2-Dimensional Algebras Application to Jordan, G-Associative and Hom-Associative Algebras

Remm E* and Goze M
Universite de Haute Alsace, LMIA, 4 rue des Freres Lumiere, 68093 Mulhouse, France

Abstract

We classify, up to isomorphism, the 2-dimensional algebras over a field \( \mathbb{K} \). We focus also on the case of characteristic 2, identifying the matrices of \( GL(2, \mathbb{F}) \) with the elements of the symmetric group \( \Sigma_3 \). The classification is then given by the study of the orbits of this group on a 3-dimensional plane, viewed as a Fano plane. As applications, we establish classifications of Jordan algebras, algebras of Lie type or Hom-Associative algebras.

Keywords: 2-Dimensional algebras; Classification; Hom-associative algebras

Introduction

An algebra \( A \) over a field \( \mathbb{K} \) is \( \mathbb{K} \)-vector space equipped with a product which corresponds to a bilinear map on \( A \) with values in \( A \). For a given dimension, one of the basic problems is the determination up to linear isomorphism of all these algebras. Sub classes of algebras where widely studied. These subclasses where often obtained setting a quadratic relation on \( \mu \). Among other examples of such classes are Lie algebras (in this case \( \mu \) is skewsymmetric and satisfies Jacobi identity), associative algebras, Lie-admissibles algebras, Pre-Lie algebras in particular. In all these examples, classifications where established in a general frame work, that is, with no other hypothesis on these classes and only in very small dimensions. For example for Lie algebras, we know the general classifications up to the dimension 6. In bigger dimension we impose additional algebraic properties if we hope to continue this classification. For example simple Lie algebras are fully classified since the work of Killing and Cartan, in any dimension. Unfortunately it is more and less the only solved case. If we consider complexe nilpotent Lie algebras, the classification is known only up to the dimension 7. It is the same for the associatives algebras. If we are only interested in general algebras, the only known cases are the dimension 2 and 3. It is true that the problem is equivalent to the classification of tensors of type (2,1) on a finite dimensional vector space. We are then facing to a basic multilinear algebra problem which is subject to a lack of informations on the tensors.

Here we reconsider this problem from the beginning, that is in dimension 2. This work is certainly not the first one of the subject. There is for example the work of Petersson. Our approach is not similar. We are not fully interested by the classification up to isomorphism but by the determination of subclasses, minimal in a certain sense, which are invariant up to isomorphism. The motivation comes from the constatation of what happen in greater dimensions for nilpotent Lie algebras for example In this case, the classification is established in dimension 7 but quasi unusable in its present forme. This means that if we have a precise example of nilpotent Lie algebra of this dimension, it is long and fastidious to recognize it in the given list because most of the time it is not adapted to the invariants used to established the classification. Moreover the length of the list can be puzzling. In greater dimensions, the number of isomorphiy classes, the need to write invariant parametrized families seems to be an unrealistic goal. Hence the idea to reduce the classification problem to a determination of invariant classes. This is the aim of this work. However we will established the link with Petersson’s work. Our approach is quite basic. In characteristic different from 2, we decompose a tensor \( \mu \) as a skewsymmetric and symmetric one. Since the skewsymmetric case is elementary, we classify those which are symmetric modulo the automorphism group of the associated skewsymmetric law. In characteristic 2, the problem is equivalent to the determination of the orbits of the Fano plane modulo the symmetric group. Finally, we use these results to describe or find again certain classes of algebras whose a direct approach is rather difficult. In particular, we determine the 2-dimensional Jordan algebras and we find again the results of ref. [1], the G-associative algebras and the Hom-associative algebras.

We have begun the study of the determination of general algebras in ref. [2] which was specially an introduction to a more precise work developed in this paper but with the same idea to describe "minimal" families invariant by isomorphism rather than a precise list for which the use is difficult. Recently, we were acquainted with the work of Pertersson, based on an Kaplansky result which permits to describe all the algebras from some unital algebras and to give isomorphism criteria. We try in this paper to look our description in a Petersson point of view. We note also a recent work, on the same subject of H. Ahmed, U. Bekbaev and I. Rakhimov [3].

Generalities

Let \( \mathbb{K} \) be a field whose characteristic will be precise later. An algebra over a field \( \mathbb{K} \) is a \( \mathbb{K} \)-vector space \( V \) with a multiplication given by a bilinear map

\[ \mu : V \times V \to V. \]

We denote by \( A=(V,\mu) \) a \( \mathbb{K} \)-algebra structure on \( V \) with multiplication \( \mu \). Throughout this paper we fix the vector space \( V \). Since we are interested by the 2-dimensional case we could assume that \( V=\mathbb{K}^2 \). Two \( \mathbb{K} \)-algebras \( A=(V,\mu) \) and \( A'=(V',\mu') \) are isomorphic if there is a linear isomorphism,

\[ \varphi : V \to V' \]

such that \( \varphi(\mu(a,b)) = \mu'(\varphi(a),\varphi(b)) \) for all \( a,b \in V \).

*Corresponding author: Elisabeth Remm, Universite de Haute Alsace, LMIA, 4 rue des Freres Lumiere, 68093 Mulhouse, France, Tel: 03 89 33 66 52; Fax: 03 89 33 66 53; E-mail: elisabeth.remm@uha.fr

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\[ f: V \rightarrow V \]
such as,

\[ f(\mu(X,Y)) = \mu(f(X),f(Y)) \]

for all \(X,Y \in V\). The classification of 2-dimensional \(\mathbb{K}\)-algebras is then equivalent to the classification of bilinear maps on \(V \otimes \mathbb{K}^2\) with values in \(V\). Let \(\{e_1,e_2\}\) be a basis of \(V\). A general bilinear map \(\mu\) has the following expression:

\[
\begin{align*}
\mu(e_1,e_1) &= \alpha_1 e_1 + \beta_1 e_2, \\
\mu(e_1,e_2) &= \alpha_2 e_1 + \beta_2 e_2, \\
\mu(e_2,e_1) &= \alpha_3 e_1 + \beta_3 e_2, \\
\mu(e_2,e_2) &= \alpha_4 e_1 + \beta_4 e_2,
\end{align*}
\]

and it is defined by 8 parameters. Let \(f\) be a linear isomorphism of \(V\). In the given basis, its matrix \(M\) is non-degenerate. If we put:

\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

Then,

\[ M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

with \(\Delta = ad - bc \neq 0\). The isomorphic multiplication:

\[ \mu = f^{-1} \circ \mu \circ (f \times f) \]

Satisfies,

\[
\begin{align*}
\mu'(e_1,e_1) &= \alpha_1' e_1 + \beta_1' e_2, \\
\mu'(e_1,e_2) &= \alpha_2' e_1 + \beta_2' e_2, \\
\mu'(e_2,e_1) &= \alpha_3' e_1 + \beta_3' e_2, \\
\mu'(e_2,e_2) &= \alpha_4' e_1 + \beta_4' e_2,
\end{align*}
\]

With,

\[
\begin{align*}
\alpha_1' &= (a' a + acc + c' c) \alpha_1 - (a' \beta_1 + ac\beta_1 + ac\beta_1 + c' \beta_1) \frac{b}{\Delta}, \\
\beta_1' &= (a' a + acc + acc) \beta_1 - (a' \beta_1 + ac\beta_1 + ac\beta_1 + c' \beta_1) \frac{a}{\Delta}, \\
\alpha_2' &= (b' a + bca + c' c) \alpha_2 - (b' \beta_1 + bc\beta_1 + bc\beta_1 + c' \beta_1) \frac{d}{\Delta}, \\
\beta_2' &= (b' a + bca + cca + c' c) \beta_2 - (b' \beta_1 + bc\beta_1 + bc\beta_1 + c' \beta_1) \frac{c}{\Delta}, \\
\alpha_3' &= (a' b + bca + cca + c' c) \alpha_3 - (a' \beta_1 + ab\beta_1 + ab\beta_1 + c' \beta_1) \frac{b}{\Delta}, \\
\beta_3' &= (a' b + bca + cca + c' c) \beta_3 - (a' \beta_1 + ab\beta_1 + ab\beta_1 + c' \beta_1) \frac{c}{\Delta}, \\
\alpha_4' &= (b' b + bca + cca + c' c) \alpha_4 - (b' \beta_1 + bc\beta_1 + bc\beta_1 + c' \beta_1) \frac{d}{\Delta}, \\
\beta_4' &= (b' b + bca + cca + c' c) \beta_4 - (b' \beta_1 + bc\beta_1 + bc\beta_1 + c' \beta_1) \frac{b}{\Delta}.
\end{align*}
\]

These formulae describe an action of the linear group \(GL(2, \mathbb{K})\) on \(\mathbb{K}^2\) parameterized by the structure constants \((\alpha_i, \beta_i)\), \(i = 1,2,3,4\) and the problem of classification consists in describing an element of each orbit.

**Algebras Over a Field of Characteristic Different from 2**

We assume in this section that \(\text{char}(\mathbb{K}) \neq 2\). We consider the bilinear map \(\mu\) and \(\mu'\) given by:

\[
\mu(X,Y) = \frac{\mu(X,Y) - \mu(Y,X)}{2}, \quad \mu(X,Y) = \frac{\mu(X,Y) + \mu(Y,X)}{2}
\]

for all \(X,Y \in V\). The multiplication \(\mu\) is skew-symmetric and it is a Lie multiplication (any skew-symmetric bilinear application in \(\mathbb{K}^2\) is a Lie bracket). It is isomorphic to one of the following:

1. \(\mu_1(e_i,e_j) = e_i\),
2. \(\mu_2 = 0\).

