Journal of Generalized Lie Theory and Applications

Remm and Goze, J Generalized Lie Theory Appl 2015, 9:2
DOI: 10.4172/1736-4337.1000235

Research Article Open Access

# A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations

**Remm E* and Goze M**

*Associate Professor, Université de Haute Alsace, 18 Rue des Frères Lumière, 68093 Mulhouse Cedex, France*

**Abstract**

There exists two types of nonassociative algebras whose associator satisfies a symmetric relation associated with a 1-dimensional invariant vector space with respect to the natural action of the symmetric group $\Sigma_3$. The first one corresponds to the Lie-admissible algebras and this class has been studied in a previous paper of Remm and Goze. Here we are interested by the second one corresponding to the third power associative algebras.

**Keywords:** Nonassociative algebras; Alternative algebras; Third power associative algebras; Operads

## Introduction

Recently, we have classified for binary algebras, Cf. [1], relations of nonassociativity which are invariant with respect to an action of the symmetric group on three elements $\Sigma_3$ on the associator. In particular we have investigated two classes of nonassociative algebras, the first one corresponds to algebras whose associator $A_\mu$ satisfies

$$A_\mu \circ (Id - \tau_{12} - \tau_{23} - \tau_{13} + c + c^2) = 0, \qquad (1)$$

and the second

$$A_\mu \circ (Id + \tau_{12} + \tau_{23} + \tau_{13} + c + c^2) = 0, \qquad (2)$$

where $\tau_{ij}$ denotes the transposition exchanging $i$ and $j$, $c$ is the 3-cycle (1,2,3).

These relations are in correspondence with the only two irreducible one-dimensional subspaces of $\mathbb{K}[\Sigma_3]$ with respect to the action of $\Sigma_3$, where $\mathbb{K}[\Sigma_3]$ is the group algebra of $\Sigma_3$. In studies of Remm [1], we have studied the operadic and deformations aspects of the first one: the class of Lie-admissible algebras. We will now investigate the second class and in particular nonassociative algebras satisfying (2) with nonassociative relations in correspondence with the subgroups of $\Sigma_3$.

**Convention:** We consider algebras over a field $\mathbb{K}$ of characteristic zero.

## $G_i$-$p^3$-associative Algebras

### Definition

Let $\Sigma_3$ be the symmetric group of degree 3 and $\mathbb{K}$ a field of characteristic zero. We denote by $\mathbb{K}[\Sigma_3]$ the corresponding group algebra, that is the set of formal sums $\sum_{\sigma\in\Sigma_3} a_\sigma \sigma$, $a_\sigma \in \mathbb{K}$ endowed with the natural addition and the multiplication induced by multiplication in $\Sigma_3$, $\mathbb{K}$ and linearity. Let $\{G_i\}_{i=1,\cdots,6}$ be the subgroups of $\Sigma_3$. To fix notations we define

$$G_1 = \{Id\}, G_2 =<\tau_{12}>, G_3 =<\tau_{23}>, G_4 =<\tau_{13}>, G_5 =<c>, G_6 = \Sigma_3,$$

where $<\sigma>$ is the cyclic group subgroup generated by $\sigma$. To each subgroup $G_i$ we associate the vector $v_{G_i}$ of $\mathbb{K}[\Sigma_3]$:

$$v_{G_i} = \sum_{\sigma\in G_i} \sigma.$$

**Lemma 1.** *The one-dimensional subspace* $\mathbb{K}\{v_{\Sigma_3}\}$ *of* $\mathbb{K}[\Sigma_3]$ *generated by*

$$v_{G_6} = v_{\Sigma_3} = \sum_{\sigma\in\Sigma_3} \sigma$$

*is an irreducible invariant subspace of* $\mathbb{K}[\Sigma_3]$ *with respect to the right action of* $\Sigma_3$ *on* $\mathbb{K}[\Sigma_3]$.

Recall that there exists only two one-dimensional invariant subspaces of $\mathbb{K}[\Sigma_3]$, the second being generated by the vector $\sum_{\sigma\in\Sigma_3} \varepsilon(\sigma)\sigma$ where $\varepsilon(\sigma)$ is the sign of $\sigma$. As we have precised in the introduction, this case has been studied in literature of Remm [1].

**Definition 2.** *A* $G_i$-$p^3$*-associative algebra is a* $\mathbb{K}$*-algebra* $(\mathcal{A},\mu)$ *whose associator*

$$A_\mu = \mu \circ (\mu \otimes Id - Id \otimes \mu)$$

*satisfies*

$$A_\mu \circ \Phi^{\mathcal{A}}_{v_{G_6}} = 0,$$

*where* $\Phi^{\mathcal{A}}_{v_{G_i}} : \mathcal{A}^{\otimes 3} \to \mathcal{A}^{\otimes 3}$ *is the linear map*

$$\Phi^{\mathcal{A}}_{v_{G_i}}(x_1 \otimes x_2 \otimes x_3) = \sum_{\sigma\in G_i} (x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}).$$

Let $\mathcal{O}(v_{G_i})$ be the orbit of $v_{G_i}$ with respect to the right action

$$\begin{array}{ccc} \Sigma_3 \times \mathbb{K}[\Sigma_3] & \to & \mathbb{K}[\Sigma_3] \\ (\sigma, \sum_i a_i\sigma_i) & \mapsto & \sum_i a_i \sigma^{-1} \circ \sigma_i \end{array}$$

