

Analysis of Temperature Distribution through Rectangular Convective Fin Using Analytical Methods

Mokhtarpour K¹ and Ganji DD^{2*}

¹Mechanical Engineering Department, Amirkabir University of Technology, Tehran, Iran

²Mechanical Engineering Department, Babol Noshirvani University of Technology, Babol, Iran

Abstract

In this paper, the power of the recently introduced method of Akbari-Ganji has been validated by solving two different nonlinear equations. In the first section, the temperature distribution model of a convective straight fin is found by solving the governing energy balance equation with Akbari-Ganji's Method. The authenticity of this method has been checked considering the fourth-order Runge-Kutta. In the second section, a linear differential equation without enough boundary conditions is converted into a nonlinear differential equation with enough boundary conditions by derivation. The precision of the AGM method has been compared with two other semi-analytical methods. Results were prepared for the ultimate solution function and its first derivative in both sections. The variational iteration method and the homotopy perturbation method were the ones which showed the lowest and the highest error amounts, respectively. The AGM method is also considered as an acceptable method with negligible error in solving different nonlinear equations.

Keywords: Rectangular convective fin; Akbari-Ganji's method (AGM); HPM; VIM; Fourth-order Runge-Kutta numerical method

Nomenclature: A_c : Cross-sectional area of the fin (m^2); b : Fin length; h : Heat transfer coefficient (W/m^2K); k : Thermal conductivity of the fin material (W/m^2K); k_a : Thermal conductivity at the ambient fluid temperature (W/m^2K); k : Thermal conductivity at the base temperature (W/m^2K); P : Fin perimeter (m); Q : Heat transfer rate (W); T_a : Temperature of surface a (K); T_b : Temperature of surface b (K); x : Distance measured from the fin tip (m); β : Dimensionless parameter describing the thermal conductivity variation; η : Fin efficiency; ζ : Dimensionless coordinate; λ : The slope of the thermal conductivity temperature curve (K^{-1}); ψ : Thermogeometric fin parameter; θ : Dimensionless temperature; AGM: Akbari-Ganji's Method; HPM: Homotopy Perturbation Method; VIM: Variational Iteration Method

Temperature Distribution Solution inside the Convective Straight Fin

In the study of heat transfer, fins are surfaces extending from an object to increase the rate of heat transfer to or from the environment by increasing convection. Increasing the temperature gradient between the object and the environment, increasing the coefficient of convection heat transfer, or increasing the surface area of the object increases the heat transfer. Sometimes it is not feasible or economical to change the first two options. Thus, adding a fin to an object increases the surface area and can sometimes be an economical solution to heat transfer problems. The sketch of the convective straight fin is depicted (Figure 1).

Analyzing the fin problem consists of two sections. One is based on thermal convection through fin surface area and the other is based on thermal conduction through fin cross section. The one-dimensional energy balance equation governing the fin problem is as below:

$$A_c \frac{d}{dx} \left[k(T) \frac{dT}{dx} \right] - Ph(T_b - T_a) = 0 \quad (1)$$

A linear temperature function for the thermal conductivity of the fin material is considered as below:

$$k(T) = k_a [1 + \lambda(T - T_a)] \quad (2)$$

Where k_a and k are the thermal conductivity at the ambient fluid temperature of the fin and the thermal conductivity variation,

respectively. Employing the following dimensionless parameters [1-2]:

$$\theta = \frac{T - T_a}{T_b - T_a}, \quad \zeta = \frac{x}{b}, \quad \beta = \lambda(T_b - T_a), \quad \psi = \left(\frac{hPb^2}{k_a A_c} \right)^{1/2} \quad (3)$$

Therefore, the governing equation will reduce to:

$$U: \frac{d^2\theta}{d\zeta^2} + \beta\theta \frac{d^2\theta}{d\zeta^2} + \beta \left(\frac{d\theta}{d\zeta} \right)^2 - \psi^2\theta = 0 \quad (4)$$

The corresponding boundary conditions are as below:

$$\begin{cases} \frac{d\theta}{d\zeta} = 0; & \zeta=0 \\ \theta = 1; & \zeta=1 \end{cases} \quad (5)$$

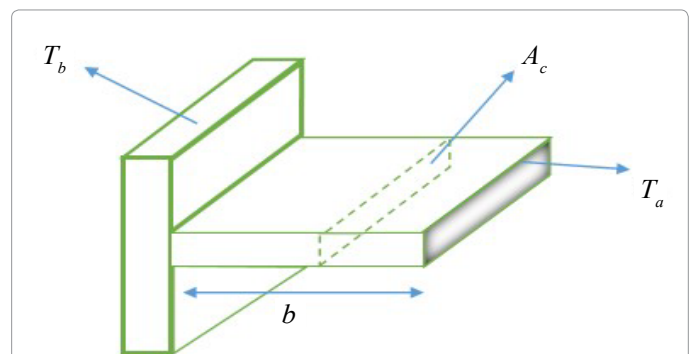


Figure 1: The sketch of the rectangular convective straight fin under discussion.

*Corresponding author: Ganji DD, Mechanical Engineering Department, Babol Noshirvani University of Technology, P.O. Box 484, Babol, Iran, Tel: +98 111 32 34 501; E-mail: ddg_davood@yahoo.com

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The heat transfer dissipation rate from the fin is expressed as below according to the Newton's law of cooling:

$$Q = \int_0^b P(T - T_\alpha) dx \tag{6}$$

The fin efficiency which is defined as the ratio of actual heat transfer from the fin surface to the other side while the whole fin surface stays at the same temperature:

$$\eta = \frac{Q}{Q_{ideal}} = \frac{\int_0^b P(T - T_\alpha) dx}{Pb(T_b - T_\alpha)} = \int_{\zeta=0}^1 \theta(\zeta) d\zeta \tag{7}$$