In fact, if \(\mu_i\) is not trivial, thus \(\mu_i(e_i,e_j) = \alpha e_i + \beta e_j\). If \(\alpha \neq 0\), we consider the change of basis:

\[ e'_1 = \alpha e_1 + \beta e_2, \quad e'_2 = \alpha' e_1 + \beta' e_2 \]

We have \(\mu_i(e'_i,e'_j) = \mu_i(\alpha e_1 + \beta e_2, \alpha' e_1 + \beta' e_2) = \mu_i(e_i,e_j) = \alpha e_i + \beta e_j = e'_i\).

If \(\alpha = 0\), then \(\beta = 0\) and we take:

\[ e'_1 = e_1, \quad e'_2 = -\beta e_1. \]

This gives \(\mu_i(e'_i,e'_j) = \mu_i(e_1,-\beta e_2,e_1) = \beta e_2 = e'_1\). In any case, if \(\mu \neq 0\), then it is isomorphic to \(\mu_2\).

**Case \(\mu_i'(e_i,e_j) = e_i\)**

An automorphism of the Lie algebra \((A,\mu_1)\) is a linear isomorphism \(f \in GL(2, \mathbb{K})\) such that:

\[ f(\mu_1(X,Y)) = \mu_1(f(X),f(Y)) \]

for every \(X,Y \in A\). The set of automorphisms of this Lie algebra is denoted by \(\text{Aut}(\mu_1)\).

**Lemma 1:** We have:

\[ \text{Aut}(\mu_1)' = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{K}, \quad a \neq 0 \cdot \]

**Proof:** In fact, assume that \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is the matrix of the automorphism \(f\) in the given basis \(\{e_1,e_2\}\). Then,

\[ f(\mu_1'(e_1,e_2)) = f(e_i) = ae_i + ce_j, \]

and,

\[ \mu_1'(f(e_1), f(e_2)) = \mu_1'(ae_1 + ce_2, be_1 + de_2) = (ad - bc)e_i. \]

Then,

\[ c = 0, a = ad. \]

But \(\det M = ad \neq 0\) so \(a = ad\) implies that \(d = 1\) This gives the lemma.

Let \(\mu\) be a general multiplication of 2-dimensional \(\mathbb{K}\)-algebra such that \(\mu_i\) is isomorphic to \(\mu_1\). It is isomorphic to the bilinear map (always denoted by \(\mu\)) whose structure constants are given by:

\[
\begin{align*}
\mu(e_i,e_j) &= \alpha_1 e_i + \beta_1 e_j, \\
\mu(e_j,e_i) &= (\alpha_1 + 1) e_i + \beta_2 e_j, \\
\mu(e_i,e_i) &= (\alpha_1 - 1) e_i + \beta_2 e_i, \\
\mu(e_i,e_j) &= \alpha_1 e_i + \beta_2 e_j.
\end{align*}
\]

The classification, up to isomorphism, of the Lie algebras \((V,\mu_i)\) such that \(\mu_i\) is isomorphic to \(\mu_1\) is equivalent to the classification up an isomorphism belonging to \(\text{Aut}(\mu_1)\) of the abelian algebras isomorphic to:

\[
\begin{align*}
\mu(e_i,e_j) &= \alpha_1 e_i + \beta_1 e_j, \\
\mu(e_j,e_i) &= \mu(e_i,e_i) = \alpha_1 e_i + \beta_2 e_j, \\
\mu(e_i,e_j) &= \alpha_1 e_i + \beta_2 e_j.
\end{align*}
\]
In this case (1) is reduced to:
\[ \begin{align*}
\alpha'_1 &= aa_1 - ab_1 \\
\beta'_1 &= a^2 \beta_1 \\
\alpha'_2 &= b a_1 + \alpha_2 - b^2 \beta_1 - b \beta_2 \\
\beta'_2 &= ab_1 + a \beta_2 \\
\alpha'_3 &= (b^2 a_1 + 2 b a_2 + \alpha_3 - b^3 \beta_1 - b \beta_2) \frac{1}{a} \\
\beta'_3 &= b^2 \beta_1 + 2 b \beta_2 + \beta_3 
\end{align*} \] (2)

1. Assume that $\beta \neq 0$.
   - Suppose that $K$ is algebraically closed and consider the isomorphism $\left( \sqrt{\beta}, \frac{a}{\sqrt{\beta}} \right)$. The isomorphic algebra is such that $\alpha'_1 = 0$ and $\beta'_1 = 1$. We deduce that in this case $\mu$ is isomorphic to:
   \[
   \begin{align*}
   \mu_1(e_1, e_1) &= e_2 \\
   \mu_2(e_1, e_2) &= \alpha_1 e_1 + \beta_1 e_2 \\
   \mu_3(e_2, e_2) &= \alpha_2 e_1 + \beta_2 e_2
   \end{align*}
   \]
   Then $\mu$ is isomorphic to:
   \[
   \begin{align*}
   \mu_{1,2,3}(e_1, e_1, e_2) &= e_2 \\
   \mu_{1,2,3}(e_1, e_2, e_2) &= (\alpha_1 + 1) e_1 + \beta_1 e_2 \\
   \mu_{1,2,3}(e_2, e_1, e_2) &= (\alpha_2 - 1) e_1 + \beta_2 e_2 \\
   \mu_{1,2,3}(e_2, e_2, e_2) &= \alpha_2 e_1 + \beta_2 e_2
   \end{align*}
   \]
   with $\alpha, \beta, \alpha_1, \beta_1 \in K$.

2. Assume $\beta = 0, \beta \neq 0$. In this case (1) is reduced to:
\[ \begin{align*}
\alpha'_1 &= aa_1 \\
\beta'_1 &= 0 \\
\alpha'_2 &= b a_1 + \alpha_2 - b \beta_2 \\
\beta'_2 &= a \beta_2 \\
\alpha'_3 &= (b^2 a_1 + 2 b a_2 + \alpha_3 - b^2 \beta_1 - b \beta_2) \frac{1}{a} \\
\beta'_3 &= 2 b \beta_2 + \beta_3 
\end{align*} \] (3)
   and taking $b = -\beta_2 / 2 \beta_3$ and $a = \beta_3^{-1}$, we see that $\mu$ is isomorphic to:
   \[
   \begin{align*}
   \mu_1(e_1, e_1) &= e_2 \\
   \mu_2(e_1, e_2) &= \alpha_1 e_1 + e_2 \\
   \mu_3(e_2, e_2) &= \alpha_2 e_1 + e_2
   \end{align*}
   \]
   We obtain the following multiplication, $K$ being algebraically closed or not:
   \[
   \begin{align*}
   \mu_{1,2,3}(e_1, e_1, e_2) &= e_2 \\
   \mu_{1,2,3}(e_1, e_2, e_2) &= (\alpha_1 + 1) e_1 + \beta_1 e_2 \\
   \mu_{1,2,3}(e_2, e_1, e_2) &= (\alpha_2 - 1) e_1 + \beta_2 e_2 \\
   \mu_{1,2,3}(e_2, e_2, e_2) &= \alpha_2 e_1 + \beta_2 e_2
   \end{align*}
   \]
3. Assume now that $\beta_1 = \beta_2 = 0, \alpha_1 \neq 0$. In this case (1) is reduced to:
\[ \begin{align*}
\alpha'_1 &= \alpha_1 \\
\beta'_1 &= 0 \\
\alpha'_2 &= 2 a \alpha_1 + 2 \\
\beta'_2 &= \alpha_2 \\
\alpha'_3 &= (b^2 a_1 + 2 b a_2 + \alpha_3 - b \beta_2) \frac{1}{a} \\
\beta'_3 &= \beta_3
   \end{align*} \]
   and taking $b = -\alpha_1 / a$ and $a = \alpha_1^{-1}$, we obtain $\alpha'_2 = 0$ and $\alpha'_1 = 1$. In this case, $\mu$ is isomorphic to:
   \[
   \begin{align*}
   \mu_{1,2,3}(e_1, e_1) &= e_1 \\
   \mu_{1,2,3}(e_1, e_2) &= \alpha_1 e_1 + e_2 \\
   \mu_{1,2,3}(e_2, e_2) &= -e_1 \\
   \mu_{1,2,3}(e_2, e_3) &= \alpha_1 e_1 + \beta_2 e_2
   \end{align*}
   \]
4. Assume now that $\beta_1 = \beta_2 = 0, \alpha_1 = 0, 2 a \alpha_2 - \beta_3 \neq 0$. In this case, considering $b = -\alpha_2 / (2 a \alpha_2 - \beta_3)$, the Lie bracket $\mu$ is isomorphic to:
\[ \begin{align*}
\mu_{1,2,3}(e_1, e_1) &= 0 \\
\mu_{1,2,3}(e_1, e_2) &= (\alpha_2 + 1) e_1 \\
\mu_{1,2,3}(e_2, e_2) &= (\alpha_2 - 1) e_1 \\
\mu_{1,2,3}(e_2, e_3) &= e_1 + 2 \alpha_2 e_2
   \end{align*} \]
5. Assume now that $\beta_1 = \beta_2 = 0, \alpha_1 = 0, 2 a \alpha_2 - \beta_4 = 0, \alpha_4 \neq 0$. Then $\mu$ is isomorphic to $\mu_{1,2,3}$, with $b = 2 a_2$.