Then putting $F_{v_{G_i}} = K(\mathcal{O}(v_{G_i}))$ we have

$$\begin{cases} \dim F_{v_{G_1}} = 6, \\ \dim F_{v_{G_2}} = \dim F_{v_{G_3}} = \dim F_{v_{G_4}} = 3, \\ \dim F_{v_{G_5}} = 2, \\ \dim F_{v_{G_6}} = 1. \end{cases}$$

***Corresponding author:** Remm E, Associate Professor, Université de Haute Alsace, 18 Rue des Frères Lumière, 68093 Mulhouse Cedex, France, Tel: +3338933652; E-mail: elisabeth.remm@uha.fr

**Received** October 15, 2015; **Accepted** November 10, 2015; **Published** November 17, 2015

**Citation:** Remm E, Goze M (2015) A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations. J Generalized Lie Theory Appl 9: 235. doi:10.4172/1736-4337.1000235

**Citation:** Remm E, Goze M (2015) A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations. J Generalized Lie Theory Appl 9: 235. doi:10.4172/1736-4337.1000235



**Proposition 3.** *Every $G_i$-$p^3$-associative algebra is third power associative.*

Recall that a third power associative algebra is an algebra $(\mathcal{A},\mu)$ whose associator satisfies $A_\mu(x, x, x) = 0$. Linearizing this relation, we obtain

$$A_\mu \circ \Phi^{\mathcal{A}}_{v_{\Sigma_3}} = 0.$$

Since each of the invariant spaces $F_{G_i}$ contains the vector $v_{\Sigma_3}$, we deduce the proposition.

**Remark.** An important class of third power associative algebras is the class of power associative algebras, that is, algebras such that any element generates an associative subalgebra.

## What are $G_i$-$p^3$-associative algebras?

(1) If $i=1$, since $v_{G_1} = Id$, the class of $G_1$-$p^3$-associative algebras is the full class of associative algebras.

(2) If $i=2$, the associator of a $G_2$-$p^3$-associative algebra $\mathcal{A}$ satisfies

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_2, x_1, x_3) = 0$$

and this is equivalent to

$$A_\mu(x, x, y),$$

for all $x, y \in \mathcal{A}$.

(3) If $i = 3$, the associator of a $G_3$-$p^3$-associative algebra $\mathcal{A}$ satisfies

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_1, x_3, x_2) = 0,$$

that is,

$$A_\mu(x, y, y),$$

for all $x, y \in \mathcal{A}$.

Sometimes $G_2$-$p^3$-associative algebras are called left-alternative algebras, $G_3$-$p^3$-associative algebras are right-alternative algebras. An alternative algebra is an algebra which satisfies the $G_2$ and $G_3$-$p^3$-associativity.

(4) If $i = 4$, we have $A_\mu(x, y, x)$ for all $x, y \in \mathcal{A}$, and the class of $G_3$-$p^3$-associative algebras is the class of flexible algebras.

(5) If $i = 5$, the class of $G_5$-$p^3$-associative algebras corresponds to $G_5$-associative algebras [2].

(6) If $i = 6$, the associator of a $G_6$-$p^3$-associative algebra satisfies

$$A_\mu(x_1,x_2,x_3)+A_\mu(x_2,x_1,x_3)+A_\mu(x_3,x_2,x_1)$$
$$+A_\mu(x_1,x_3,x_2)+A_\mu(x_2,x_3,x_1)A_\mu(x_3,x_1,x_2)=0.$$

If we consider the symmetric product $x\star y=\mu(x,y)+\mu(y,x)$ and the skew-symmetric product $[x,y]=\mu(x,y)-\mu(y,x)$, then the $G_6$-$p^3$-associative identity is equivalent to

$$[x\star y,z]+[y\star z,x]+[z\star x,y]=0.$$

**Definition 4.** *A* ([ , ],$\star$)*-admissible-algebra is a* $\mathbb{K}$*-vector space* $\mathcal{A}$ *provided with two multiplications:*

(a) *a symmetric multiplication* $\star$,

(b) *a skew-symmetric multiplication* [,] *satisfying the identity*

$$[x\star y,z]+[y\star z,x]+[z\star x,y]=0$$

*for any* $x, y \in \mathcal{A}$.

Then a $G_6$ - $p^3$-associative algebra can be defined as ([ , ], $\star$)-admissible algebra.

**Remark: Poisson algebras.** A $\mathbb{K}$-Poisson algebra is a vector space $\mathcal{P}$ provided with two multiplications, an assocative commutative one $x \bullet y$ and a Lie bracket $[x, y]$, which satisfy the Leibniz identity

$$[x\bullet y,z]-x\bullet[y,z]-[x,z]\bullet y=0$$

In studies of Remm [3], it is shown that these conditions are equivalent to provide $\mathcal{P}$ with a nonassociative multiplication $x \cdot y$ satisfying

$$x\cdot(y\cdot z)=(x\cdot y)\cdot z-\frac{1}{3}\{(x\cdot z)\cdot y+(y\cdot z)\cdot x-(y\cdot x)\cdot z-(z\cdot x)\cdot y\}$$

If we denote by $A^R_\cdot(x,y,z)=x\cdot(y\cdot z)$ and $A^L_\cdot(x,y,z)=(x\cdot y)\cdot z$ then the previous identity is equivalent to

$$A^R_\cdot\circ\Phi^{\mathcal{P}}_{w_1}+A^L_\cdot\circ\Phi^{\mathcal{P}}_{w_2}=0$$

where $w_1 = 3Id$ and $w_2=-3Id+\tau_{23}+c_1-c_2-\tau_{12}$. In fact the class of Poisson algebras is a subclass of a family of nonassociative algebras defined by conditions on the associator. The product satisfies

$$A_\cdot(x,y,z)+A_\cdot(y,z,x)+A_\cdot(z,x,y)=0$$

and

$$A_\cdot(x,y,z)+A_\cdot(z,y,x)=0,$$

so it is a subclass of the class of algebras which are flexible and $G_5$-$p^3$-associative [1].