AGM approach

Description of the method: To elucidate on, consider the nonlinear differential equation p that is a function of the parameter u (a function of x) and its derivatives as below:

$$p_k : f(u, u', \dots, u^{(m)}) = 0; u = u(x) \tag{8}$$

The corresponding boundary conditions are as below:

$$\begin{cases} u(0) = u_0, u'(0) = u_1, \dots, u^{(m-1)}(0) = u_{m-1} \\ u(L) = u_{L0}, u'(L) = u_{L1}, \dots, u^{(m-1)}(L) = u_{Lm-1} \end{cases} \tag{9}$$

In order to solve the first differential equation with respect to the boundary conditions in $x=L$ (Eq. 9), the corresponding answer is considered as follows:

$$u(x) = \lim_{n \rightarrow \infty} \sum_{i=0}^n a_i x^i = \lim_{n \rightarrow \infty} (a_0 + a_2 x^1 + a_2 x^2 + \dots + a_n x^n) \tag{10}$$

By means of increasing the number of series in Eq. 10, the obtained solution is closer to the real answer and shows more precision. Approximately, five or six sentences are enough for obtaining answers with negligible errors in applied problems. The boundary conditions are applied to the functions as below:

- a) The application of the boundary conditions for the answer of differential Eq. 11 is in the form of:

If $x=0$

$$\begin{cases} u(0) = a_0 = u_0 \\ u'(0) = a_1 = u_1 \\ \dots \end{cases} \tag{11}$$

And if $x=L$

$$\begin{cases} u(L) = a_0 + a_1 L + a_2 L^2 + \dots + a_n L^n = u_{L0} \\ u'(L) = a_1 + 2a_2 L + 3a_3 L^2 + \dots + n a_n L^{n-1} = u_{L1} \end{cases} \tag{12}$$

- b) After substituting Eq. 12 into Eq. 8, the boundary conditions are applied on Eq. 8 according to the procedure below:

$$\begin{cases} p_0 : f(u(0), u'(0), \dots, u^{(m)}(0)) \\ p_1 : f(u(L), u'(L), \dots, u^{(m)}(L)) \end{cases} \tag{13}$$

With regard to the choice of n ; $n < m$ sentences from Eq. 10 and in order to make a set of equations which is consisted of $(n+1)$ equations and $(n+1)$ unknowns, we confront with a number of additional unknowns which are indeed the same coefficients of Eq. 10. Therefore, to remove this problem, we should derive m times from Eq. 8 according to the additional unknowns in the afore-mentioned sets of differential equations and then applying the boundary conditions on them.

$$\begin{cases} p'_k : f'(u', u'', u''', \dots, u^{(m+1)}) \\ p'' : f''(u'', u''', \dots, u^{(m+2)}) \end{cases} \tag{14}$$

- b) Application of the boundary conditions on the derivatives of the differential equation P_k in Eq. 14 is done in the form below:

$$p'_k : \begin{cases} f(u'(0), u''(0), u'''(0), \dots, u^{(m+1)}(0)) \\ f(u'(L), u''(L), u'''(L), \dots, u^{(m+1)}(L)) \end{cases} \tag{15}$$

$$p''_k : \begin{cases} f(u''(0), u'''(0), \dots, u^{(m+2)}(0)) \\ f(u''(L), u'''(L), \dots, u^{(m+2)}(L)) \end{cases} \tag{16}$$

$(n+1)$ equations can be made from Eq. 11-16 so those $(n+1)$ unknown coefficients of Eq. 10, $a_0, a_1, a_2, \dots, a_n$ can be computed. The answer of the nonlinear differential Eq. 8 will be obtained by computing coefficients of Eq. 10 [3-11].

Applying AGM to the given equation: The answer of the energy balance equation governing the fin problem can be considered as a finite series of polynomials with constant coefficients:

$$\frac{d^2 \theta}{d\zeta^2} + \beta \theta \frac{d^2 \theta}{d\zeta^2} + \beta \left(\frac{d\theta}{d\zeta} \right)^2 - \psi^2 \theta = 0, \theta(\zeta) = \sum_i^5 a_i \zeta^i \tag{17}$$

The aforementioned unknown coefficients are capable of being computed by applying the boundary conditions.

- a) Applying the boundary conditions on Eq. 17:

$$\frac{d\theta}{d\zeta}(\zeta=0) = 0 \rightarrow a_1 = 0 \tag{18}$$

$$\theta(\zeta=1) = 1 \rightarrow a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 1 \tag{19}$$

- b) Applying the boundary conditions on the main differential Eq. 4 and its derivatives:

$$\theta(\theta(\zeta)) \rightarrow U(\theta(B.C)) = 0, U'(\theta(B.C)) = 0, \dots \tag{20}$$

$$\begin{cases} U(0) = 0 \rightarrow \beta a_1^2 - \psi^2 a_0 + 2a_2 = 0 \\ U(1) = 0 \rightarrow 20a_5 + 12a_4 + 6a_3 + 2a_2 + \beta(20a_5 + 12a_4 + 6a_3 + 2a_2) + \beta(5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1)^2 - \psi^2(a_5 + a_4 + a_3 + a_2 + a_1) = 0 \end{cases} \tag{21}$$

$$\begin{cases} U'_0 = 0 \rightarrow 4\beta a_1 a_2 - \psi^2 a_1 + 2\beta a_2 + 6a_3 = 0 \\ U'_0 = 1 \rightarrow 60a_5 + 24a_4 + 6a_3 + \beta(20a_5 + 12a_4 + 6a_3 + 2a_2) + \beta(60a_5 + 24a_4 + 6a_3)^2 + 2\beta(5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1) - (20a_5 + 12a_4 + 6a_3 + 2a_2) - \psi^2(5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1) = 0 \end{cases} \tag{22}$$

By solving a set of algebraic equations, six unknown coefficients are computed according to the existing six equations. By entering the values of the coefficients, the ultimate answer of AGM method is obtained [12-20].

Results

The Temperature Distribution tables and figures are depicted below for two cases in which the dimensionless parameter β describing variation of thermal conductivity is considered zero which corresponds to a constant thermal conductivity through the fin's material (Tables 1 and 2; Figures 2 and 3).