**Theorem 2:** Any 2-dimensional non-commutative algebras isomorphic to one of the following algebras:

- If $K$ is algebraically closed:
\[
\begin{align*}
\mu_{1,2,3}(e_1, e_1) &= e_2 \\
\mu_{1,2,3}(e_1, e_2) &= \alpha_1 e_1 + e_2 \\
\mu_{1,2,3}(e_2, e_2) &= \alpha_2 e_1 + e_2
   \end{align*}
\]
- If $K$ is not algebraically closed (for example if $K$ is a finite field), let $K'$ be the multiplicative subgroup of elements $a^2$ with $a \in K$. In this case $\mu$ is isomorphic to a Lie bracket belonging to the 4 parameters family:
   \[
   \begin{align*}
   \mu_{1,2,3}(e_1, e_1) &= e_2 \\
   \mu_{1,2,3}(e_1, e_2) &= (\alpha_1 + 1) e_1 \\
   \mu_{1,2,3}(e_2, e_2) &= (\alpha_2 - 1) e_1 \\
   \mu_{1,2,3}(e_2, e_3) &= e_1 + 2 \alpha_2 e_2
   \end{align*}
   \]
with $\alpha, \beta \in \mathbb{K}$.

- If $\mathbb{K}$ is not algebraically closed:
  \[
  \begin{align*}
  \varphi_{\alpha,\beta}^2(e_i, e_j) &= \lambda_i e_i, \\
  \varphi_{\alpha,\beta}^2(e_i, e_j) &= (\alpha_1 + 1)e_i + \beta e_j, \\
  \varphi_{\alpha,\beta}^2(e_i, e_j) &= (\alpha_1 - 1)e_i + \beta e_j, \\
  \varphi_{\alpha,\beta}^2(e_i, e_j) &= \alpha_1 e_i + \beta e_j, \\
  \varphi_{\alpha,\beta}^2(e_i, e_j) &= \alpha_1 e_i + \beta^* e_j, \\
  \varphi_{\alpha,\beta}^2(e_i, e_j) &= \alpha_1 e_i + \beta e_j.
  \end{align*}
  \]

\[\alpha_1, \beta \in \mathbb{K}, \lambda \in \mathbb{K}/(\mathbb{K})^2.\]

Let us make the link with the results of Peterson [4]. The main idea of this work is to construct unital algebras from unital algebra. Recall that an algebra $A=(\mathbb{V}, \mu)$ is called unital if there exists $1 \in \mathbb{V}$ such that $\mu(1, X) = \mu(X, 1) = X$ for any $X \in \mathbb{V}$ for any $X \in \mathbb{V}$.

**Lemma 3:** If $\mu_1$ is not trivial, then $A$ is not unital.

Proof: Assume that there exists 1 satisfying $\mu(1, X) = \mu(1, X) = X$, then:
\[
0 = \mu(1, X) - \mu(1, X) = \mu(a(1, X) - \mu(a(1, X) = 2\mu(a(1, X)
\]
for any $X \in \mathbb{V}$. Then $\mu(1, X) = 0$ for any $X$ and 1 is in the center of $A = \mathbb{V}$, $\mu$). But if $\mu_1$ is not trivial, the center of $A$ is reduced to $\{0\}$. The algebra $A$ cannot be unital.

The algebra $A=(\mathbb{V}, \mu)$ is called regular if there exists $U, T \subseteq \mathbb{V}$ such that the linear applications:
\[
L_u : X \rightarrow \mu(U, X), \quad R_t : X \rightarrow \mu(X, T)
\]
are linear isomorphisms. From ref. [5], for any regular algebra $A=(\mathbb{V}, \mu)$ there exist a unique, up an isomorphism, unital algebra $B=(\mathbb{V}, \mu)$ and two linear isomorphisms $f, g$ of $V$ such that:
\[
\mu(X, Y) = \mu(f(X), g(Y))
\]
for any $X, Y \in \mathbb{V}$. The algebra $B$ is called the unital heart of $A$. To compare Theorem 2 with the Peterson results, we have to determine the regular algebras. Let us consider the first family. The application $L_u$ is not regular for any $U$ if only if its determinant is identically null that:
\[
\alpha_2 = -1, \quad \alpha_4 = -2\beta_1, \quad \beta_4 = \beta_2^2.
\]
Likewise $R_t$ is not regular for any $T$ if only if its determinant is identically null that:
\[
\alpha_2 = 1, \quad \alpha_4 = 2\beta_1, \quad \beta_4 = \beta_2^2.
\]

We deduce that any algebra $A_{\alpha,\beta} = (\mathbb{V}, \mu_{\alpha,\beta})$ is regular except the algebras given by:
\[
\begin{align*}
\mu_{\alpha,\beta}^1(e_i, e_j) &= e_i, \\
\mu_{\alpha,\beta}^2(e_i, e_j) &= \alpha_1 e_i + \beta e_j, \\
\mu_{\alpha,\beta}^3(e_i, e_j) &= 2\alpha_1 e_i + \beta e_j, \\
\mu_{\alpha,\beta}^4(e_i, e_j) &= 2\beta_1 e_i + \beta_2 e_j.
\end{align*}
\]

Let us note that $A_{\alpha,\beta}^2$ is left-singular but right-regular and $A_{\alpha,\beta}^4$ is right-singular and left-regular. An algebra which is left and right singular is called bi-singular. We can summarize the results in the following array:

1. $A_{\alpha,\beta}^1$ is regular except $A_{\alpha,\beta}^2$ and $A_{\alpha,\beta}^3$.
2. $A_{\alpha,\beta}^4$ is left-singular and right-regular.
3. $A_{\alpha,\beta}^5$ is right-singular and left-regular.
4. $A_{\alpha,\beta}^6$ is regular.
5. $A_{\alpha,\beta}^7$ is regular except $A_{\alpha,\beta}^8$.
6. $A_{\alpha,\beta}^9$ is bisingular.
7. $A_{\alpha,\beta}^{10}$ is regular except $A_{\alpha,\beta}^{11}, A_{\alpha,\beta}^{12}$.
8. $A_{\alpha,\beta}^{13}$ is bisingular.
9. $A_{\alpha,\beta}^{14}$ is left-singular and right-regular as soon as $\beta \neq 0$.
10. $A_{\alpha,\beta}^{15}$ is left-regular and right-singular as soon as $\beta \neq 0$.
11. $A_{\alpha,\beta}^{16}$ is regular except for $\alpha = 0, 1$ or $-1$.
12. $A_{\alpha,\beta}^{17}$ is bisingular.
13. $A_{\alpha,\beta}^{18}$ is left-singular and right-regular as soon as $\beta \neq 0$.
14. $A_{\alpha,\beta}^{19}$ is left-regular and right-singular as soon as $\beta \neq 0$.

We deduce.

**Proposition 4:** We consider the following algebras,
\[
1. A_{\alpha,\beta}^{20} = (\mathbb{V}, \mu_{\alpha,\beta}) \; \text{with} \; (\alpha, \beta) \in \{(-1, \beta, -2\beta, \beta^2) \} \; \text{or} \; (1, \beta, 2\beta, \beta^2)
\]
\[
2. A_{\alpha,\beta}^{21} = (\mathbb{V}, \mu_{\alpha,\beta}) \; \text{with} \; (\alpha, \beta) \neq (0, \beta) \; \text{or} \; (-1, \beta)
\]
\[
3. A_{\alpha,\beta}^{22} = (\mathbb{V}, \mu_{\alpha,\beta}) \; \text{with} \; (\alpha, \beta) \neq (\alpha, 0) \; \text{or} \; (1, \beta)
\]
\[
4. A_{\alpha,\beta}^{23} = (\mathbb{V}, \mu_{\alpha,\beta}) \; \text{with} \; \alpha \neq 0, 1, -1.
\]

For any of these algebras $A$, there exists an unital $\mathbb{K}$ algebra $B=(\mathbb{V}, \mu_{\alpha,\beta})$ and linear endomorphisms $f, g$ such that the multiplication of $A$ is given by:
\[
\mu_{\alpha,\beta}(X, Y) = \mu_{\alpha,\beta}(f(X), g(Y)).
\]

This unital algebra $B_{\alpha,\beta}$ is called the unital heart of $A$. Since $B_{\alpha,\beta}$ is unital, then [5] it is an etale algebra, that is $B_{\alpha,\beta} \otimes \mathbb{K} = \mathbb{K}$ is the algebraic closure of $B_{\alpha,\beta}$ or $B_{\alpha,\beta}$ is isomorphic to the dual algebra defined by:
\[
\mu_{\alpha,\beta}(e_i, e_j) = \mu_{\alpha,\beta}(e_i, e_j) = e_i, \; \beta = 1, \beta_2
\]
and $\mu_{\alpha,\beta}(e_i, e_j) = 0$. To find this heart algebra we use the Kaplansky’s Trick. If $A$ is regular, we consider $U$ and $V$ such that $L_u$ and $R_t$ are non singular and $f = L_{\alpha,\beta}^{-1}, g = R_{\alpha,\beta}^{-1}$. The multiplication $\mu_{\alpha,\beta}$ of the heart $B_{\alpha,\beta}(X, Y) = \mu_{\alpha,\beta}(f(X), g(Y))$ and the identity of $B_{\alpha,\beta}$ is $1 = \mu(U, T)$.