## The Operads $G_i$-$p^3\mathcal{A}ss$ and their Dual

For each $i\in\{1,\cdots,6\}$, the operad for $G_i$-$p^3$-associative algebras will be denoted by $G_i$-$p^3\mathcal{A}ss$. The operads $\{G_i\text{-}p^3\mathcal{A}ss\}_{i=1,\cdots,6}$ are binary quadratic operads, that is, operads of the form $\mathcal{P} = \Gamma(E)/(R)$, where $\Gamma(E)$ denotes the free operad generated by a $\Sigma_2$-module $E$ placed in arity 2 and $(R)$ is the operadic ideal generated by a $\Sigma_3$-invariant subspace $R$ of $\Gamma(E)(3)$. Then the dual operad $\mathcal{P}^!$ is the quadratic operad $\mathcal{P}^!:=\Gamma(E^\vee)/(R^\perp)$, where $R^\perp\subset\Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the pairing

$$\begin{cases} <(x_i\cdot x_j)\cdot x_k,(x_{i'}\cdot x_{j'})\cdot x_{k'}>=0, \text{ if } (i,j,k)\neq(i',j',k'),\\ <(x_i\cdot x_j)\cdot x_k,(x_i\cdot x_j)\cdot x_k>=(-1)^{\varepsilon(\sigma)},\\ \qquad \text{with } \sigma=\begin{pmatrix}1&2&3\\ i&j&R\end{pmatrix},\\ <x_i\cdot(x_j\cdot x_k),x_{i'}\cdot(x_{j'}\cdot x_{k'})>=0, \text{ if } (i,j,k)\neq(i',j',k'),\\ <x_i\cdot(x_j\cdot x_k),x_i\cdot(x_j\cdot x_k)>=-(-1)^{\varepsilon(\sigma)},\\ \qquad \text{with } \sigma=\begin{pmatrix}1&2&3\\ i&j&R\end{pmatrix},\\ <(x_i\cdot x_j)\cdot x_k,x_{i'}\cdot(x_{j'}\cdot x_{k'})>=0,\end{cases} \quad (3)$$

and $(R^\perp)$ is the operadic ideal generated by $R^\perp$. For the general notions of binary quadratic operads [4,5]. Recall that a quadratic operad $\mathcal{P}$ is Koszul if the free $\mathcal{P}$-algebra based on a $\mathbb{K}$-vector space $V$ is Koszul, for any vector space $V$. This property is preserved by duality and can be studied using generating functions of $\mathcal{P}$ and of $\mathcal{P}^!$ [4,6]. Before studying the Koszulness of the operads $G_i$-$p^3\mathcal{A}ss$, we will compute the homology of an associative algebra which will be useful to look if $G_i$-$p^3\mathcal{A}ss$ are Koszul or not.

Let $A_2$ be the two-dimensional associative algebra given in a basis $\{e_1, e_2\}$ by $e_1e_1 = e_2$, $e_1e_2 = e_2e_1 = e_2e_2 = 0$. Recall that the Hochschild homology of an associative algebra is given by the complex $(C_n(\mathcal{A},\mathcal{A}),d_n)$ where $C_n(\mathcal{A},\mathcal{A})=\mathcal{A}\otimes\mathcal{A}^{\otimes n}$ and the differentials $d_n:C_n(\mathcal{A},\mathcal{A})\to C_{n-1}(\mathcal{A},\mathcal{A})$ are given by

**Citation:** Remm E, Goze M (2015) A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations. J Generalized Lie Theory Appl 9: 235. doi:10.4172/1736-4337.1000235



$$d_n(a_0,a_1,\cdots,a_n) = \quad (a_0a_1,a_2,\cdots,a_n)+\sum \quad (-1)\ (a_0,a_1,\cdots,a_ia_{i\ 1},\cdots,a_n)$$
$$+(-1)\ (a_na_0,a_1, \quad ,a_{n\ 1}).$$

Concerning the algebra $A_2$, we have

$$d_1(e_i,e_j)=e_ie_j-e_je_i=0,$$

for any $i, j = 1,2$. Similarly we have

$$\begin{cases} d_2(e_1,e_1,e_1)=2(e_2,e_1)-(e_1,e_2), \\ d_2(e_1,e_1,e_2)=d_2(e_1,e_2,e_1)=-d_2(e_2,e_1,e_1)=(e_2,e_2) \end{cases}$$

and 0 in all the other cases. Then dim $Im\ d_2 = 2$ and dim $Ker\ d_1 = 4$. Then $H_1(A_2, A_2)$ is isomorphic to $A_2$. We have also

$$\begin{cases} d_3(e_1,e_1,e_1,e_1) &= -(e_1,e_2,e_1)+(e_1,e_1,e_2), \\ d_3(e_1,e_1,e_1,e_2) &= (e_2,e_1,e_2)-(e_1,e_2,e_2), \\ d_3(e_1,e_1,e_2,e_1) &= -d_2(e_2,e_1,e_1,e_1)=(e_2,e_2,e_1)-(e_2,e_1,e_2), \\ d_3(e_1,e_2,e_1,e_1) &= (e_1,e_2,e_2)-(e_2,e_2,e_1), \\ d_3(e_1,e_1,e_2,e_2) &= -d_3(e_1,e_2,e_2,e_1)=-d_3(e_2,e_1,e_1,e_2)=d_3(e_2,e_2,e_1,e_1) \\ &= (e_2,e_2,e_2) \end{cases}$$

and $d_3 = 0$ in all the other cases. Then dim $Im\ d_3 = 4$ and dim $Ker\ d_2 = 6$. Thus $H_2(A_2, A_2)$ is non trivial and $A_2$ is not a Koszul algebra.