We compared the results of the AGM approach with an accurate numerical solution, using fourth-order Runge-Kutta with absolute error of $1e-10$ as demonstrated in Tables 1 and 2; for two cases of $\Psi=0.5$ and $\Psi=1$ with $\beta=0$. An excellent agreement between the results is observed which confirms the validity of the AGM approach [3].

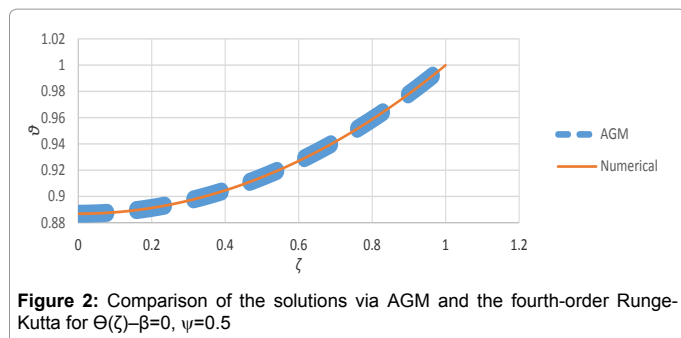


Figure 2: Comparison of the solutions via AGM and the fourth-order Runge-Kutta for $\Theta(\zeta)$ – $\beta=0, \psi=0.5$

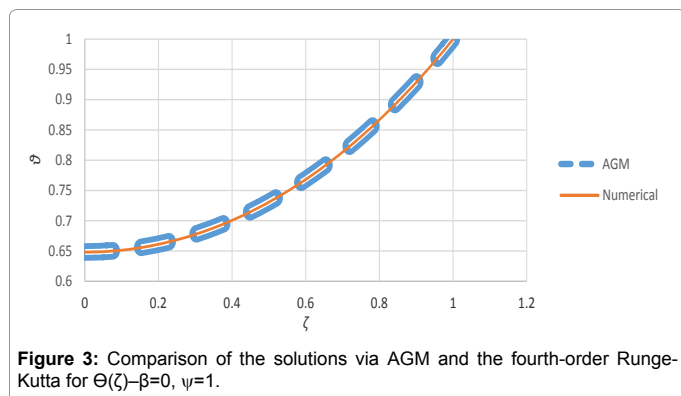


Figure 3: Comparison of the solutions via AGM and the fourth-order Runge-Kutta for $\Theta(\zeta)$ – $\beta=0, \psi=1$.

ζ	$\beta=0, \psi=0.5$		
	AGM	Numerical	Error
0	0.648414986	0.648054499	0.000360487
0.1	0.651659539	0.651297468	0.000362071
0.2	0.661423401	0.661058833	0.000364568
0.3	0.677799078	0.677436294	0.000362783
0.4	0.700944784	0.700593758	0.000351026
0.5	0.731087896	0.730762995	0.000324901
0.6	0.768528415	0.768245949	0.000282466
0.7	0.813642421	0.813417762	0.000224659
0.8	0.866885533	0.866730522	0.000155011
0.9	0.928796369	0.928717806	7.86E-05
1	1	1	0

Table 1: The results of AGM and the fourth-order Runge-Kutta for $\Theta(\zeta)$.

Validation of the AGM Approach Considering an Ordinary Differential Equation

In this section, we consider the following differential equation and solve it by HPM, VIM and AGM semi-analytical approaches to analyze the precision of AGM.

$$h(x); \frac{d^2 f}{dx^2} + \frac{df}{dx} + f^2 + f^4 = 0 \quad (23)$$

In this problem, the number of boundary conditions is one more than the order of the presented equation. The given equation should be derived so that the number of boundary conditions and the order of the differential equation will be equaled. Therefore, we will have:

$$g(x); \frac{d^2 f}{dx^2} + \frac{df}{dx} + 2f \frac{df}{dx} + 4f^3 \frac{df}{dx} = 0 \quad (24)$$

The corresponding boundary conditions are as below:

$$f(0) = 0, f(1) = 1, f'(1) = 0 \quad (25)$$

Applying HPM approach to the given equation

Considering Eq. 23 and 24, we construct the homotopy function as below:

$$F(x, p) = (1-p) \left(\frac{d^3 f}{dx^3} \right) + p \left(\frac{d^3 f}{dx^3} + \frac{d^2 f}{dx^2} + 2f \frac{df}{dx} + 4f^3 \frac{df}{dx} \right) = 0 \quad (26)$$

Assuming $f(x)$ as a summation of a power series of parameter p , we will have:

$$f(x) = \sum_{i=0}^4 p^i f_i(x) \quad (27)$$

Substituting Eq. 27 into Eq. 26 and rearranging the answer by powers of p , the multipliers of each power are obtained. Solving the obtained answers according to the given boundary condition which is constant for all of them, the consecutive terms of HPM solution are gained:

$$f_0(x) = -x^2 + 2x \quad (28)$$

$$f_1(x) = -\frac{1}{90}x^{10} + \frac{1}{9}x^9 + \frac{3}{7}x^8 + \frac{16}{21}x^7 - \frac{17}{30}x^6 + \frac{1}{5}x^5 - \frac{1}{3}x^4 + \frac{1}{3}x^3 + \frac{2}{315}x^2 - \frac{23}{315}x \quad (29)$$

$$f_2(x) = -\frac{1}{6885}x^{18} + \frac{2}{765}x^{17} - \frac{43}{2100}x^{16} + \frac{428}{4725}x^{15} - \frac{14081}{57330}x^{14} + \frac{341}{819}x^{13} - \frac{169}{378}x^{12} + \frac{12731}{34650}x^{11} - \frac{32}{81}x^{10} + \frac{509}{1134}x^9 - \frac{809}{2940}x^8 + \frac{533}{13230}x^7 + \frac{1}{1890}x^6 + \frac{61}{1050}x^5 - \frac{223}{3780}x^4 - \frac{2}{945}x^3 + \frac{456325}{12864852}x^2 - \frac{7498577}{482431950}x \quad (30)$$

The answers of the next terms and the ultimate answer of Eq. 25 are presented in Appendix 1.

Applying VIM approach to the given equation

To apply the construction of the variational iteration method, we have to find the first sentence to start the loop.