1. Let be $A_{\alpha,\beta}^{20}$. If $\alpha \neq 1$ or $-1$ then $L_u$ and $R_t$ are not singular. In fact,
\[
L_u = \begin{pmatrix} \alpha_2 + 1 \\ \beta \end{pmatrix}, \quad R_t = \begin{pmatrix} \alpha_2 - 1 \\ \beta \end{pmatrix}
\]

Thus,
\[
f = \begin{pmatrix} -1 \\ \alpha_2 + 1 \end{pmatrix}, \quad g = \begin{pmatrix} -1 \\ \alpha_2 - 1 \end{pmatrix}
\]

Then the identity element of $B_{\alpha,\beta}$ is $e_i$, and
\[
\mu_{\alpha,\beta}(e_i, e_j) = \mu_{\alpha,\beta}(f(e_i), g(e_j)) = \frac{1}{\alpha_2 - 1}(\beta \beta_{\alpha,\beta}^2 - e_i)^2
\]
and $B_{\alpha,\beta}$ is etale. If $\alpha = 1$, then we can take $U = e_{\alpha,\beta}^2$ and $V = \alpha - e_i$ as soon as $\alpha \beta \neq 2\beta$. If not we take $U = e_{\alpha,\beta}^2$ and $V = e_i$. We have the same calculc for $\alpha = -1$.

2. Let be $A_{\alpha,\beta}^{21}$. This algebra is regular. If $\alpha \neq 0$, then $L_u$ and $R_t$ are not singular and $B_{\alpha,\beta}$ is etale.
Case $\mu_\lambda(e_1, e_2) = 0$

The multiplication $\mu$ is symmetric. The group of automorphisms of $\mu_\lambda$ is $\text{GL}(2, \mathbb{K})$. Moreover the multiplication writes:

\[
\begin{align*}
\mu(e_1, e_1) &= \alpha e_1 + \beta e_2, \\
\mu(e_1, e_2) &= \alpha e_1 + \beta e_2, \\
\mu(e_2, e_1) &= \alpha e_2 + \beta e_1, \\
\mu(e_2, e_2) &= \alpha e_2 + \beta e_1.
\end{align*}
\]

We assume that there exists two independent idempotent vectors. If $e_1$ and $e_2$ are these vectors, then:

\[
\begin{align*}
\mu(e_1, e_1) &= e_1, \\
\mu(e_1, e_2) &= e_1 + \beta e_2, \\
\mu(e_2, e_1) &= \alpha e_1 + e_2, \\
\mu(e_2, e_2) &= \alpha e_1 + \beta e_2.
\end{align*}
\]

Remark that if any element is idempotent, thus $\mu(e_1, e_1) = \mu(e_2, e_2) = 0$. In fact:

\[
\begin{align*}
\mu(e_1 + e_2, e_1 + e_2) &= e_1 + e_2 + \mu(e_1, e_2) + \mu(e_2, e_1) + 2\mu(e_1, e_2) \\
&= e_1 + e_2 + 2\mu(e_1, e_2).
\end{align*}
\]

In the general case, if $ae_1 + be_2$ is an idempotent with $ab \neq 0$, then $a$ and $b$ satisfy the system:

\[
\begin{align*}
2abx &= a, \\
2b^2 &= b.
\end{align*}
\]

If $4\alpha \beta = 1$, then the system has solutions as soon as $\xi = \eta = 1/2$. In this case we obtain the multiplication $\mu_{\xi, \eta}$ and for any $\alpha$, the vectors $ae_1 + (1-\alpha)e_2$ are idempotent. If $4\alpha \beta 
eq 1$, the vector:

\[
v = \frac{1-2\xi}{1-4\alpha \beta} e_1 - \frac{1-2\eta}{1-4\alpha \beta} e_2
\]

is an idempotent and the only idempotents are $e_1, e_2$ and $v$. The changes of basis $[e_1, v]$ or $[e_2, v]$ do not simplify the number of independent parameters.

We assume that there exists only one idempotent vector. If $e_1$ is this vector, thus $\mu(e_1, e_1) = e_1$. If we consider a vector $v = xe_1 + ye_2$ such that $\mu(x, v) = v$, then $x$ and $y$ have to satisfy:

\[
\begin{align*}
x^2 + 2xy\alpha + y^2\alpha &= x, \\
2xy\beta + y^2\beta &= y.
\end{align*}
\]

If we assume that $x \neq 0$, the second equation gives as soon as $\beta \neq 0$:

\[
x = \frac{2y^2 \beta}{2y^2 \beta - y^2 \beta} = \frac{2y^2 \beta}{2y^2 \beta - y^2 \beta} = 2y^2 \beta
\]

and thus:

\[
y^2 (\beta^2 - 4\alpha \beta \beta + 4\beta^2 \alpha) + y(4\alpha \beta \beta + 2\beta \beta - 2\beta) + 1 - 2\beta = 0.
\]

Let us consider a change of basis which preserves $e_1$ that is,:

\[
\begin{align*}
e_1' &= e_1, \\
e_2' &= ae_1 + de_2,
\end{align*}
\]

with $ad \neq 0$. Since in this new basis we have $\beta_1 = 2b\beta_2 + d\beta_2$, we can find $b$ such that $\beta_1 = 0$. Then we can assume that $\beta_1 = 0$.

Moreover $\alpha \neq 0$, taking $d = \alpha_2^{-1}$, we obtain $\alpha' = 1$ and we have the algebra:

\[
\begin{align*}
\mu(e_1', e_1') &= e_1, \\
\mu(e_1', e_2') &= e_1 + \beta_1 e_2, \\
\mu(e_2', e_1') &= \alpha_1 e_1 + e_2, \\
\mu(e_2', e_2') &= \alpha_1 e_1 + \beta_1 e_2.
\end{align*}
\]

Equation (6) simplifies as:

\[
y^2 (4\beta^2 \alpha) + 4\beta \gamma + 1 - 2\beta = 0.
\]

If we assume that $\mathbb{K}$ is algebraically closed, then this equation has in general two roots. It has no root if $\beta_1 = 0$ which is excluded. Then to have only one idempotent, $0$ must be the only root which is equivalent to $\alpha = 0$ and $\beta_1 = 1/2$. We obtain the following algebra:

\[
\begin{align*}
\mu(e_1', e_1') &= e_1, \\
\mu(e_1', e_2') &= e_1 + \frac{1}{2} \gamma e_2, \\
\mu(e_2', e_1') &= 0.
\end{align*}
\]

If $\mathbb{K}$ is not algebraically closed, then we have no idempotent other than $0$ if $\alpha = 0$ and $\beta_1 = 1/2$ and we obtain the previous algebra $\mu'$ or if $y^2 (4\beta^2 \alpha) + 4\beta \gamma + 1 - 2\beta$ is irreducible in $\mathbb{K}$. We obtain:

\[
\begin{align*}
\mu'(e_1', e_1') &= e_1, \\
\mu'(e_1', e_2') &= e_1 + \beta_1 e_2, \\
\mu'(e_2', e_1') &= \alpha_1 e_1, \\
\mu'(e_2', e_2') &= \alpha_1 e_1 + \beta_1 e_2.
\end{align*}
\]

with $y^2 (4\beta^2 \alpha) + 4\beta \gamma + 1 - 2\beta$ irreducible in $\mathbb{K}$ (so $\alpha 
eq 0$).

If $\alpha_1 = 0$ and if $\mathbb{K}$ is algebraically closed, we consider in the change of basis (7) defined above, $b = 0$ and $d = \sqrt{-\beta_1}$ if $\alpha_1 = 0$:

\[
\begin{align*}
\mu(e_1', e_1') &= e_1, \\
\mu(e_1', e_2') &= \beta_1 e_2, \\
\mu(e_2', e_1') &= e_1.
\end{align*}
\]

There exits only one idempotent if and only if $\beta_1 = 1/2$. We obtain the following algebra:

\[
\begin{align*}
\mu(e_1', e_1') &= e_1, \\
\mu(e_1', e_2') &= \beta_1 e_2, \\
\mu(e_2', e_1') &= 0.
\end{align*}
\]

If $\alpha_1 = \alpha_1 = 0$, we have only one idempotent if and only if $2\beta_1 = 1$. We obtain:

\[
\begin{align*}
\mu_1(e_1', e_1') &= e_1, \\
\mu_1(e_1', e_2') &= \beta_1 e_2, \\
\mu_1(e_2', e_1') &= 0.
\end{align*}
\]

Assume $\mathbb{K}$ not algebraically closed and $\alpha_1 = 0$. If the equation $d\alpha_1$ has a root in $\mathbb{K}$, we find $\mu_1$. If not, let $\lambda \in \mathbb{K}/(\mathbb{K})^*$ such that $d\alpha_1 = \lambda$. In this case we have only one idempotent if and only if $(2\beta_1 = 1)$ or $1 - 2\beta_1 \notin (\mathbb{K}/(\mathbb{K})^*)^1$. We obtain:

\[
\begin{align*}
\mu_1(e_1', e_1') &= e_1, \\
\mu_1(e_1', e_2') &= \frac{1}{2} \gamma e_2, \\
\mu_1(e_2', e_1') &= \lambda e_1.
\end{align*}
\]

and,
\[
\begin{align*}
\mu^0_\alpha(e_1,e_2) &= e_1, \\
\mu^1_\alpha(e_1,e_2) &= \beta_\alpha e_2, \quad 1 - 2\beta_\alpha \notin (\mathbb{K}^2)^2, \\
\mu^2_\alpha(e_1,e_2) &= \lambda_\alpha e_1.
\end{align*}
\]