Now we will study all the operads $G_i$-$p^3\mathcal{A}ss$.

## The operad ($G_1$-$p^3\mathcal{A}ss$)

Since $G_1$-$p^3\mathcal{A}ss$=$\mathcal{A}ss$, where $\mathcal{A}ss$ denotes the operad for associative algebras, and since the operad $\mathcal{A}ss$ is selfdual, we have

$$(G_1\text{-}p^3\mathcal{A}ss)^! = \mathcal{A}ss^! = G_1\text{-}p^3\mathcal{A}ss.$$

We also have

$$\widetilde{G_1\text{-}p^3\mathcal{A}ss} = \widetilde{\mathcal{A}ss} = \mathcal{A}ss,$$

where $\widetilde{\mathcal{P}}$ is the maximal current operad of $\mathcal{P}$ defined in [7,8].

## The operad ($G_2$-$p^3\mathcal{A}ss$)

The operad $G_2$-$p^3\mathcal{A}ss$ is the operad for left-alternative algebras. It is the quadratic operad $\mathcal{P} = \Gamma(E) / (R)$, where the $\Sigma_3$-invariant subspace $R$ of $\Gamma(E)(3)$ is generated by the vectors

$$(x_1\cdot x_2)\cdot x_3 - x_1\cdot(x_2\cdot x_3)+(x_2\cdot x_1)\cdot x_3 - x_2\cdot(x_1\cdot x_3).$$

The annihilator $R^\perp$ of $R$ with respect to the pairing (3) is generated by the vectors

$$\begin{cases} (x_1\cdot x_2)\cdot x_3 - x_1\cdot(x_2\cdot x_3), \\ (x_1\cdot x_2)\cdot x_3 + (x_2\cdot x_1)\cdot x_3. \end{cases} \qquad (4)$$

We deduce from direct calculations that dim $R^\perp = 9$ and

**Proposition 5.** *The* ($G_2$-$p^3\mathcal{A}ss$)$^!$*-algebras are associative algebras satisfying*

$$abc = -bac.$$

Recall that ($G_2\mathcal{A}ss$)$^!$-algebras are associative algebras satisfying

$$abc = bac.$$

and this operad is classically denoted $\mathcal{P}erm$.

**Theorem 6.** *The operad* ($G_2$-$p^3\mathcal{A}ss$)$^!$ *is not Koszul* [9].

*Proof*. It is easy to describe ($G_2$-$p^3\mathcal{A}ss$)$^!$ ($n$) for any $n$. In fact ($G_2$-$p^3\mathcal{A}ss$)$^!$ (4) corresponds to associative elements satisfying

$$x_1x_2x_3x_4 = -x_2x_1x_3x_4 = -x_2(x_1x_3)x_4 = x_1x_3x_2x_4 = -x_1x_2x_3x_4$$

and ($G_2$-$p^3\mathcal{A}ss$)$^!$ (4) = {0}. Let $\mathcal{P}$ be ($G_2$-$p^3\mathcal{A}ss$). The generating function of $\mathcal{P}^!$ = ($G_2$-$p^3\mathcal{A}ss$)$^!$ is

$$g_{\mathcal{P}^!}(x) = \sum_{a\geq 1}\frac{1}{a!}\dim(G_2\text{-}p^3\mathcal{A}ss)^!(a)x^a = x+x^2+\frac{x^3}{2}.$$

But the generating function of $\mathcal{P}$ = ($G_2$-$p^3\mathcal{A}ss$) is

$$g_{\mathcal{P}}(x) = x+x^2+\frac{3}{2}x^3+\frac{5}{2}x^4+O(x^5)$$

and if ($G_2$-$p^3\mathcal{A}ss$) is Koszul, then the generating functions should be related by the functional equation

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x$$

and it is not the case so both ($G_2$-$p^3\mathcal{A}ss$) and ($G_2$-$p^3\mathcal{A}ss$)$^!$ are not Koszul.

By definition, a quadratic operad $\mathcal{P}$ is Koszul if any free $\mathcal{P}$-algebra on a vector space $V$ is a Koszul algebra. Let us describe the free algebra $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ when dim $V$ =1 and 2.

A ($G_2$-$p^3\mathcal{A}ss$)$^!$-algebra $\mathcal{A}$ is an associative algebra satisfying

$$xyz = -yxz,$$

for any $x, y, z \in \mathcal{A}$. This implies $xyzt = 0$ for any $x, y, z \in \mathcal{A}$. In particular we have

$$\begin{cases} x^3 = 0, \\ x^2y = 0, \end{cases}$$

for any $x, y \in \mathcal{A}$. If dim $V$ =1, $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ is of dimension 2 and given by

$$\begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_2e_2 = 0. \end{cases}$$