By considering the linear sentence with maximum power of derivative $\frac{d^3}{dx^3} f_0(x)$ which is also the multiplier of power p^0 in HPM method) and applying the corresponding boundary conditions, $f_0(x)$ can be found.

$$\begin{cases} \frac{d^3}{dx^3} f(x) + \frac{d^2}{dx^2} f(x) + 2f(x) \left(\frac{d}{dx} f(x) \right) + 4f(x) \left(\frac{d}{dx} f(x) \right) = 0 \\ \rightarrow f_0(x) = -x^2 + 2x \\ f(0) = 0, f(1) = 1, f'(1) = 0 \end{cases} \quad (31)$$

Based on the structure of VIM method, we construct the following formulation:

$$f_{n+1}(x) = f_n(x) + \lambda \int_0^x \left(\frac{d^3}{d\tau^3} f_n(\tau) + \frac{d^2}{d\tau^2} f_n(\tau) + 2f_n(\tau) \left(\frac{d}{d\tau} f_n(\tau) \right) + 4f_n(\tau) \left(\frac{d}{d\tau} f_n(\tau) \right) \right) d\tau \quad (32)$$

Considering an adequate Lagrange multiplier and setting n from 0 to 3, four terms of $f(x)$ are calculated, the last cycle of this answer is known as the ultimate solution of the given equation.

$$f_1(x) = \frac{1}{3}x^3 + 2x - \frac{1}{90}x^{10} + \frac{1}{9}x^9 - \frac{3}{7}x^8 + \frac{16}{21}x^7 - \frac{17}{30}x^6 + \frac{1}{5}x^5 = \frac{1}{3}x^4 - x^2 \quad (33)$$

The answers of the next terms and the ultimate answer of Eq. 25 are presented in Appendix 2.

Applying AGM approach to the given equation

We first consider the given equation in the form below:

$$U: \frac{d^3}{dx^3} f(x) + \frac{d^2}{dx^2} f(x) + 2f(x) \left(\frac{d}{dx} f(x) \right) + 4f(x)^3 \left(\frac{d}{dx} f(x) \right) = 0 \quad (34)$$

As it was discussed in the previous section, the answer of the differential equation is considered as a finite series of polynomials with constant coefficients:

$$f(x) = \sum_{i=0}^5 a_i \cdot x^i = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \quad (35)$$

The aforementioned unknown coefficients are capable of being computed by applying the boundary conditions.

a) Applying the boundary conditions on Eq. 36:

$$f(0) = 0 \rightarrow a_0 = 0 \quad (36)$$

$$f(1) = 1 \rightarrow a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1 \quad (37)$$

b) Applying the boundary conditions on the main differential Eq. 35 and its derivatives:

$$U(f(x)) \rightarrow U(f(BC)) = 0, U'(f(BC)) = 0, \dots \quad (38)$$

$$U(f(0), f(1)) = \begin{cases} U(0) = 0 \\ U(1) = 0 \end{cases} \rightarrow 4a_0^3 a_1 + 2a_0 a_1 + 2a_2 + 6a_3 = 0 \quad (39)$$

$$\begin{aligned} &80a_5 + 36a_4 + 12a_3 + 2a_2 + 2(a_5 + a_4 + a_3 + a_2 + a_1 + a_0) \\ &(5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1) + 4(a_5 + a_4 + a_3 + a_2 + a_1 + a_0)^3 \\ &(5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1) = 0 \end{aligned} \quad (40)$$

$$U'(f(0), f(1)) = \begin{cases} U'(0) = 0 \\ U'(1) = 0 \end{cases} \rightarrow 8a_0^2 a_2 + 12a_0 a_1^2 + 4a_2 a_1 + 2a_1^2 + 6a_3 + 24a_4 = 0 \quad (41)$$

$$\begin{aligned} &180a_5 + 48a_4 + 6a_3 + 2(5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1)^2 + \\ &2(a_5 + a_4 + a_3 + a_2 + a_1 + a_0)(20a_5 + 12a_4 + 6a_3 + 2a_2) + \\ &12(a_5 + a_4 + a_3 + a_2 + a_1)^2 (5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1)^2 + \\ &4(a_5 + a_4 + a_3 + a_2 + a_1 + a_0)^3 + (20a_5 + 12a_4 + 6a_3 + 2a_2) = 0 \end{aligned} \quad (42)$$

By solving a set of algebraic equations, six unknown coefficients are computed according to the existing six equations. By entering the values of the coefficients, the ultimate answer of AGM method is obtained.

Analogy between the methods is shown in Tables 3-6; Figures 4-15.

Conclusion

In this paper, we have successfully developed semi-analytical methods HPM, VIM & AGM to compute the given equation. We have utilized the Maple Package for our calculations. AGM & HPM methods were the most precise methods in this matter, so that they can be applied to the numerous questions arising in the fields of science and engineering day in day out [20-26].

Appendix 1

Description of the HPM approach: To elucidate on, consider the following equation:

$$A(u) - f(r) = 0; r \in \Omega \quad (43)$$

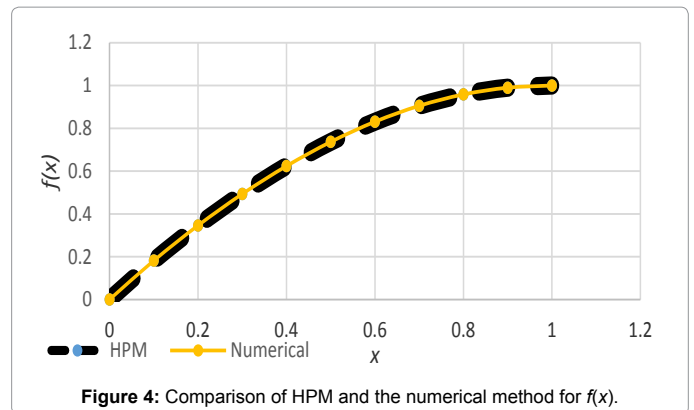


Figure 4: Comparison of HPM and the numerical method for $f(x)$.