Assume now that $\beta_\alpha = 0$. Then (5) implies $\gamma^2\beta_\alpha = y$. If $\beta_\alpha = 0$, then $y = 0$ and we have:

\[
\begin{align*}
\mu(e_1,e_2) &= e_1, \\
\mu(e_1,e_2) &= \alpha e_2, \\
\mu(e_1,e_2) &= \alpha e_1.
\end{align*}
\]

The change of basis $e'_1 = e_1, e'_2 = ke_1 + de_2$ gives $\alpha' \equiv d\alpha, \alpha_e = d^2 \alpha_e$. We obtain:

\[
\begin{align*}
\mu^{10}(e_1,e_2) &= e_1, \\
\mu^{10}(e_1,e_2) &= e_2, \\
\mu^{10}(e_1,e_2) &= \alpha e_1.
\end{align*}
\]

if $\alpha_e = 0$. Assume now that $\alpha_e = 0$ and $\alpha_e \neq 0$. If $\mathbb{K}$ is algebraically closed, we obtain:

\[
\begin{align*}
\mu^{11}(e_1,e_2) &= e_1, \\
\mu^{11}(e_1,e_2) &= 0, \\
\mu^{11}(e_1,e_2) &= \alpha e_1 + \beta e_2,
\end{align*}
\]

with $\lambda_\alpha \in \mathbb{K}(\mathbb{K}^2)^2$. If $\alpha_e = 0$,

\[
\begin{align*}
\mu^{12}(e_1,e_2) &= e_1, \\
\mu^{12}(e_1,e_2) &= e_2, \\
\mu^{12}(e_1,e_2) &= 0.
\end{align*}
\]

No vector is idempotent. If there exists $v$ with $\mu(v,v) \neq 0$, thus we can consider that $\mu(e_1,e_2) = e_i$ that is,

\[
\begin{align*}
\mu(e_1,e_2) &= e_1, \\
\mu(e_1,e_2) &= \mu(e_1,e_2) = \alpha e_i + \beta e_i, \\
\mu(e_1,e_2) &= \alpha e_i + \beta e_i.
\end{align*}
\]

1. If $\alpha_e = 0$, that is $\mu(e_1,e_2) = \beta e_i$, then the vector $e'_i = \beta_\alpha e_i$ is idempotent as soon as $\beta_\alpha \neq 0$. Then the hypothesis implies $\beta_\alpha = 0$. Let be $v = xe_1 + ye_2$. The equation $\mu(v,v) = v$ is equivalent to:

\[
x^2 e_1 + 2xe_1(\alpha e_1 + \beta e_2) = 2xy \alpha e_1 + (x^2 + 2xy) \beta_\alpha e_1 = xe_1 + ye_2.
\]

that is,

\[
2xy \alpha e_1 = x, \quad x^2 + 2xy \beta_\alpha = y.
\]

If $\alpha_e = 0$, then $x = y = 0$, and no elements are idempotent. We obtain the algebra, corresponding to $\beta_\alpha \neq 0$ or $\beta_\alpha = 0$.

\[
\begin{align*}
\mu^{10}(e_1,e_2) &= e_1, \\
\mu^{10}(e_1,e_2) &= e_2, \\
\mu^{10}(e_1,e_2) &= 0.
\end{align*}
\]

If $\alpha_e \neq 0$, then $y = 2(2\alpha_e)^{-1}$ then $x$ satisfies the equation:

\[
x^2 + \frac{2\beta_\alpha}{\alpha_e} x - \frac{1}{2\alpha_e} = 0
\]

(9)

If $\mathbb{K}$ is algebraically closed, such equation admits a non trivial solution. This is not compatible with our hypothesis. Assume that $\mathbb{K}$ is not algebraically closed. If $\beta_\alpha \neq 0$, the change of basis $e'_1 = \beta_\alpha e_1$ and $e'_2 = \beta_\alpha e_2$ permits to consider $\beta_\alpha = 1$ and the (9) becomes,

\[
x^2 + \frac{1}{\alpha_e} x - 1 = 2\alpha_e
\]

This equation has a non solution if $1 + 2\delta_\alpha \notin (\mathbb{K})^2$ where $(\mathbb{K})^2 = \langle \lambda \rangle$. We obtain the algebra:

\[
\begin{align*}
\mu^{10}(e_1,e_2) &= e_2, \\
\mu^{10}(e_1,e_2) &= \alpha e_1 + e_2, \\
\mu^{10}(e_1,e_2) &= 0,
\end{align*}
\]

and,

\[
\begin{align*}
\mu^{11}(e_1,e_2) &= e_1, \\
\mu^{11}(e_1,e_2) &= \alpha e_1, \\
\mu^{11}(e_1,e_2) &= 0.
\end{align*}
\]

2. If $\alpha_e \neq 0$ the vector $v = xe_1 + ye_2$ is idempotent if and only if:

\[
\begin{align*}
2\chi\alpha_2 + y^2\alpha_4 &= x, \\
x^2 + 2\chi\beta \lambda + y^2\beta_4 &= y.
\end{align*}
\]

Then $x = \frac{1}{2\lambda_\alpha}$. Let us note that $1 - 2\chi\alpha_2 \neq 0$ implies $y\eta_0 = 0$ that is $y = 0$ and in this case $x = 0$ and $v = 0$. We deduce that $y$ is a root of the equation:

\[
\frac{y^2\alpha_4}{1 - 2\chi_\alpha} + 2\frac{y^2\alpha_4}{1 - 2\chi_\alpha} y\beta_4 + y^2\beta_4 - y = 0
\]

that is:

\[
1 + y(4\alpha_2 + \beta_4) + y^2(2\alpha_2\beta_4 - 4\alpha_4\beta_4) + y^3(4\alpha_4 - 4\alpha_2\beta_4 + 4\alpha_4\beta_4) = 0.
\]

If $\mathbb{K}$ is algebraically closed, this equation admits always a solution except if:

\[
\begin{align*}
4\alpha_2 + \beta_4 &= 0, \\
2\alpha_2\beta_4 - 4\alpha_4\beta_4 &= 0, \\
4\alpha_4 - 4\alpha_2\beta_4 + 4\alpha_4\beta_4 &= 0.
\end{align*}
\]

Then $\beta_4 = 4\alpha_2, \alpha_4 = 4\alpha_2, \beta_4 = -8\alpha_4$. We note that $\beta_4 \neq 0$, if the characteristic of $\mathbb{K}$ is not 3, $\alpha_2 = \alpha_4 = 0$. From hypothesis, we can assume that $\beta_4 \neq 0$ and the change of basis $e'_1 = ke_1, e'_2 = k'e_2$, which preserves the condition $e_1, e_2$ changes $\beta_4$ in $k\beta_4$, and we can take $\beta_4 = 3$. Then $\alpha_2 = -2\alpha_4, \alpha_4 = 4\alpha_2, \beta_4 = -8\alpha_4$, then $\alpha_e = -2$ and $\alpha_e = 4, \beta_4 = 8$ and we obtain the algebra:

\[
\begin{align*}
\mu^{10}(e_1,e_2) &= e_1, \\
\mu^{10}(e_1,e_2) &= -2e_1 + 5e_2, \\
\mu^{10}(e_1,e_2) &= -8e_1 + 8e_2.
\end{align*}
\]
Let us note that if the characteristic of \( K \) is 3, then \( \alpha \beta = 0 \) and \( \beta^2 = 0 \). This gives \( \alpha (\alpha + \beta) = 0 \) and \( \alpha^2 + 4 \alpha \beta \beta = 0 \). Since \( \alpha = 0 \) implies \( \alpha = 0 \) and \( \beta = 0 \), we obtain \( \beta = 2 \alpha \) and \( \alpha^2 = 2 \alpha \beta = \alpha^2 \). By a change of basis, we can take \( \alpha = 1 \) and we obtain the algebra:

\[
\begin{align*}
\mu_{13}^{(1)} (e_1, e_2) &= e_1, \\
\mu_{13}^{(1)} (e_1, e_3) &= e_1, \\
\mu_{13}^{(1)} (e_2, e_3) &= e_1 + 2e_2.
\end{align*}
\]

which correspond to \( \mu_{13} \) in characteristic 3.