In fact if $V = \mathbb{K}\{e_1\}$ thus in $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ we have $e_1^3 = 0$. We deduce that $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V) = A_2$ and $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ is not Koszul. It is easy to generalize this construction. If dim $V = n$, then $\dim\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V) = \frac{n(n^2+n+2)}{2}$ and if $\{e_1,\cdots,e_n\}$ is a basis of $V$ then $\{e_i, e_i^2, e_ie_j, e_le_me_p\}$ for $i, j = 1,\cdots,n$ and $l, m, p = 1,\cdots,n$ with $m > l$, is a basis of $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$. For example, if $n = 2$, the basis of $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$ is

$$\{v_1 = e_1, v_2 = e_2, v_3 = e_1^2, v_4 = e_2^2, v_5 = e_1e_2, v_6 = e_2e_1, v_7 = e_1e_2e_1, v_8 = e_1e_2^2\}$$

and the multiplication table is

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | $v_3$ | $v_5$ | 0 | $v_8$ | 0 | $v_7$ | 0 | 0 |
| $v_2$ | $v_6$ | $v_4$ | $-v_7$ | 0 | $-v_8$ | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_5$ | $v_7$ | $v_8$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_6$ | $-v_7$ | $-v_8$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

For this algebra we have

$$\begin{cases} d_1(v_1, v_2) = v_5 - v_6, \\ \frac{1}{2} d_1(v_1, v_6) = v_7 = -d_1(v_1, v_5) = d_1(v_2, v_3), \\ \frac{1}{2} d_1(v_2, v_5) = -v_8 = d_1(v_6, v_2) = -d_1(v_1, v_4), \end{cases}$$

and $Ker\, d_1$ is of dim 64. The space $Im\, d_2$ doesn't contain in particular the vectors $(v_i, v_i)$ for $i = 1, 2$ because these vectors $v_i$ are not in the derived subalgebra. Since these vectors are in $Ker\, d_1$ we deduce that the second space of homology is not trivial.

**Proposition 7.** *The current operad of $G_2$-$p^3\mathcal{A}ss$ is*

$$\widetilde{G_2 - p^3\mathcal{A}ss} = \mathcal{P}erm.$$

This is directly deduced of the definition of the current operad [7].

## The operad ($G_3$-$p^3\mathcal{A}ss$)

It is defined by the module of relations generated by the vector

$$(x_1x_2)x_3 - x_1(x_2x_3) + (x_1x_3)x_2 - x_1(x_3x_2),$$

and $R^\perp$ is the linear span of

$$\begin{cases} (x_1x_2)x_3 - x_1(x_2x_3), \\ (x_1x_2)x_3 + (x_1x_3)x_2. \end{cases}$$

**Proposition 8.** *A ($G_3$-$p^3\mathcal{A}ss$)$^!$-algebra is an associative algebra $\mathcal{A}$ satisfying*

$$abc = -acb,$$

*for any $a, b, c \in \mathcal{A}$.*

Since ($G_3$-$p^3\mathcal{A}ss$)$^!$ is basically isomorphic to ($G_3$-$p^3\mathcal{A}ss$)$^!$ we deduce that ($G_3$-$p^3\mathcal{A}ss$) is not Koszul.

## The operad ($G_4$-$p^3\mathcal{A}ss$)

Remark that a ($G_4$-$p^3\mathcal{A}ss$)-algebra is generally called flexible algebra. The relation

$$A_\mu(x_1, x_2, x_3) + A_\mu(x_3, x_2, x_1) = 0$$

is equivalent to $A_\mu(x, y, x) = 0$ and this denotes the flexibility of ($\mathcal{A}, \mu$).

**Proposition 9.** *A ($G_4$-$p^3\mathcal{A}ss$)$^!$-algebra is an associative algebra satisfying*

$$abc = -cba.$$

This implies that dim ($G_4$-$p^3\mathcal{A}ss$)$^!$ (3) = 3. We have also $x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} = (-1)^{\varepsilon(\sigma)} x_1x_2x_3x_4$ for any $\sigma \in \Sigma_4$. This gives

dim ($G_4$-$p^3\mathcal{A}ss$)$^!$ (4) = 1. Similarly

$$\begin{aligned} x_1x_2(x_3x_4x_5) &= -x_3(x_4x_5x_2)x_1 = x_1(x_4x_5(x_2x_3)) = -x_1x_2(x_3x_5x_4) \\ &= (x_1x_2(x_4x_5))x_3 = -(x_4x_5)(x_2x_1)x_3 = (x_3x_2x_1)x_4x_5 \\ &= -x_1x_2x_3x_4x_5 \end{aligned}$$

(the algebra is associative so we put some parenthesis just to explain how we pass from one expression to another). We deduce ($G_4$-$p^3\mathcal{A}ss$)$^!$ (5) = {0} and more generally ($G_4$-$p^3\mathcal{A}ss$)$^!$ ($a$) = {0} for $a \geq 5$.

The generating function of ($G_4$-$p^3\mathcal{A}ss$)$^!$ is

$$f(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{12}.$$

Let $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V)$ be the free ($G_4$-$p^3\mathcal{A}ss$)$^!$-algebra based on the vector space $V$. In this algebra we have the relations

$$\begin{cases} a^3 = 0, \\ aba = 0, \end{cases}$$

for any $a, b \in V$. Assume that dim $V$ =1. If $\{e_1\}$ is a basis of $V$, then $e_1^3 = 0$ and $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V) = \mathcal{F}_{(G_2 - p^3\mathcal{A}ss)^!}(V)$. We deduce that $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V)$ is not a Koszul algebra.

**Proposition 10.** *The operad for flexible algebra is not Koszul.*

Let us note that, if dim $V$=2 and $\{e_1, e_2\}$ is a basis of $V$, then $\mathcal{F}_{(G_4 - p^3\mathcal{A}ss)^!}(V)$ is generated by $\{e_1, e_2, e_1^2, e_2^2, e_1e_2, e_2e_1, e_1e_2^2, e_1^2e_2, e_2e_1^2, e_2^2e_1, e_1^2e_2^2, e_2^2e_1^2\}$ and is of dimension 12.