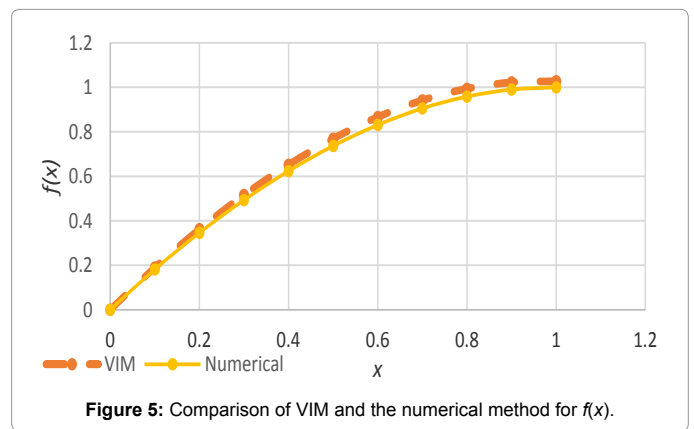


Figure 5: Comparison of VIM and the numerical method for $f(x)$.

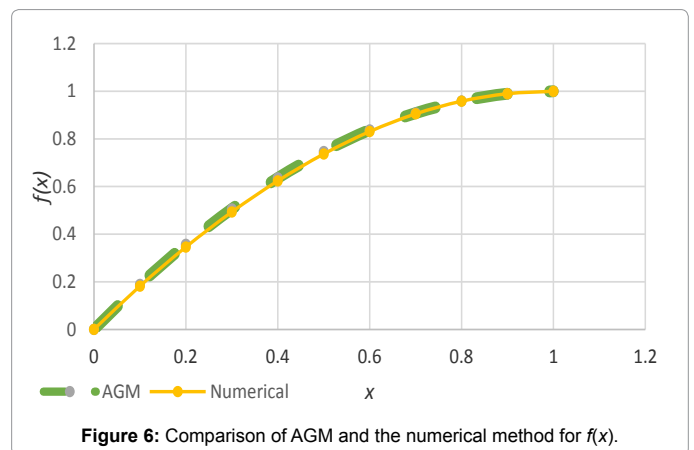


Figure 6: Comparison of AGM and the numerical method for $f(x)$.

The boundary conditions are:

$$B \left(u, \frac{\partial u}{\partial n} \right) = 0; r \in \Gamma \quad (44)$$

A: General differential operator, B: Boundary operator, $f(r)$: Known analytical function

Γ : Boundary of the domain Ω

The operator A consists of linear and nonlinear parts, so the Eq. 1 can be rewritten in the form below [3,5]:

$$L(u) + N(u) - f(r) = 0; r \in \Omega \quad (45)$$

$$L(u) + N(u) - f(r) = 0; r \in \Omega$$

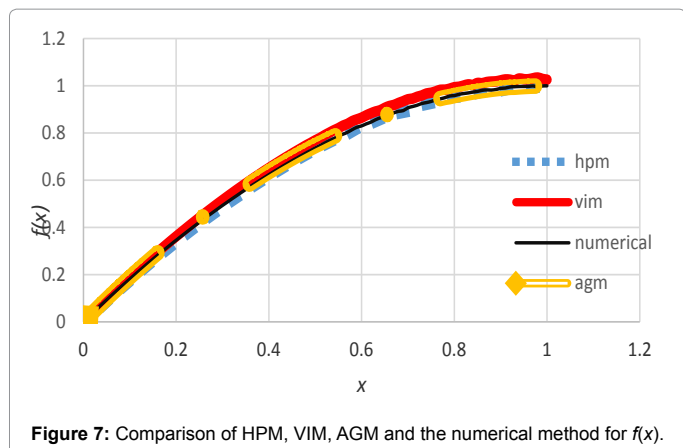


Figure 7: Comparison of HPM, VIM, AGM and the numerical method for $f(x)$.

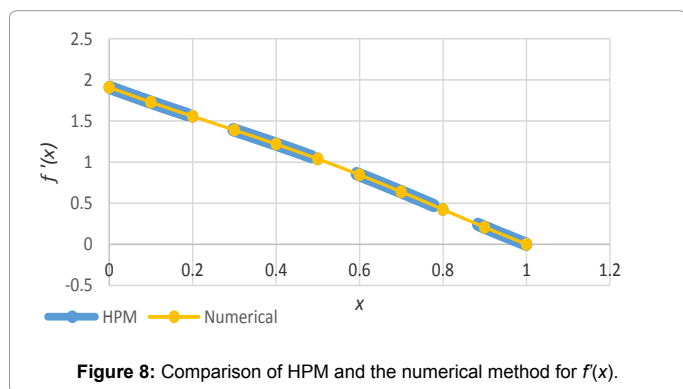


Figure 8: Comparison of HPM and the numerical method for $f'(x)$.

ζ	$\beta=0, \psi=1$		
	AGM	Numerical	Error
0	0.886827894	0.886818904	8.99E-06
0.1	0.887936656	0.887927655	9.00E-06
0.2	0.891265668	0.891256693	8.98E-06
0.3	0.896823156	0.896814335	8.82E-06
0.4	0.904622907	0.904614478	8.43E-06
0.5	0.914684345	0.914676629	7.72E-06
0.6	0.927032599	0.927025948	6.65E-06
0.7	0.941698575	0.941693312	5.26E-06
0.8	0.958719023	0.958715403	3.62E-06
0.9	0.978136613	0.978134783	1.83E-06
1	1	1	9.99E-16

Table 2: The results of AGM and the fourth-order Runge-Kutta for $\Theta(\zeta)$.

x	$f(x)$			
	HPM	VIM	AGM	Numerical
0	0	0	0	0
0.1	0.181728983	0.190293926	0.1886	0.181730225
0.2	0.34588862	0.36205851	0.357	0.345891022
0.3	0.493193118	0.515992529	0.5059	0.493196085
0.4	0.623637046	0.651986985	0.6358	0.623639949
0.5	0.736564905	0.769194539	0.7464	0.736567288
0.6	0.830836697	0.866221325	0.8376	0.830838346
0.7	0.905118449	0.941472686	0.9083	0.90511938
0.8	0.958272246	0.993612536	0.9587	0.958272631
0.9	0.989767158	1.022026176	0.9897	0.989767241
1	1.000000004	1.027139019	1	1

Table 3: Comparing the obtained charts by HPM, VIM, AGM and the numerical approach for $f(x)$.