If \( K \) is not algebraically closed, we have to consider all the algebras for which the polynomial:

\[ P(y) = y(y + \beta)(y^2 + 4y + 4\alpha \beta + 4\alpha^2 \beta^2) \]

has no root, which is equivalent to say that \( P(y) \) is irreducible. If we consider the coefficient of \( y^3 \), that is \( q_1(A) = \alpha^2 - 4\alpha \beta \alpha + 4\alpha^2 \beta^2 \), it is equal to the discriminant of the determinant of the endomorphism \( L_y \), that is \( q_1(A) = \text{Disc}(det(L_y)) \). We deduce:

**Proposition 5:** The algebra \( A \) is regular if and only if \( P_A(y) \equiv 0 \) is strictly of degree 3.

It remains to examine the case \( \mu(y)v = 0 \) for any \( v \). That is:

\[
\begin{align*}
\mu_1(e_1, e_2) &= 0, \\
\mu_2(e_1, e_2) &= \alpha e_1 + \beta e_2, \\
\mu_3(e_1, e_2) &= 0.
\end{align*}
\]

If \( \alpha \beta \neq 0 \) we can find some idempotents. In all the others cases, we have no idempotent. We obtain:

\[
\begin{align*}
\mu_1^{(1)} (e_1, e_2) &= 0, \\
\mu_2^{(1)} (e_1, e_2) &= e_1, \\
\mu_3^{(1)} (e_1, e_2) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\mu_1^{(2)} (e_1, e_2) &= 0, \\
\mu_2^{(2)} (e_1, e_2) &= e_1, \\
\mu_3^{(2)} (e_1, e_2) &= 0.
\end{align*}
\]

**Theorem 6:** Any commutative 2-dimensional algebra over an algebraically closed isomorphic to one of the following:

\[
\begin{align*}
\mu_1^{(3)} (e_1, e_2) &= e_1, \\
\mu_2^{(3)} (e_1, e_2) &= \frac{1}{2} e_2, \\
\mu_3^{(3)} (e_1, e_2) &= 0.
\end{align*}
\]

If \( K \) is not algebraically closed, we have also the following algebras where \( \lambda \in \mathbb{C} / (\mathbb{K}^*) \):

\[
\begin{align*}
\mu_1^{(4)} (e_1, e_2) &= e_1, \\
\mu_2^{(4)} (e_1, e_2) &= \beta e_2, \\
\mu_3^{(4)} (e_1, e_2) &= 2 \alpha e_2 + 2 \beta e_2.
\end{align*}
\]

1. The algebras \( A' = (V, \mu') \), \( A'' = (V, \mu'') \), \( A''' = (V, \mu''') \), and \( A'''' = (V, \mu''''') \) are regular.
2. \( A'''' = (V, \mu''''') \) is regular if \( \beta \neq 0 \).
3. The algebras \( A' = (V, \mu'), A'' = (V, \mu''), A''' = (V, \mu''') \), and \( A'''' = (V, \mu''''') \) are isomorphic.

**Algebras Over A Field of Characteristic 2**

Let \( \mathbb{F} \) be a field of characteristic 2. Assume that \( \mathbb{F} = \mathbb{F}_2 \). If \( A \) is a 2-dimensional \( \mathbb{F} \)-algebra, then the values of \( A' \) and \( A'' \) are determined by the basis \( \{ e_1, e_2, e_3 \} \) by one of the following matrices:

\[
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Each of these matrices corresponds to a permutation of the finite set \( \{ e_1, e_2, e_3 \} \). If we have the correspondence:

\[
\begin{align*}
GL(A) &\cong \mathfrak{S}_3, \\
M_1 &= \text{Id}, \\
M_2 &= \tau_{12}, \\
M_3 &= \tau_{23}, \\
M_4 &= c
\end{align*}
\]

where \( \tau_{ij} \) is the transposition between \( i \) and \( j \) and \( c \) the cycle [231]. In fact, the matrix \( M_1 \) corresponds to the linear transformation \( f(e_1) = e_2, f(e_2) = e_3 \) and in the set \( \{ e_1, e_2, e_3 \} \) we have the transformation whose image is \( \{ e_1, e_3, e_2 \} \) that is the transposition \( \tau_{12} \). The matrix \( M_2 \) corresponds to the linear transformation \( f(e_1) = e_2, f(e_2) = e_3 \), which corresponds to the permutation \( \{ e_1, e_3, e_2 \} \), that is \( \tau_{23} \). For all other matrices we have similar results. We deduce:

**Theorem 7:** There is a one-to-one correspondence between the change of \( \mathbb{F} \)-basis in \( A \) and the group \( \Sigma_3 \).

If we want to classify all these products of \( A \), we have to consider all the possible results of these products and to determine the orbits of the action of \( \Sigma_3 \). More precisely the product \( \mu(e_1, e_2) \) is in values in the
set \( (e_1, e_2, e_3 = e_1 + e_2) \). If we write \( \mu(e_1, e_2) = a e_1 + b e_2 + c e_3 \), thus the matrix \((a, b, c)\) is one of the following:
\[
R_0 = (0, 0, 0) = 0, R_1 = (1, 0, 0), R_2 = (0, 1, 0), R_3 = (0, 0, 1)
\]

Let us consider the following sequence:
\[
\mu(e_1, e_1), \mu(e_2, e_1), \mu(e_3, e_1), \mu(e_4, e_1), \mu(e_5, e_1), \mu(e_6, e_1), \mu(e_7, e_1)
\]

As \( \mu(e_1, e_1) = \mu(e_2, e_1) = R_i \) and \( \mu(e_2, e_2) = R_i \) then \( \mu(e_2, e_2) = R_i \) with the relations:
\[
R_i + R_i = 0, R_i + R_i = R_i,
\]
for \( i, j, k \) all different and non zero. Thus the four first terms of this sequence determine all the other terms. More precisely, such a sequence writes:
\[
(R_i, R_j, R_k, R_i + R_j, R_i + R_k, R_i + R_j + R_k, R_i + R_j + R_k + R_i)
\]

**Consequence:** We have \( 4^3 \times 2^5 = 256 \) sequences, each of these sequences corresponds to a 2-dimensional \( F \)-algebra.

Let us denote by \( S \) the set of these sequences. We have an action of \( S \times S \) on \( S \) so that \( s \tau \sigma = \tau s \sigma \). Let us define:
\[
\mu(e_1, e_1) = R_i, \mu(e_2, e_2) = R_j, \mu(e_3, e_3) = R_k
\]

where \( R_i = 0, R_j = 0, R_k = 0 \) or \( 0, 0, 0 \).

The classification of the 2-dimensional \( F \)-algebras corresponds to the determination of the orbits of this action. Recall that the subgroups of \( S \) are:
\[
G_1 = \{e\}, G_2 = \{id, \tau_{12}\}, G_3 = \{id, \tau_2, \tau_1, \tau_{12}\}, G_4 = \{id, e, c\}, G_5 = \{id, e, e\}
\]

1. The isotropy subgroup is \( S \). In this case we have the following sequence (we write only the 4 first terms which determine the algebras):
\[
s_1 = (0, 0, 0, 0)
\]

2. The isotropy subgroup is \( G_1 = \{id, e, c\} \): we have only one orbit:
\[
s = \{0, 0, 0, 0\}
\]

3. The isotropy subgroup is order 2.
\[
s = \{0, 0, 0, 0\}
\]

The isotropy subgroup is trivial. In this case any orbit contains 6 elements. As there are 256–46–46=210 elements having \( S \) as isotropy group, we deduce that we have 35 distinguished non isomorphic classes.

**Conclusion**

We have 52 classes of non isomorphic algebras of dimension 2 on the field \( F \).

**Applications : 2-dimensional G-associative and Jordan algebras**

**G-associative commutative algebras**

The notion of G-associativity has been defined in ref. [4]. Let \( G \) be a subgroup of the symmetric group \( S_3 \). An algebra whose multiplication is denoted by \( \mu \) is G-associative if we have:
\[
\sum_{\sigma \in S_3} \sigma \mu(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \mu(x_{\sigma(1)}, \mu(x_{\sigma(2)}, x_{\sigma(3)})) = 0
\]

where (\( e(\sigma) \)) is the signum of the permutation. Since we assume that \( \mu \) is commutative, all these notions are trivial or coincide with the simple associativity. Now, if the algebra is of dimension 2, then the associativity is completely determined by the identities:
\[
\mu(e_1, e_2) = \mu(e_2, e_1) = 0, \mu(e_1, e_1) = 0, \mu(e_2, e_2) = 0
\]

We deduce that the only associative commutative 2-dimensional algebras are:

- \( \mu^4 \) for \( (\alpha, \beta) \in \{(0, 1), (1, 0), (0, 0)\} \),
- \( \mu^4 \) for \( \beta = 0 \) or 1,
- \( \mu^4, \mu^4\), \( \mu^4 \).

If \( K = \mathbb{R} \), \( \mu^4 \) for \( \beta = 1 \) and \( \lambda = -1 \).

We find again the classical list [6].

**G-associative noncommutative algebras**

Let us consider now the noncommutative case. From Theorem 2, the multiplication \( \mu \) is isomorphic to some \( \mu^i = 1, \cdots, 5 \) (we consider here that \( K \) is algebraically closed). Let \( \mathcal{A} \) be the associator of \( \mu \), that is \( A_{\mu} = \mu - (\mu \circ id) - (id \circ \mu) \) and \( \mu \) is associative if and only if \( A_{\mu} = 0 \). The examination of this list allows to find the classification of the 2-dimensional noncommutative associative algebras: these algebras are isomorphic to one of the following:

1. \( \mu^4 \) that is:
   \[
   e_1 e_1 = 0, \quad e_2 e_2 = 0, \quad e_1 e_2 = -e_2 e_1
   \]

2. \( \mu^4 \) that is:
   \[
   e_1 e_1 = 0, \quad e_2 e_2 = 2e_1, \quad e_1 e_2 = 0, \quad e_2 e_2 = 2e_2
   \]

Now, for any nonassociative algebra, we examine the \( G_i \)-associativity. Note that all these algebras are Lie-admissible, that is \( \mathcal{S}_3 \)-associative. We focus essentially on the \( G_2 \)-associativity, \( G_2 = \{id, \tau_{12}\} \), because we deduce immediately the affine structures on the associated Lie algebra \( \mathcal{G}_2 \). Then we compute for any algebra \( A_j(e_1, e_2, e_3) = A_j(e_1, e_2, e_3) \) and \( A_j(e_1, e_2, e_3) = A_j(e_1, e_2, e_3) \). We deduce that \( \mu^4 \) is \( G_2 \)-associative if and only if \( \beta_2 = \alpha_2 = 0 \) and \( \alpha_2 = -1, \beta_2 = -4 \). The algebras \( \mu^2 \)
and $\mu'$ are never $G_2$-associative, $\mu'_{i-1,0}$ is $G_2$-associative for $\alpha_2=-1$ or $(\beta_2=-1)$. Likewise, $\mu'_{i,0}$ is $G_2$-associative for $\alpha_2=-1$ or $\alpha_3=-1$.