**Proposition 11.** *We have*

$$\widetilde{G_4\text{-}p^3\mathcal{A}ss} = (G_4\text{-}\mathcal{A}ss)^!.$$

This means that a $\widetilde{G_4\text{-}p^3\mathcal{A}ss}$ is an associative algebra $\mathcal{A}$ satisfying $abc = cba$, for any $a, b, c \in \mathcal{A}$.

## The operad ($G_5$-$p^3\mathcal{A}ss$)

It coincides with ($G_5$-$\mathcal{A}ss$) and this last has been studied by Remm [2].

## The operad ($G_6$-$p^3\mathcal{A}ss$)

A ($G_6$-$p^3\mathcal{A}ss$)-algebra ($\mathcal{A}, \mu$) satisfies the relation

$$\begin{aligned} &A_\mu(x_1, x_2, x_3) + A_\mu(x_2, x_1, x_3) + A_\mu(x_3, x_2, x_1) \\ &+ A_\mu(x_1, x_3, x_2) + A_\mu(x_2, x_3, x_1) + A_\mu(x_3, x_1, x_2) = 0. \end{aligned}$$

The dual operad ($G_6$-$p^3\mathcal{A}ss$)$^!$ is generated by the relations

$$\begin{cases} (x_1x_2)x_3 = x_1(x_2x_3), \\ (x_1x_2)x_3 = (-1)^{\varepsilon(\sigma)}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}, \text{ for all } \sigma \in \Sigma_3. \end{cases}$$

We deduce

**Proposition 12.** *A ($G_6$-$p^3\mathcal{A}ss$)$^!$-algebra is an associative algebra $\mathcal{A}$ which satisfies*

$$abc = -bac = -cba = -acb = bca = cab,$$

for any $a, b, c \in \mathcal{A}$. In particular

$$\begin{cases} a^3 = 0, \\ aba = aab = baa = 0. \end{cases}$$

**Lemma 13.** *The operad ($G_6$ - $p^3\mathcal{A}ss$)$^!$ satisfies ($G_6$ - $p^3\mathcal{A}ss$)$^!$(4) = {0}.*

*Proof.* We have in ($G_6$-$p^3\mathcal{A}ss$)$^!$(4) that

$$x_1(x_2x_3)x_4 = x_2(x_3x_4)x_1 = -x_1(x_3x_4x_2) = x_1x_3x_2x_4 = -x_1x_2x_3x_4$$

so $x_1x_2x_3x_4 = 0$. We deduce that the generating function of ($G_6$-$p^3\mathcal{A}ss$)$^!$ is

$$f^!(x) = x + x^2 + \frac{x^3}{6}.$$

If this operad is Koszul the generating function of the operad ($G_6$-$p^3\mathcal{A}ss$) should be of the form

$$f(x) = x + x^2 + \frac{11}{6}x^3 + \frac{25}{6}x^4 + \frac{127}{12}x^5 + \cdots$$

**Citation:** Remm E, Goze M (2015) A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations. J Generalized Lie Theory Appl 9: 235. doi:10.4172/1736-4337.1000235



But if we look the free algebra generated by $V$ with $\dim V = 1$, it satisfies $a^3 = 0$ and coincides with $\mathcal{F}_{(G_2-p^3\mathcal{A}ss)^!}(V)$. Then ($G_6$-$p^3\mathcal{A}ss$) is not Koszul.

**Proposition 14.** *We have*

$$\widetilde{G_6 - p^3\mathcal{A}ss} = \mathcal{L}ieAdm^!$$

*that is the binary quadratic operad whose corresponding algebras are associative and satisfying*

$$abc = acb = bac.$$

## Cohomology and Deformations

Let $(\mathcal{A},\mu)$ be a $\mathbb{K}$-algebra defined by quadratic relations. It is attached to a quadratic linear operad $\mathcal{P}$. By deformation of $(\mathcal{A},\mu)$, we mean [10]

- A $\mathbb{K}^*$ non archimedian extension field of $\mathbb{K}$, with a valuation $v$ such that, if $A$ is the ring of valuation and $\mathcal{M}$ the unique ideal of $A$, then the residual field $A/\mathcal{M}$ is isomorphic to $\mathbb{K}$.
- The $A/\mathcal{M}$ vector space $\widetilde{\mathcal{A}}$ is $\mathbb{K}$-isomorphic to $\mathcal{A}$.
- For any $a, b \in \mathcal{A}$ we have that

$$\tilde{\mu}(a,b) - \mu(a,b)$$

belongs to the $\mathcal{M}$-module $\widetilde{\mathcal{A}}$ (isomorphic to $\mathcal{A} \otimes \mathcal{M}$).

The most important example concerns the case where $A$ is $\mathbb{K}[[t]]$, the ring of formal series. In this case $\mathcal{M} = \{\sum_{i\geq 1} a_i t^i, a_i \in K\}$, $\mathbb{K}^* = \mathbb{K}((t))$ the field of rational fractions. This case corresponds to the classical Gerstenhaber deformations. Since $A$ is a local ring, all the notions of valued deformations coincides [11].

We know that there exists always a cohomology which parametrizes deformations. If the operad $\mathcal{P}$ is Koszul, this cohomology is the "standard"-cohomology called the operadic cohomology. If the operad $\mathcal{P}$ is not Koszul, the cohomology which governs deformations is based on the minimal model of $\mathcal{P}$ and the operadic cohomology and deformations cohomology differ [12].