L: Linear part, N: Nonlinear part

The introduced structure of homotopy perturbation method is as below:

$$H(v, p) = (1 - p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0 \quad (46)$$

While,

$$v(r, p): \Omega \times [0, 1] \rightarrow R \quad (47)$$

$p \in [0, 1]$: An embedding parameter, u_0 : First approximation satisfying the boundary condition

If the equation (4) be rewritten as a power series in p as below:

$$v = v_0 + pv_1 + p^2v_2 \quad (48)$$

The best approximation of the solution is considered as below:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 \quad (49)$$

Answer terms of the HPM approach to the Problem 2:

$$f_3(x) = -\frac{45391772903}{23717560741875}x + \frac{24547}{5542425}x^{23} - \frac{215183}{10319400}x^{22} + \frac{30009089}{438574500}x^{21} - \frac{266716803}{16665831000}x^{20} + \frac{2486051657}{9166207050}x^{19} - \frac{1503854581}{4341887550}x^{18} + \frac{182146189}{482431950}x^{17} - \frac{15942559}{37837800} + \frac{19616641}{425675250}x^{15} - \frac{61279531}{158918760}x^{14} + \frac{377021}{1719900}x^{13} - \frac{336601}{2619540}x^{12} + \frac{206711}{1559250}x^{11} - \frac{1467612833}{14472958500}x^{10} + \frac{10777231}{868377510}x^9 - \frac{204740047}{4502698200}x^8 - \frac{8932307}{241215975}x^7 + \frac{133508}{19348875}x^6 + \frac{1028171}{321621300}x^5 + \frac{1016179}{192972780}x^4 + \frac{481504330859}{60371972797500}x^2 - \frac{456325}{38594556}x^3 - \frac{7}{3442500}x^{26} + \frac{91}{1721250}x^{25} - \frac{9239}{14779800}x^{24} \quad (50)$$

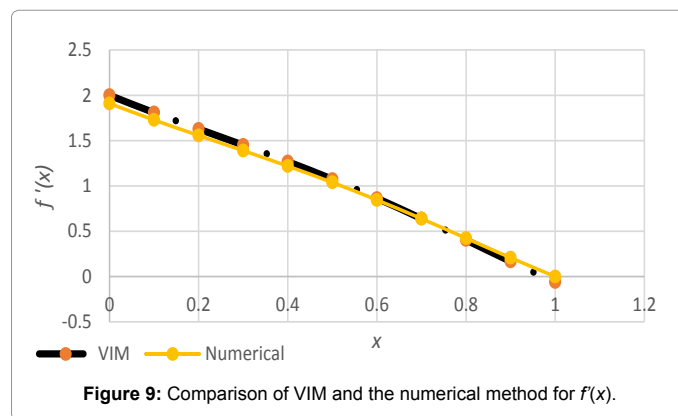


Figure 9: Comparison of VIM and the numerical method for $f(x)$.

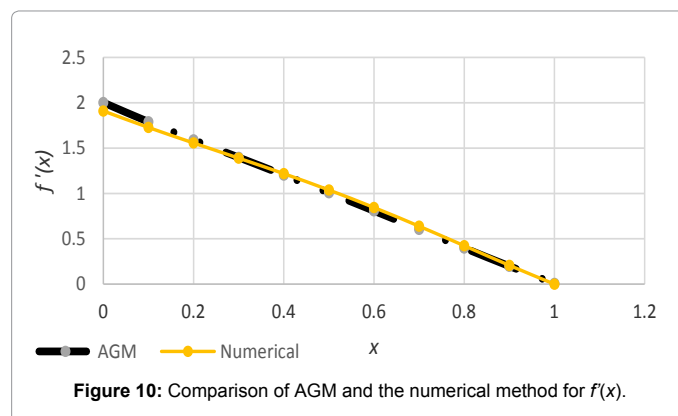


Figure 10: Comparison of AGM and the numerical method for $f(x)$.

$$f_3(x) = -\frac{405658497511479907}{3312868296642072918750}x + \frac{39836300049677}{116880139335960}x^{23} - \frac{51789939743231}{127043629713000}x^{22} + \frac{1768611653621}{3849806961000}x^{21} - \frac{193670612903}{434188755000}x^{20} + \frac{109811128723}{303932128500}x^{19} - \frac{320226806923}{1148188041000}x^{18} + \frac{61472796976}{258342309225}x^{17} - \frac{1845420119}{10131070950}x^{16} + \frac{83412542789}{1013107095000}x^{15} - \frac{26719578911}{13828911846750}x^{14} + \frac{3499052681}{263407844700}x^{13} - \frac{24226459}{17144889300}x^{12} - \frac{375595963}{39800635875}x^{11} - \frac{1673831481319}{90557959196250}x^{28} + \frac{103051068431}{1990284817500}x^{27} - \frac{2554138}{16567891875}x^{31} - \frac{204178760771}{197457204262500}x^{30} + \frac{33212328043}{6581906808750}x^{29} - \frac{199036157989699}{664091700725000}x^{10} - \frac{1748074812487}{51083976982500}x^9 - \frac{21217737858113}{1033031534535000}x^8 - \frac{19566395921389}{2789185143244500}x^7 + \frac{44325301833991}{19922751023175000}x^6 - \frac{1552062491257}{603719727975000}x^5 + \frac{27132296319721}{7969100409270000}x^4 - \frac{511}{17381182500}x^{34} + \frac{43992766081460126129}{43540554755867244075000}x^2 - \frac{481504330859}{18111591832500}x^3 + \frac{511}{511211250}x^{33} - \frac{465757}{29454030000}x^{32} - \frac{176523923566853}{1569671292735000}x^{26} + \frac{37694149682417}{195321088462500}x^{25} - \frac{463766994431}{1702799232750}x^{24} - \frac{1702799232750}{1702799232750}x^{24}$$

