[\frac{1}{\beta}\right]. \] \hspace{1cm} (3.17)
Therefore, \( \beta_s = \left( \sum_{i=1}^{s} x_i \right) / \beta \) and \( \beta \) parameters \( a, b \) and \( \beta \) are similar to those of Exponential distribution with parameter \( \beta \).

Parameter estimation: If \( X_1, X_2, X_3, \ldots, X_m \) be a random sample drawn from the distribution (3.12) and parameters \( a, b \) are assumed known, then the distribution contains only one unknown parameter \( \alpha \). So, \( \hat{\alpha} = \frac{1}{m} \sum_{i=1}^{m} x_i \), and \( m = \sum_{m} X \). Therefore, \( \hat{\alpha} = \frac{1}{m} \sum_{i=1}^{m} x_i \).

Definition 3.6

If a random variable \( X \) has the density function
\[
f(x,a,b,\alpha,\beta) = \int_{0}^{\infty} abr^{a-1}e^{-br}\left(\frac{\alpha}{\alpha+\beta}\right)dr, x > 0
\]
then it is said to have a Weibull mixture of Erlang distribution with parameters \( a, b, \alpha, \beta \) since
\[
\int_{0}^{\infty} f(x,a,b,\alpha,\beta)dx = 1
\]
The characteristic function as well as the moments is stated in the following theorem.

Theorem 3.5

If \( X \) has Weibull mixture of Erlang distributions with parameters \( a, b, \alpha, \beta \) and then its characteristic function is given by
\[
\phi(t) = ab \left( 1 - \frac{it}{\alpha+\beta} \right)^{-a} \int_{0}^{\infty} e^{-b\left(\frac{\alpha}{\alpha+\beta}\right)dr, x > 0
\]
Mean
\[
= \frac{1}{\alpha} \left( \frac{1}{\alpha + \beta} \right) \left( \frac{1}{\frac{\alpha}{\alpha+\beta}} \right)
\]
Variance
\[
= \left( \frac{1}{\alpha+\beta} \right) \left( \frac{1}{\alpha + \beta} \right) + \frac{3}{2} \left( \frac{1}{\alpha + \beta} \right) - \left( \frac{1}{\alpha} \right) \left( \frac{1}{\alpha + \beta} \right)
\]
\[
\beta = \left[ \frac{2a^{2} + 2a^{2} + 3a^{2} + 3a^{2} + 3a^{2} + 3a^{2}}{\alpha + \beta} \right]^{2}
\]
Remark: When \( a = b = 0 \), then all of \( \Pi(\alpha,\beta,\mu,\mu,\mu,\beta) \) and \( \beta_s \) are similar to those of Exponential distribution with parameter \( \beta \).

\[
\int f(x; a, b, \alpha, \beta)dx = 1
\]  
(3.24)

**Theorem 3.7**

If \( X \) follows weibull mixture of beta distributions of first kind with parameters \( a, b, \alpha \) and \( \beta \), then its \( s^{\text{th}} \) moment about origin is given by

\[
\phi_s(t) = \int_0^\infty ab^{s-1}e^{-a(x/b)^{1/r}}B(a + s + r, \beta - s) \frac{dr}{B(a + r, \beta)}
\]

(3.25)

**Remark:** For \( a = b = 0 \), all the values of \( \mu', \mu', \mu' \) and \( \mu_2 \) are true for Beta distribution of 1st kind with parameters \( a \) and \( \beta \).

**Definition 3.9**

A random variable \( X \) having the density function

\[
f(x; a, b, \alpha, \beta) = \int_0^\infty ab^{s-1}e^{-a(x/b)^{1/r}}B(a + s + r, \beta - s) \frac{dr}{B(a + r, \beta)}
\]

(3.26)

is called a weibull mixture of Beta distribution of 2nd kind with parameters \( a, b \) and \( \beta \). Moreover,

\[
\int_0^\infty f(x; a, b, \alpha, \beta)dx = 1
\]

(3.27)

Next theorem presents some properties of the same distribution.

**Theorem 3.8**

If \( X \) follows weibull mixture of beta distribution of second kind with parameters \( a, b, \alpha \) and \( \beta \), then its \( s^{\text{th}} \) moment about origin is given by

\[
\sum_{i=1}^{n} \int_0^\infty ab^{s-1}e^{-a(x/b)^{1/r}}B(a + s + r, \beta - s) \frac{dr}{B(a + r, \beta)}
\]

(3.28)

**Remark:** Putting \( a = b = 0 \) then all the values of \( \mu', \mu', \mu' \) and \( \mu_2 \) are true for Beta distribution of 2nd kind with parameters \( a \) and \( \beta \).