**Proposition 8:** Any 2-dimensional noncommutative $G_2$-associative algebra is isomorphic to one of the following:

1. $\mu_{1,2}$ or $\mu'_{0,1}$, that is $\mu$ is associative,

\[
\begin{align*}
\mu & = e_1 e_2 = e_2 e_1 = 0, \\
\mu & = e_1 e_2 = -2 e_1, \\
\mu & = e_1 e_2 = -4 e_1.
\end{align*}
\]

2. $\mu_{i,0,0,0}$, that is $\mu = e_1 e_2 = (\alpha_2+1)e_1$,

\[
\begin{align*}
\mu & = e_1 e_2 = (\alpha_3-1)e_1, \\
\mu & = e_1 e_2 = (\alpha_3+1)e_2.
\end{align*}
\]

3. $\mu_{i,0}$, that is $\mu = e_1 e_2 = 0$,

\[
\begin{align*}
\mu & = e_1 e_2 = 2 e_1, \\
\mu & = e_1 e_2 = 0, \\
\mu & = e_1 e_2 = -2 e_1, \\
\mu & = e_1 e_2 = e_1 + 2 e_2.
\end{align*}
\]

4. $\mu_{i,0,-1}$, that is $\mu = e_1 e_2 = 0$,

\[
\begin{align*}
\mu & = e_1 e_2 = 2 e_1, \\
\mu & = e_1 e_2 = 0, \\
\mu & = e_1 e_2 = -2 e_1, \\
\mu & = e_1 e_2 = e_1 - 2 e_2.
\end{align*}
\]

5. $\mu_{i}$, that is $\mu = e_1 e_2 = 0$,

\[
\begin{align*}
\mu & = e_1 e_2 = 2 e_1, \\
\mu & = e_1 e_2 = 0, \\
\mu & = e_1 e_2 = -2 e_1, \\
\mu & = e_1 e_2 = e_1 - 2 e_2.
\end{align*}
\]

6. $\mu_{i}$, that is $\mu = e_1 e_2 = 0$,

\[
\begin{align*}
\mu & = e_1 e_2 = 2 e_1, \\
\mu & = e_1 e_2 = 0, \\
\mu & = e_1 e_2 = -2 e_1, \\
\mu & = e_1 e_2 = e_1 - 2 e_2.
\end{align*}
\]

**Jordan algebras**

In a Jordan algebra, the multiplication $\mu$ satisfies:

\[
\begin{align*}
\mu(vw) & = \mu(wv), \\
\mu(\mu(vw), \mu(vv)) & = \mu(v, \mu(w, \mu(vv)))
\end{align*}
\]

for all $v, w$. We assume in this section that $K$ is algebraically closed and that the Jordan algebra are of dimension 2. Thus the multiplication $\mu$ is isomorphic to $\mu_i$ for $i=1,\ldots,16$. To simplify the notation, we will write $vw$ in place of $\mu(vw)$. If $v$ is an idempotent, thus $v^2 = v$ and the Jordan identity gives:

\[
v(vw) = v(vw)
\]

for any $w$, that is, this identity is always satisfied.

**Lemma 9:** If $v_1$ and $v_2$ are idempotent vectors, thus:

\[
(\mu(v_1v_2))(v_1 + v_2) = (v_1 + v_2)(\mu(v_1v_2))
\]

for any $w$.

**Proof.** In the Jordan identity, we replace $v$ by $v_1 + v_2$. We obtain:

\[
\mu(v_1v_2) + (v_1v_2)(v_1 + v_2) + v_1v_2(w) + (v_1v_2)w = v_1v_2(v_1 + v_2) + (v_1 + v_2)(\mu(v_1v_2))
\]

Since $v_1$ and $v_2$ are idempotent, this equation reduces:

\[
(\mu(v_1v_2))(v_1 + v_2) = (v_1 + v_2)(\mu(v_1v_2)).
\]

**Proposition 10:** If $v_1$ and $v_2$ are idempotent vectors such that $v_1v_2$ and $v_1 + v_2$ are independent, thus the Jordan algebra is associative.

**Proof.** Let $x$ and $y$ be two vectors of the algebra. Thus, by hypothesis, $x = x_1v_1 + x_2(v_1 + v_2)$ and $y = y_1v_1 + y_2(v_1 + v_2)$. Thus:

\[
x(yw) = x(y_1v_1w + x_1v_1v_2 + y_2(v_1 + v_2)) + (x_1v_1 + y_1v_1)(v_1v_2w) + x_2y_2(v_1 + v_2)(v_1v_2w)
\]

and:

\[
x(yw) = y(xw)
\]

By commutativity we obtain:

\[
x(yw) = x(\mu(vw)) = y(xw) = (xw)y
\]

this proves that the algebra is associative.

If $\mu$ is given by:

\[
\begin{align*}
\mu(e_1, e_1) & = e_1, \\
\mu(e_1, e_2) & = \alpha e_1 + \beta e_2, \\
\mu(e_2, e_2) & = e_2
\end{align*}
\]

the Jordan algebra admits two idempotents $e_1$ and $e_2$. Since $e_1e_2 = \alpha e_1 + \beta e_2$, the vectors $e_1$, $e_2$ and $e_1 + e_2$ are independent if and only if $\alpha \neq \beta$. In this case the algebra can be associative and we obtain the following associative Jordan algebra corresponding to:

1. $\alpha_2 = 1$, $\beta_2 = 0$
2. $\alpha_2 = 0$, $\beta_2 = 1$

These Jordan algebras are isomorphic. This gives the following Jordan algebra:

\[
J_1 = \begin{cases}
\alpha e_1 + \beta e_2, \\
\alpha e_1 + \beta e_2 = e_2, \\
\alpha e_1 + \beta e_2 = e_1 + 2e_2.
\end{cases}
\]

If $e_1$ and $e_1 + e_2$ are dependent, that is $e_1e_2 = \lambda e_1$, then $\lambda = 1$ or 0. If $e_1e_2 = 0$, the product is not a Jordan product. If $\lambda = 1$ the product is never a Jordan product. If $\lambda = 1$, we obtain the following Jordan algebra,

\[
J_1 = \begin{cases}
\alpha e_1 + \beta e_2, \\
\alpha e_1 + \beta e_2 = e_2, \\
\alpha e_1 + \beta e_2 = e_1 + 2e_2.
\end{cases}
\]

$\mu$ is given by:

\[
\begin{align*}
\mu(e_1, e_1) & = e_1, \\
\mu(e_1, e_2) & = \beta e_2, \\
\mu(e_2, e_2) & = 0.
\end{align*}
\]

This product is a Jordan product if $\beta = 1$ or 0. We obtain:

\[
J_2 = \begin{cases}
\alpha e_1 + \beta e_2, \\
\alpha e_1 + \beta e_2 = e_2, \\
\alpha e_1 + \beta e_2 = e_1 + 2e_2.
\end{cases}
\]

If $\mu = \mu_{i1}$, we have also a Jordan structure,
implies. In this case also, if we compute $f(e_1)\neq f(e_2)$, we obtain $a=0$ in the first case and $a=\beta=0$ in the second case. If $\alpha=0$, then we have the trivial Jordan algebra.

\[\begin{align*}
J_1 &= \begin{cases}
e_1^1 = e_1, \\
e_1e_2 = e_2e_1 = e_1, \\
e_2e_2 = 0
\end{cases}
\end{align*}\]

$\mu=0$, we have the trivial Jordan algebra.

• If $\mathbb{K}$ is not algebraically closed, we consider,

\[\begin{align*}
\mu^{\mathbb{K}}(e_1,e_1) &= e_1, \\
\mu^{\mathbb{K}}(e_1,e_2) &= \beta_2 e_2, \\
\mu^{\mathbb{K}}(e_2,e_1) &= \lambda e_1,
\end{align*}\]

We obtain a Jordan structure:

\[\begin{align*}
e_1^1 &= e_1, \\
e_1e_2 &= e_2e_1, \\
e_2e_2 &= \lambda e_2.
\end{align*}\]

We find the list established in ref. [1].