(51)

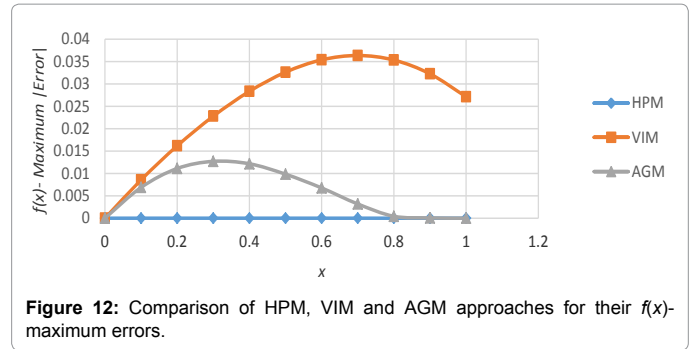


Figure 12: Comparison of HPM, VIM and AGM approaches for their $f(x)$ -maximum errors.

$$f(x) = \sum_{i=0}^{i=4} p_i' f_i(x) = \frac{72744466563256185043862}{38097985411383838565625}x + \frac{201769768723177}{584400696679800}x^{23} - \frac{27219544428883}{63521814856500}x^{22} + \frac{2032031436863}{3849806961000}x^{21} - \frac{2500130731321}{4124793172500}x^{20} + \frac{3652623989647}{5774710441500}x^{19} - \frac{6462716064487}{10333692369000}x^{18} + \frac{319374971831}{516684618450}x^{17} - \frac{25285944119}{40524283800}x^{16} - \frac{642057235969}{43540554755867244075000}x^{15} + \frac{3183909058579}{2789185143244500}x^{14} - \frac{40978926958}{19922751023175000}x^{13} + \frac{1013107095000}{5028695217000}x^{12} - \frac{65851961175}{2461033684}{x^{11}} + \frac{3904847398}{7969100409270000}x^{10} - \frac{1673831481319}{90557959196250}x^9 - \frac{4286222325}{1990284817500}x^8 + \frac{7960127175}{16567891875}x^7 - \frac{204178760771}{197457204262500}x^6 + \frac{103051068431}{1990284817500}x^5 - \frac{2554138}{16567891875}x^4 - \frac{204178760771}{197457204262500}x^3 + \frac{33212328043}{6581906808750}x^2 - \frac{3171738108384851}{664091700725000}x^{10} + \frac{21022637795657}{39064217692500}x^9 - \frac{329398467503081}{329398467503081}x^8 + \frac{2114611134289421}{11097224706015809}x^7 - \frac{11097224706015809}{11097224706015809}x^6 - \frac{516515767267500}{1718146300726423}x^5 + \frac{2789185143244500}{3057404630291279}x^4 - \frac{511}{17381182500}x^{34} + \frac{664091700725000}{4132843788240228922691}x^2 - \frac{315511458093463}{996137551158750}x^3 - \frac{511}{511211250}x^{33} - \frac{43540554755867244075000}{465757}x^{32} - \frac{176523923566853}{1569671292735000}x^{26} + \frac{640976092443559}{3320458503862500}x^{25} - \frac{29454030000}{1859325724679}x^{24} - \frac{6811196931000}{6811196931000}x^{24}$$

(52)

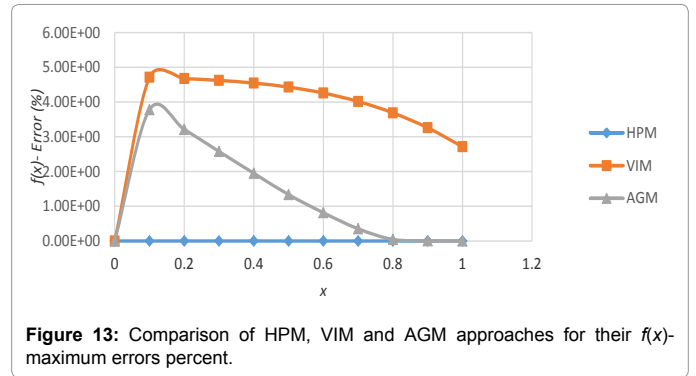


Figure 13: Comparison of HPM, VIM and AGM approaches for their $f(x)$ -maximum errors percent.

x	f(x)-Maximum Error		
	HPM	VIM	AGM
0	0	0	0
0.1	1.24E-06	0.008563701	0.006869775
0.2	2.40E-06	0.016167488	0.011108978
0.3	2.97E-06	0.022796445	0.012703915
0.4	2.90E-06	0.028347035	0.012160051
0.5	2.38E-06	0.032627251	0.009832712
0.6	1.65E-06	0.035382979	0.006761654
0.7	9.31E-07	0.036353306	0.00318062
0.8	3.86E-07	0.035339905	0.000427369
0.9	8.31E-08	0.032258935	6.72E-05
1	4.00E-09	0.027139019	0

Table 4: Comparing the obtained charts by HPM, VIM and AGM for their $f(x)$ -errors in accordance to the numerical approach.

Appendix 2

Description of the VIM approach: To clarify, note the equation below:

$$A(u) = g(x), A(u) = L(u) + N(u) \quad (53)$$

Where L & N represent the linear and nonlinear parts of the general differential operator A. $g(x)$ is the inhomogeneous term of the

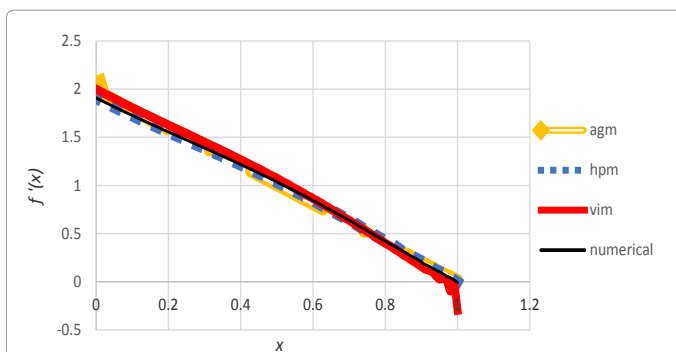


Figure 11: Comparison of HPM, VIM, AGM and the numerical method for $f(x)$.

