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Linear Stability Conditions for a First Order 4-Dimensional Discrete Dynamic

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Abstract

Linear stability conditions for a first order 4-dimensional discrete dynamic are derived in terms of the trace, sum of minors, sum of their minors, and the determinant of the Jacobian evaluated at the equilibrium.

Keywords: Stability; Discrete; Dynamic

Motivations

A standard method used to determine the linear stability of a first order discrete dynamic equates to determining whether or not the eigenvalues of the Jacobian evaluated at the equilibrium all have magnitude less than one. This is often quite algebraically cumbersome in many mathematical models where the elements of the Jacobian are expressed as a combination of many different parameters. In those cases the direct calculation of the eigenvalues is not at all practical and the Gershgorin theorem is too harsh as it only guarantees linear stability if all the discs happen to lie inside the unit circle. In the case of a non-linear 2-dimensional discrete dynamical system one can calculate the trace, tr (J), and determinant, det (J), of the Jacobian evaluated at the equilibrium and if use the set of inequalities.

$$det(J) < 1,$$
$$|tr(J)| < 1 + det(J)$$

that describe Thompson's [1] stability triangle. This set of two in equalities is necessary and sufficient for linear stability of the 2-dimensional discrete dynamic, that is, satisfying both conditions equates to both eigenvalues of the 2×2 Jacobian having magnitude less than 1. Linear stability conditions of a first order 3- dimensional discrete dynamic have been derived [2]. In the case of a non-linear 3-dimensional discrete dynamical system one can calculate the trace, tr (J), sum of principle minors, Σ M_i(J), and determinant, det(J), of the Jacobian evaluated at the equilibrium and if use the set of two inequalities

$$|\mathrm{tr(J)} + \mathrm{det(J)}| < 1 + \Sigma \mathrm{M_i(J)},$$
$$\Sigma \mathrm{M_i(J)} - \mathrm{tr(J)}\,\mathrm{det(J)} + (\mathrm{det(J)})^2 < 1$$

This set of two inequalities is necessary and sufficient for linear stability of the 3-dimensional discrete dynamic, that is, satisfying both conditions equates to all of the eigenvalues of the 3×3 Jacobian having magnitude less than 1. Note that the |det(J)|<1 condition in the 3-dimensional case [2] is always satisfied if the above two inequalities are satisfied. These conditions proved very useful in my dynamic evolutionary game theory research and generated much interest as a tool for other researchers as well. The 3-dimensional conditions have aided in researchers in economics [3,4], and finance [5]. Could an analogous approach be effective with a 4-dimensional discrete dynamic? As in the 3-dimensional case necessary and sufficient conditions are needed that equate to linear stability. These conditions will be derived in terms of the coefficient of the characteristic equation of the Jacobian such as the determinant, trace, principle sum of minors of the Jacobian, and sum of the determinants of the principal 2×2 submatrices of the Jacobian. These four elements are the magnitudes of the coefficients of the characteristic polynomial [6].

Stability Conditions

Linear stability conditions for a first order 4-dimensional discrete dynamic, $\overrightarrow{X_{t+1}} = \langle x_{t+1}, y_{t+1}, z_{t+1}, w_{t+1} \rangle = F(x_t, y_t, z_t, w_t)$, are equivalent to the necessary and sufficient conditions that the Jacobian, J, evaluated at the equilibrium has eigenvalues of magnitude less than one. These conditions will be expressed in terms of the determinant of the Jacobian, det(J), trace of the Jacobian, tr(J), sum of the determinants of the principal 2×2 submatrices of the Jacobian, Σ S_i(J), and the sum of principle minors of the Jacobian, Σ M_i(J), all evaluated at the equilibrium. The det(J), tr(J), Σ S_i(J) and Σ M_i (J) are all functions of the four eigenvalues, λ_1, λ_2, λ_3, λ_4 of the Jacobian.

$$\begin{aligned}
&\det(\mathrm{J}) = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \\
&tr(\mathrm{J}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\
&\sum S_i(\mathrm{J}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \\
&\sum M_i(\mathrm{J}) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4
\end{aligned} \tag{2.1}$$

To calculate Σ S_i(J), the sum of the determinants of the principal 2×2 submatrices of the Jacobian, one must add the six determinants of the six 2×2 principle submatrices that result from removing two rows and two columns of the Jacobian [6]. The determinant will provide the first necessary stability condition (C1), namely,

$$|\det(\mathrm{J})| = |\lambda_1 \lambda_2 \lambda_3 \lambda_4| < 1 \tag{2.2}$$

Following the same methods as the 3-dimensional case [2] we combine the four equations (2.1) to get a 6 degree polynomial, $P_2(x)$, in terms of $x = \lambda_i \lambda_j$.