**2-dimensional Hom-algebra**

The notion of Hom-algebra was introduced to generalized form of Hom-Lie algebra which appeared naturally when we are interested by the notion of $q$-derivation on the Witt algebra. In dimension 2, this notion is equivalent to the classical notion of Lie algebra. In dimension 3, we have shown that any skew-symmetric algebra is a Hom-Lie algebra. Then our interest concerns Hom-associative algebra [7,8], that is algebra $A=(V,\alpha)$ such that there exists $f\in End(V)$ satisfying the Hom-Ass identity:

\[\mu_{\alpha}(X,Y,f(Z)) = \mu(f(X),\mu(Y,Z))\]

for any $X,Y,Z\in V$. Using previous notations, we consider the algebras $A^{\text{ad}}\alpha$ and its opposite $A^{\text{ad}}\alpha$. Their multiplication law are respectively defined by:

\[\mu_{\alpha}(X,Y) = \mu(X,f(Y)), \quad \mu_{\alpha}(X,Y) = \mu(f(X),Y)\]

and the Hom-Ass identity can be written:

\[\mu_{\alpha} \circ (\mu \otimes Id) - \mu_{\alpha} \circ (Id \otimes \mu) = 0.\]

Assume now that the algebra $A$ is regular. In this case, assuming that the field is algebraically closed, there exists an unital algebra whose product is denoted $XY$ and two endomorphisms $u$ and $v$ of $V$ such that:

\[\mu(X,Y) = u(X)\cdot v(Y)\]

Then,

\[\mu_{\alpha}(X,Y) = u(X)\cdot v(f(Y)), \quad \mu_{\alpha}(X,Y) = u\circ f(X)\cdot v(Y).\]

Then the Hom-Ass identity becomes:

\[u(u(X)\cdot v(f(Z)))\cdot v(f(Y)) - u\circ f(X)\cdot v(u(Y)\cdot v(Z)) = 0.\]

Maybe, it is better to look the Hom-Ass identity from the previous list. Assume that $A$ is non commutative.

1. $A = A^{\text{ad}}\alpha$, let $f$ be an endomorphism of $V$ satisfying the Hom-Ass identity. To simplify notations we write $XY$ for $\mu(X,Y)$ and $[X,Y]$ for $\mu(X,Y)$. We have in particular:

\[\begin{align*}
(e_1)f(e_1) - f(e_1)(e_1)e_1 &= \left[e_2, f(e_1)\right] = 0\cdot
\end{align*}\]

We deduce $f(e_1)=\alpha e_1$. Likewise we have $[e_1,f(e_1)]=0$ and $f(e_1)=k(e_1+\beta e_2)$. Other identities give:

(a) $(e_1\cdot f(e_1) - f(e_1)\cdot e_1) = 0$ implies $a=0$ or $e_1\cdot e_1 = 0$.

(b) If $a=0$, then $(e_1\cdot f(e_1)\cdot e_1)=0$ implies $f(e_1)\cdot e_1 = 0$ and $(e_1\cdot e_1)\cdot f(e_1) = 0$. We deduce $f(e_1)\cdot e_1 = 0$ and $f(e_1)\cdot e_1 = k e_1$. This gives $0=f(e_1)\cdot e_2 = b e_2$ that is $f=0$ or $e_1\cdot e_1 = 0$. But we have seen that $f(e_1) = k(e_1 + \beta e_2)$, then in all the cases, $f=0$.

(c) If $a\neq 0$, then $(e_1\cdot f(e_1)\cdot e_1) = 0$. We deduce that $(e_1\cdot f(e_1)\cdot e_1) = e_1\cdot e_1 = a$, $a\neq 0$.

We deduce that the algebra $A_{\gamma_1,\gamma_2,\alpha_1,\alpha_2}$ is not a Hom-associative algebra.

2. $A = A^{\text{ad}}\alpha$. With similar simple computation we can look also this algebra is not a Hom-Ass algebra.

3. $A = A^{\text{ad}}\alpha$. In this case also, if we compute $(e_1\cdot f(e_1) - f(e_1)\cdot e_1) = [e_1, f(e_1)] = 0$, we obtain $f(e_1) = k e_1$. We also have $(e_1\cdot f(e_1) - f(e_1)\cdot e_1) = 2 k e_1 = 0$ and $f(e_1) = 0$. We deduce $e_1\cdot f(e_1) = 0$ and $f(e_1)\cdot e_1 = 0$. Thus $f=0$ and $A$ is not a Hom-associative algebra.

4. $A = A^{\text{ad}}\alpha$. If $\beta \neq 0$, then the Hom-Ass condition implies $\alpha_2 = 1$ or $-1$. We obtain the following Hom-Ass algebras:

\[\begin{align*}
\mu^{\mathbb{K}}(e_1,e_1) &= 0, \\
\mu^{\mathbb{K}}(e_1,e_2) &= 0, \\
\mu^{\mathbb{K}}(e_2,e_1) &= 0, \\
\mu^{\mathbb{K}}(e_2,e_2) &= 0.
\end{align*}\]

In each of these two cases, $f$ is a diagonal endomorphism. These algebras are for $\beta\neq 2$ or $-2$, not associative.

5. $A = A^{\text{ad}}\alpha$. If $\alpha_2=0$, any linear endomorphism with values in $\mathbb{K}[e_1]$ satisfies the Hom-Ass identity. Then the following algebra is Hom-associative:

\[\begin{align*}
\mu^{\mathbb{K}}(e_1,e_1) &= 0, \\
\mu^{\mathbb{K}}(e_1,e_2) &= 0, \\
\mu^{\mathbb{K}}(e_2,e_1) &= e_1, \\
\mu^{\mathbb{K}}(e_2,e_2) &= e_1.
\end{align*}\]

Assume now that $\alpha_2 \neq 0$. If $\alpha_2 \neq \pm 1$, then any endomorphism satisfying the Hom-Ass identity is trivial. If $\alpha_2 = 1$ or $-1$, we have non trivial solution and the following algebras are Hom-associative algebras:

\[\begin{align*}
1^{\text{ad}}(e_1,e_1) &= 0, \\
1^{\text{ad}}(e_1,e_2) &= 0, \\
1^{\text{ad}}(e_2,e_1) &= 0, \\
1^{\text{ad}}(e_2,e_2) &= e_1 + 2 e_2
\end{align*}\]

with $f = \left(\begin{array}{cc} -4x & x \\ 2 & 0 \end{array}\right)$ in the first case and $f = \left(\begin{array}{cc} 4x & -x \\ 0 & 2 \end{array}\right)$ in the second case.

Then we have the list of noncommutative Hom-associative algebras. The commutative case can be established in the same way. In this case the Hom-Ass identity is reduced to:

\[\begin{align*}
\mu^{\mathbb{K}}(e_1,e_1) &= 0, \\
\mu^{\mathbb{K}}(e_1,e_2) &= 0, \\
\mu^{\mathbb{K}}(e_2,e_1) &= 0, \\
\mu^{\mathbb{K}}(e_2,e_2) &= e_1 + 2 e_2
\end{align*}\]
\[(e_1e_1)f(e_1) - (e_2e_2)f(e_2) = 0, \quad (e_1e_2)f(e_1) - (e_2e_1)f(e_1) = 0.\]

Then \(f\) is in the kernel of the linear system whose matrix is:
\[
H_{A_i} = \begin{pmatrix}
-a_1a_1 - \beta_1 \beta_2 & -a_1^2 - \beta_3 \beta_4 & a_1^2 + \beta_1 \alpha_2 & a_2 \alpha_2 + \beta \alpha_2 \\
a_2 \beta_1 - \beta_2 & a_1 \beta_1 - \beta_2 & a_2 \beta_1 + \beta_1 \beta_2 & a_2 \beta_2 + \beta_1 \\
a_1 \beta_1 - \beta_2 & -a_1 \beta_1 - \beta_2 & a_1 \beta_2 + \beta_2 \beta_1 & a_2 \beta_2 + \beta_2 \\
-a_1 \beta_1 - \beta_2 & -a_1 \beta_1 - \beta_2 & a_1 \beta_2 + \beta_2 \beta_1 & a_2 \beta_2 + \beta_2 \\
\end{pmatrix}
\]

Then \(A\) is a Hom-associative algebra if and only if \(H(A) = \text{det}(H_{A_i}) = 0\).

We deduce that the set of 2-dimensional commutative Hom-associative algebra can be provided with an algebraic hypersurface embedded in the affine variety \(\mathbb{k}^6\). From Theorem 6, when \(\mathbb{k}\) is algebraically closed, we obtain:

1. \(H(A^*) = a_2 \beta_1 \left(1 - a_1 - \beta_1 - 3a_1 \beta_3 + 2a_1 \beta_1 + a_1^2 \beta_2 + 2a_1 \beta_1 \beta_2 + 2 \beta_2 \beta_1 \right)\). Its equal to 0 for \(a_1 = 0\) or \(\beta_1 = 0\) or \(a_1 = 1 - \beta_2\) or \(a_1 = -\frac{3\beta_2 - \beta_2^2 + (1 - \beta_2) \sqrt{\beta_2(1 + \beta_2)}}{2\beta_2}\).

2. \(H(A^*) = -\frac{1}{4}\) and \(A^*\) is not a Hom-associative algebra.

3. \(H(A^*) = -\frac{9}{64}\) and \(A^*\) is not a Hom-associative algebra.

4. \(H(A^*) = 0\) for \(i = 9, 10, 11, 12, 13, 14, 15, 16, 17\) and \(A^*, A^9, A^{10}, A^{11}, A^{12}, A^{13}, A^{14}, A^{15}, A^{16}, A^{17}\) are a Hom-associative algebras.

References