x	f(x)			
	HPM	VIM	AGM	Numerical
0	1.909404547	2	2.0025	1.909413569
0.1	1.727633922	1.808441316	1.7894	1.727647369
0.2	1.556743091	1.628011282	1.5901	1.556752052
0.3	1.389328156	1.450478213	1.3971	1.389330476
0.4	1.218431025	1.267967474	1.2034	1.218427709
0.5	1.03817714	1.0738019	1.0077	1.03817046
0.6	0.844968441	0.863996832	0.80784	0.844968053
0.7	0.638698783	0.638722634	0.6042	0.638692269
0.8	0.423436239	0.402993554	0.3987	0.423431972
0.9	0.207100456	0.165978303	0.19825	0.20709865
1	-0.000000009	-0.061143663	0.0077515	0

Table 5: Comparing the obtained charts by HPM, VIM, AGM and the numerical approach for $f(x)$.

equation. The introduced structure of the VIM method is as below [3]:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(\tau) + Nu_n(\tau) - g(\tau)) d\tau \quad (54)$$

λ is the general Lagrange multiplier which can be obtained by disparate ways including variational theory, Laplace method and etc. the subscript n represents the n^{th} approximation of the solution, The appellation of \tilde{u}_n is justified by the fact that $\delta u_n \equiv 0$. The ultimate solution is given as:

$$u(x) = \lim_{n \rightarrow \infty} u_n(x) \quad (55)$$

Answer terms of the VIM approach of Problem 2:

$$f_2(x) = \frac{1}{3}x^3 + 2x - \frac{170347}{311850}x^{12} + \frac{7706}{17325}x^{11} - \frac{11}{1492627500}x^{40} + \frac{41}{447788250}x^{39} - \frac{902149}{1130033835000}x^{38} + \frac{154157}{29737732500}x^{37} - \frac{153389}{5907360375}x^{36} + \frac{7631233}{74388982500}x^{35} - \frac{3186179767}{9819345690000}x^{34} - \frac{2880696397}{343535142000}x^{33} - \frac{5686447603}{325561194000}x^{32} + \frac{1252554187}{406951492500}x^{31} - \frac{18124598909}{3806965575000}x^{30} + \frac{569117}{81033750}x^{29} - \frac{284583223}{26254935000}x^{28} + \frac{1300956379}{73138747500}x^{27} - \frac{2227520147}{77395500000}x^{26} + \frac{126270101}{29767500000}x^{25} - \frac{34726441}{625117500}x^{24} + \frac{510276677}{7531177500}x^{23} - \frac{43463209}{509355000}x^{22} + \frac{18378074}{156279375}x^{21} - \frac{188379067}{1131165000}x^{20} + \frac{12350041}{56558250}x^{19} - \frac{37887727}{151814250}x^{18} + \frac{1159157}{4498200}x^{17} - \frac{1110959}{3969000}x^{16} + \frac{208499}{595350}x^{15} - \frac{4735399}{10319400}x^{14} + \frac{408601}{737100}x^{13} - \frac{1}{112980420000}x^{12} + \frac{1}{2690010000}x^{11} - \frac{1244}{2835}x^{10} + \frac{1291}{2268}x^9 - \frac{1871}{2520}x^8 + \frac{187}{210}x^7 - \frac{29}{45}x^6 + \frac{4}{15}x^5 - \frac{5}{12}x^4 - x^2$$

$f_3(x), f_3(x)$; too many sentences.

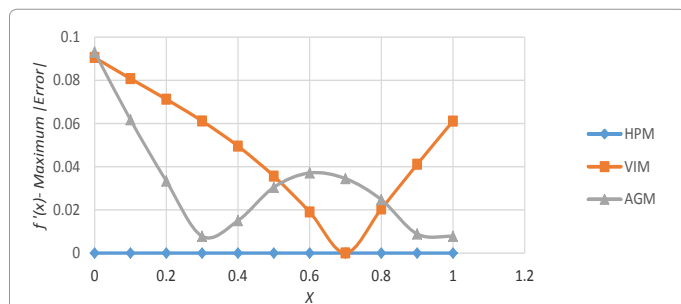


Figure 14: Comparison of HPM, VIM and AGM approaches for their $f'(x)$ -maximum errors.

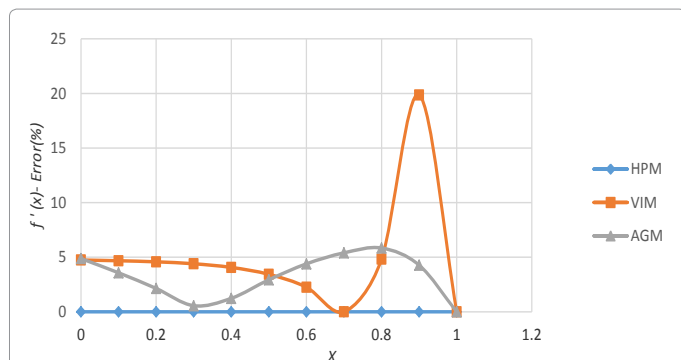


Figure 15: Comparison of HPM, VIM and AGM approaches for their $f'(x)$ -maximum errors percent.

x	F'(x)-Maximum Error		
	HPM	VIM	AGM
0.1	9.02E-06	0.090586431	0.093086431
0.2	1.34E-05	0.080793947	0.061752631
0.3	8.96E-06	0.07125923	0.033347948
0.4	2.32E-06	0.061147737	0.007769524
0.5	3.32E-06	0.049539765	0.015027709
0.6	6.68E-06	0.03563144	0.03047046
0.7	7.59E-06	0.01903598	0.037120853
0.8	6.51E-06	3.04E-50	0.034492269
0.9	4.27E-06	0.020438418	0.024731972
1	1.81E-06	0.041120346	0.00884865

Table 6: Comparing the obtained charts by HPM, VIM and AGM for their $f'(x)$ -errors in accordance to the numerical approach.

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