$$\begin{aligned}
&P_2(x) = x6 - \left(\sum S_i(J)\right)x^5 + \left[\sum M_i(J)tr(J) - \det(J)\right]x^4 \\
&+\left[2\left(\sum S_i(J)\right)\det(J) - \left(\sum M_i(J)\right)^2 - \det(J)(tr(J))^2\right]x^3 \\
&+\left[\sum M_i(J)tr(J) - \det(J))^2\right]x^2 \\
&-\left[\left(\sum S_i(J)\right)(\det(J))^2\right]x + (\det(J))^3
\end{aligned} \tag{2.3}$$

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The second stability condition (C2) is $P_2(-1)>0$ and $P_2(1)>0$ which can be expressed as

$$|[1-\det(J)^2\left(\sum S_i(J)\right)+\left(\sum M_i(J)\right)^2+\det(J)(tr(J))^2|$$
$$<1+\left[tr(J)\sum M_i(J)-\det(J)\right](1+\det(J))+(\det(J))^3$$

Note that in the instances where two or four of the four eigenvalues of the Jacobian are complex the conditions (C1) and (C2) are necessary and sufficient to guarantee the magnitudes of the four eigenvalues are all less than one. This can be seen by noting that in the cases where two or four of the eigenvalues are complex condition (C2) requires that the two real roots of $P_2(x)$ must be both inside the interval (-1, 1) or both outside that interval. If both real roots of $P_2(x)$ are outside the interval (-1, 1) then condition (C1) cannot be met. If (C1) and (C2) are satisfied then at least 2 of the 4 eigenvalues are inside the unit circle. Next consider the characteristic polynomial of the Jacobian,

$$C(x)=x^4-tr(J)x^3+\sum S_i(J)x^2-\sum M_i(J)x+\det(J) \quad (2.4)$$

which has the four eigenvalues as roots. The connection between the coefficients of the characteristic polynomial and the eigenvalues are derived in Brooks et al., [6]. In order that this fourth degree characteristic polynomial to have at most two real roots outside the interval (-1, 1) we require $0 < C(-1)$ and $0 < C(1)$ in addition to conditions (C1) and (C2). This third necessary stability condition (C3) can be expressed as

$$|tr(J)+\sum M_i(J)|<1+\sum M_i(J)+\det(J) \quad (2.5)$$

If condition (C3) is violated then there exists at least one real eigenvalue outside the interval (-1, 1) which implies linear instability of the equilibrium. Lastly we will require that all the critical points of the characteristic polynomial be in the interval (-1, 1). This last condition is needed because condition (C4) can be met despite a pair of real eigenvalues both greater than 1(and less than -1). Thus the roots of the cubic

$$\frac{C'(x)}{4}=x^3-\frac{3}{4}tr(J)x^2+\frac{1}{2}\sum S_i(J)x-\frac{1}{4}\sum M_i(J)$$

must all be in the interval (-1, 1): If conditions (C1) (C2) and (C3) are met then at least one of the roots of the cubic must be in the interval (-1, 1): Using the results from Brooks et al., [2] which derives the conditions for a cubic to have roots of magnitude less than one we obtain the last necessary conditions (C4) and (C5) for the 4 dimensional case:

$$\frac{1}{2}\sum S_i(J)-\frac{3}{16}tr(J)\sum M_i(J)+\frac{1}{16}\left(\sum M_i(J)\right)^2<1 \quad (2.6)$$

$$|3tr(J)+\sum M_i(J)|<4+2\sum S_i(J) \quad (2.7)$$

Strictly applying the results from Brooks et al., [2] will result in additional redundant condition $\sum M_i(J)|<4$: Thus conditions (C3), (C4) and (C5) are met if and only if all the real eigenvalues are inside the interval (-1, 1): The conditions described in inequalities (C1), (C2), (C3), (C4) and (C5) are necessary and sufficient for all eigenvalues of the Jacobian to have magnitude less than 1. It can be shown that conditions (C3) (C4) and (C5) are satisfied when the set of four eigenvalues contain a complex conjugate pair and all the eigenvalues have magnitude less than one, whether the set is comprised of two complex conjugate pairs or one complex conjugate pair and two real eigenvalues.