<table>
<thead>
<tr>
<th>Sl.</th>
<th>Name of the distribution</th>
<th>Probability density function ( f(x) )</th>
<th>Support</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Weibull mixed Normal</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
<td>( a, b )</td>
</tr>
<tr>
<td>2</td>
<td>Weibull mixed Lognormal</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( x &gt; 0 )</td>
<td>( a, b )</td>
</tr>
<tr>
<td>3</td>
<td>Weibull mixed Gamma</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( x &gt; 0 )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>4</td>
<td>Weibull mixed Exponential</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( x &gt; 0 )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>5</td>
<td>Weibull mixed Erlang</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( x &gt; 0 )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>6</td>
<td>Weibull mixed Rectangular</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( 0 &lt; x &lt; m )</td>
<td>( a, b, m )</td>
</tr>
<tr>
<td>7</td>
<td>Weibull mixed Beta 1st kind</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( 0 &lt; x &lt; 1 )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>8</td>
<td>Weibull mixed Beta 2nd kind</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( x &gt; 0 )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>9</td>
<td>Weibull mixed Chi-square</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( x^2 &gt; 0 )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>10</td>
<td>Weibull mixed ( t )</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( \infty &lt; t &lt; \infty )</td>
<td>( a, b, \alpha )</td>
</tr>
<tr>
<td>11</td>
<td>Weibull mixed ( F )</td>
<td>( \int ab^{s-1}e^{-a(x/b)^{1/r}} \frac{dr}{B(a + s + r, \beta - s)} )</td>
<td>( F &gt; 0 )</td>
<td>( a, b, \alpha, \beta )</td>
</tr>
</tbody>
</table>

Table 1: Comparison of density functions of different Weibull mixture distributions. \( \chi^2 > 0 \).
**Definition 3.11**

If \( t \) as a random variable has the density function
\[
f(t; a, b, n) = \int_{-\infty}^{\infty} abr^{-n}e^{-\beta t} dr, -\infty < t < \infty
\]
then it is said to have a weibull mixture of t distribution with parameters \( a, b \) and \( n \) if
\[
\int_{-\infty}^{\infty} f(t; a, b, n)dt = 1
\]

The following theorem expresses some of the properties of the distribution.

**Theorem 3.10**

If \( t \) is weibull mixture of t distribution with parameters \( a, b \) and \( n \) then the \( 2s \)th moment about origin is given by
\[
n!a^{b} \int_{-\infty}^{\infty} \frac{r^{s}}{\beta} e^{-\frac{r^2}{\beta}} dr
\]
and the \((2s+1)\)th moment about origin...
is zero, \( \beta = 0, \beta_2 = \frac{n - 4}{n} \left[ \frac{3 + 8a^2}{1 + \frac{1}{b}} + 4a^2 \left( 1 + \frac{2}{b^2} \right) \right] \). 

Remark: If \( a = b = 0 \) then all the values of \( \mu_{a}, \mu_{b}, \mu, \mu_{a} \mu_{b} \mu_{c} \beta \) and \( \beta_2 \) are true for t distribution with parameter \( n \).

**Definition 3.12**

A random variable \( F \) having the density function

\[
   f(F; a, b, n_1, n_2) = \frac{n!}{n_1! n_2!} F^{n_1-1} (1 + x)^{n_2} dr, \quad F > 0
\]

is said to have a weibull mixture of \( F \) distribution with parameters \( a, b, n_1 \) and \( n_2 \), if

\[
   \int_0^\infty f(F; a, b, n_1, n_2) dF = 1 \quad (3.35)
\]

The following theorem presents the characteristic function and moments of this distribution.

**Theorem 3.11**

If \( F \) follows weibull mixture of \( F \) distribution with parameters \( a, b, n_1 \) and \( n_2 \) then its characteristic function is given by

\[
   \phi(t) = \int_0^\infty ab \exp^{b t} e^{-a} \sum_{n=0}^{\infty} \left( \frac{n!}{n_1! n_2!} \right) \left( \frac{n_2}{n_2} + \frac{r}{x} \right) \left( \frac{n_2}{n_2} - x \right) dr
\]

and the \( \mu \)th moment about origin in \( \left( \frac{n_2}{n_2} \right) \int_0^\infty ab \exp^{b t} e^{-a} \left( \frac{n_2}{n_2} + \frac{r}{x} \right)^{\mu - 1} \left( \frac{n_2}{n_2} - x \right)^{\mu - 1} dr
\]

Remark: For \( a = b = 0 \) all the values of \( \phi_1(t) \mu, \mu_2, \mu_3 \) and \( \mu_4 \) are true for \( F \) distribution with parameters \( n_1 \) and \( n_2 \).

**Comparison**

A Comparison among various features of the different weibull mixture distributions is shown in the following table 1 and table 2.

**References**


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