In this section we are interested by the case of left-alternative algebras, that is, by the operad ($G_2$-$p^3\mathcal{A}ss$) and also by the classical alternative algebras.

### Deformations and cohomology of left-alternative algebras

A $\mathbb{K}$-left-alternative algebra $(\mathcal{A},\mu)$ is a $\mathbb{K}$-($G_2$-$p^3\mathcal{A}ss$)-algebra. Then $\mu$ satisfies

$$A_\mu(x_1,x_2,x_3) + A_\mu(x_2,x_1,x_3) = 0.$$

A valued deformation can be viewed as a $\mathbb{K}[[t]]$-algebra ($A \otimes \mathbb{K}[[t]]$, $\mu_t$) whose product $\mu_t$ is given by

$$\mu_t = \mu + \sum_{i\geq 1} t^i \varphi_i.$$

**The operadic cohomology:** It is the standard cohomology $H^*_{(G_2-p^3\mathcal{A}ss)}(\mathcal{A},\mathcal{A})_{st}$ of the ($G_2$-$p^3\mathcal{A}ss$)-algebra $(\mathcal{A},\mu)$. It is associated to the cochains complex

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^3_{st}} \cdots$$

where $\mathcal{P} = (G_2\text{-}p^3\mathcal{A}ss)$ and

$$\mathcal{C}^p_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} = Hom(\mathcal{P}^!(p) \otimes_{\Sigma_p} \mathcal{A}^{\otimes p}, \mathcal{A}).$$

Since $(G_2\text{-}p^3\mathcal{A}ss)^!(4) = 0$, we deduce that

$$H^p_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} = 0 \text{ for } p \geq 4,$$

because the cochain complex is a short sequence

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^2_{st}} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{0} 0.$$

The coboundary operator are given by

$$\begin{cases} \delta^1 f(a,b) &= f(a)b + af(b) - f(ab), \\ \delta^2\varphi(a,b,c) &= \varphi(ab,c) + \varphi(ba,c) - \varphi(a,bc) - \varphi(b,ac) \\ & \quad \varphi(a,b)c + \varphi(b,a)c - a\varphi(b,c) - b\varphi(a,c). \end{cases}$$

**The deformations cohomology:** The minimal model of ($G_2$-$p^3\mathcal{A}ss$) is a homology isomorphism

$$(G_2 - p^3\mathcal{A}ss, 0) \xrightarrow{\rho} (\Gamma(E), \partial)$$

of dg-operads such that the image of $\partial$ consists of decomposable elements of the free operad $\Gamma(E)$. Since $(G_2\text{-}p^3\mathcal{A}ss)(1) = \mathbb{K}$, this minimal model exists and it is unique. The deformations cohomology $H^*(\mathcal{A},\mathcal{A})_{defo}$ of $\mathcal{A}$ is the cohomology of the complex

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^3} \cdots$$

where

$$\begin{cases} \mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(\mathcal{A},\mathcal{A}), \\ \mathcal{C}^k_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(\oplus_{q\geq 2} E_{k-2}(q) \otimes_{\Sigma_q} \mathcal{A}^{\otimes q}, \mathcal{A}). \end{cases}$$

The Euler characteristic of $E(q)$ can be read off from the inverse of the generating function of the operad ($G_2$-$p^3\mathcal{A}ss$)

$$g_{G_2 - p^3\mathcal{A}ss}(t) = t + t^2 + \frac{3}{2}t^3 + \frac{5}{2}t^4 + \frac{53}{12}t^5$$

which is

$$g(t) = t - t^2 + \frac{t^3}{2} + \frac{13}{3}t^5 + O(t^6).$$

We obtain in particular

$$\chi(E(4)) = 0.$$

Each one of the modules $E(p)$ is a graded module $(E_*(p))$ and

$$\chi(E(p)) = \dim E_0(p) - \dim E_1(p) + \dim E_2(p) + \cdots$$

We deduce

- $E(2)$ is generated by two degree 0 bilinear operations $\mu_2 : V \cdot V \to V$,
- $E(3)$ is generated by three degree 1 trilinear operations $V^{\otimes} \to V$,
- $E(4) = 0$.

Considering the action of $\Sigma_n$ on $E(n)$ we deduce that $E(2)$ is generated by a binary operation of degree 0 whose differential satisfies

$$\partial(\mu_2) = 0,$$

$E(3)$ is generated by a trilinear operation of degree one such that

$$\partial(\mu_3) = \mu_2 \circ_1 \mu_2 - \mu_2 \circ_2 \mu_2 + \mu_2 \circ_1 (\mu_2 \cdot \tau_{12}) - (\mu_2 \circ_2 \mu_2) \cdot \tau_{12}.$$

(we have $(\mu_2 \circ_2 \mu_2) \cdot \tau_{12}(a,b,c) = b(ac)$).

Since $E(4) = 0$ we deduce

**Proposition 15.** *The cohomology* $H^*(\mathcal{A},\mathcal{A})_{defo}$ *which governs deformations of right-alternative algebras is associated to the complex*

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^1} \mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^2} \mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \xrightarrow{\delta^3} \mathcal{C}^4_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} \to \cdots$$

*with*

$$\mathcal{C}^1_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 1},V),$$
$$\mathcal{C}^2_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 2},V),$$
$$\mathcal{C}^3_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 3},V),$$
$$\mathcal{C}^4_{\mathcal{P}}(\mathcal{A},\mathcal{A})_{defo} = Hom(V^{\otimes 5},V)\oplus\cdots\oplus Hom(V^{\otimes 5},V),$$

*In particular any* 4*-cochains consists of* 5*-linear maps.*

## Alternative algebras

Recall that an alternative algebra is given by the relation

$$A_\mu(x_1,x_2,x_3) = -A_\mu(x_2,x_1,x_3) = A_\mu(x_2,x_3,x_1).$$

**Theorem 16.** *An algebra* $(\mathcal{A},\mu)$ *is alternative if and only if the associator satisfies*

$$A_\mu \circ \Phi^{\mathcal{A}}_v = 0,$$

*with* $v = 2Id + \tau_{12} + \tau_{13} + \tau_{23} + c_1$.