To illustrate that each of the 5 conditions are necessary consider the case where the set of 4 eigenvalues of the Jacobian is $\frac{-465643561413}{500000000000}$, $\frac{429427177933}{500000000000}$, $\frac{687903492167}{500000000000}$, $\frac{1099444795477}{1000000000000}$.

This set will satisfy all stability conditions but C1. The set $\frac{40893981047}{500000000000}$, $\frac{96797253401}{250000000000}$, $\frac{244820359039}{62500000000}$, $\frac{601948647317}{200000000000}$ will satisfy all stability conditions but C2. The set $\frac{120431726877}{125000000000}$, $\frac{-143395695753}{250000000000}$, $\frac{398545776023}{500000000000}$, $\frac{313670975509}{250000000000}$ will satisfy all stability conditions but C3. The set

$\frac{187563689653}{250000000000}$, $\frac{-290004475873}{500000000000}$, $\frac{782236145093}{500000000000}$, $\frac{1230376756597}{1000000000000}$. will satisfy all stability conditions but C4. The set $\frac{32077947}{50000000}$, $\frac{7350716667}{62500000000}$, $\frac{231172538573}{200000000000}$, $\frac{3519315154111}{1000000000000}$ will satisfy all stability conditions but C5.

The goal of deriving necessary and sufficient conditions for linear stability of a first order 4-dimensional discrete dynamic has been met. These conditions can be applied without a priori knowledge of whether the set of eigenvalues contain complex conjugate pairs.

Theorem 1: The equilibrium of a 4-dimensional discrete dynamic of the form $\overrightarrow{X_{t+1}}=F\left(\overrightarrow{X_t}\right)$ is linearly stable if and only if the det (J), tr (J), ΣS_i (J) and ΣM_i (J) of the Jacobian evaluated at that equilibrium satisfy the following set of five inequalities

$$|\det(J)|<1,$$
$$|[1-\det(J)^2\left(\sum S_i(J)\right)+\left(\sum M_i(J)\right)^2+\det(J)(tr(J))^2|$$
$$<1+\left[tr(J)\sum M_i(J)-\det(J)\right](1+\det(J))+(\det(J))^3,$$
$$|tr(J)+\sum M_i(J)|<1+\sum S_i(J)+\det(J),$$
$$\frac{1}{2}\sum S_i(J)-\frac{3}{16}tr(J)\sum M_i(J)+\frac{1}{16}\left(\sum M_i(J)\right)^2<1$$
$$|3tr(J)+\sum M_i(J)|<4+2\sum S_i(J)$$

Theorem 2: If one of the 4 eigenvalues of the Jacobian is zero, that is, det(J) = 0, then the conditions resemble those for the stability in a 3-dimensional discrete dynamic. The set of inequalities becomes

$$|tr(J)+\sum M_i(J)|<1+\sum S_i(J),$$
$$\sum S_i(J)-tr(J)\sum M_i(J)+\left(\sum M_i(J)\right)^2<1$$

which match the stability conditions derived in Brooks et al., [2] where $\sum M_i(J)$ replaces the determinant and $\sum S_i(J)$ replace the sum of principle minors. Note that the $|\det(J)|<1$ condition in the 3-dimensional case [2] is always satisfied if the other two conditions are satisfied.

Theorem 3: If two of the eigenvalues are zero, that is, $\det(J)=\sum M_i(J)=0$, then the conditions resemble those for stability in a 2-dimensional discrete dynamic. The set of inequalities becomes

$$\sum S_i(J)<1,$$
$$|tr(J)|<1+\sum S_i(J)$$

and the resulting stability triangle is familiar. It resembles Thompson's

[1] stability triangle that results when the eigenvalues of a 2×2 Jacobian are required to be of magnitude less than 1. That triangle differs only by replacing $\sum S_i(J)$ by the determinant.

Conclusion

In mathematical models where the elements of the Jacobian are combinations of many parameters the inequalities derived in this paper provide necessary and sufficient conditions that allow stability conclusions to be made without the actual calculation of the eigenvalues. It is interesting to note the fact that in a 1-dimensional discrete dynamic, $x_{t+1}=F(x_t)$, there is one linear stability condition, $\left|\left|\frac{dF}{dx}\right| x_{equilibrium}\right| < 1$. In a 2-dimensional discrete dynamic, there are two linear stability conditions [1]. In a 3-dimensional discrete dynamic, there are two linear stability conditions [2]. This paper shows that there are five linear stability conditions in the 4-dimensional discrete dynamic.

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