*Proof.* The associator satisfies $A_\mu \circ \Phi^{\mathcal{A}}_{v_1} = A_\mu \circ \Phi^{\mathcal{A}}_{v_2}$ with $v_1 = Id + \tau_{12}$ and $v_2 = Id + \tau_{23}$. The invariant subspace of $\mathbb{K}[\Sigma_3]$ generated by $v_1$ and $v_2$ is of dimension 5 and contains the vector $\sum_{\sigma\in\Sigma_3}\sigma$. From literature of Remm [1], the space is generated by the orbit of the vector $v$.

**Proposition 17.** *Let* $\mathcal{A}lt$ *be the operad for alternative algebras. Its dual is the operad for associative algebras satisfying*

$$abc - bac - cba - acb + bca + cab = 0.$$

**Remark.** The current operad $\widetilde{\mathcal{A}lt}$ is the operad for associative algebras satisfying $abc = bac = cba = acb = bca$, that is, 3-commutative algebras so

$$\widetilde{\mathcal{A}lt} = \mathcal{L}ie\mathcal{A}dm^!.$$

In literature of Dzhumadil'daev and Zusmanovich [9], one gives the generating functions of $\mathcal{P} = \mathcal{A}lt$ and $\mathcal{P}^! = \mathcal{A}lt^!$

$$g_{\mathcal{P}}(x) = x + \frac{2}{2!}x^2 + \frac{7}{3!}x^3 + \frac{32}{4!}x^4 + \frac{175}{5!}x^5 + \frac{180}{6!}x^6 + O(x^7),$$
$$g_{\mathcal{P}^!}(x) = x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \frac{12}{4!}x^4 + \frac{15}{5!}x^5.$$

and conclude to the non-Koszulness of $\mathcal{A}lt$.

The operadic cohomology is the cohomology associated to the complex

$$\mathcal{C}^p_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} = (Hom(\mathcal{A}lt^!(p)\otimes_{\Sigma_p}\mathcal{A}^{\otimes p},\mathcal{A}),\delta_{st}).$$

Since $\mathcal{A}lt^!(p) = 0$ for $p \geq 6$ we deduce the short sequence

$$\mathcal{C}^1_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} \xrightarrow{\delta^1_{st}} \mathcal{C}^2_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} \to \cdots \to \mathcal{C}^5_{\mathcal{A}lt}(\mathcal{A},\mathcal{A})_{st} \to 0.$$

But if we compute the formal inverse of the function $-g_{\mathcal{A}lt}(-x)$ we obtain

$$x + x^2 + \frac{5}{6}x^3 + \frac{1}{2}x^4 + \frac{1}{8}x^5 - \frac{11}{72}x^6 + O(x^7).$$

Because of the minus sign it can not be the generating function of the operad $\mathcal{P}^! = \mathcal{A}lt^!$. So this implies also that both operad are not Koszul. But it gives also some information on the deformation cohomology. In fact if $\Gamma(E)$ is the free operad associated to the minimal model, then

$$\begin{cases} \dim \chi(E(2)) = -2, \\ \dim \chi(E(3)) = -5, \\ \dim \chi(E(4)) = -12, \\ \dim \chi(E(5)) = -15, \\ \dim \chi(E(6)) = +110. \end{cases}$$

Since $\chi(E(6)) = \sum_i (-1)^i \dim E_i(6)$, the graded space $E(6)$ is not concentred in degree even. Then the 6-cochains of the deformation cohomology are 6-linear maps of odd degree.

## References


1. Goze M, Remm E (2007) A class of nonassociative algebras. Algebra Colloquium 14: 313-326.
2. Goze M, Remm E (2004) Lie-admissible algebras and operads. Journal of Algebra 273: 129-152.
3. Markl M, Remm E (2006) Algebras with one operation including Poisson and other Lie-admissible algebras. Journal of Algebra 299: 171-189.
4. Ginzburg V, Kapranov M (1994) Koszul duality for operads. Duke Math Journal 76: 203-272.
5. Markl M, Shnider S, Stasheff J (2002) Operads in algebra, topology and physics. American Mathematical Society, Providence, RI.
6. Markl M, Remm E (2015) (Non-) Koszulness of operads for n-ary algebras, galgalim and other curiosities. Journal of Homotopy and Related Structures 10: 939-969.
7. Goze M, Remm E (2011) On algebras obtained by tensor product. Journal of Algebra 327: 13-30.
8. Goze M, Remm E (2014) Rigid current Lie algebras. Algebra, geometry and mathematical physics, Springer Proc Math Stat 85: 247-258.
9. Dzhumadil'daev A, Zusmanovich P (2011) The alternative operad is not Koszul. Experimental Mathematics Formerly 20: 138-144.
10. Goze M, Remm E (2004) Valued deformations. Journal of Algebra and Applications 3: 345-365.
11. Fialowsky A, Schlichenmaier M (2003) Global deformations of the Witt algebra of Krichever-Novikov type. Communications in Contemporary Mathematics 5: 921-945.
12. Markl M (2004) Homotopy algebras are homotopy algebras. Forum Math 16: 129-160.