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A formula for the number of Gelfand-Zetlin patterns

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Abstract

In this article, we give a formula for the number of Gelfand-Zetlin patterns, using dimensions of the symmetry classes of tensors.

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1 Introduction

Suppose $\Lambda = (a_1, a_2, \ldots, a_{n-1})$ is a sequence of decreasing non-negative integers. A Gelfand-Zetlin pattern based on Λ is an array of integers:

$$\begin{array}{ccccccccccc} a_1 & & a_2 & & a_3 & & \cdots & & a_{n-2} & & a_{n-1} \\ & b_1 & & b_2 & & b_3 & \cdots & & & b_{n-2} & \\ & & c_1 & & c_2 & & \cdots & & c_{n-3} & & \\ & & & & & & \cdots & & & & \end{array}$$

such that for all i,

$$\begin{aligned} & a_i \geq b_i \geq a_{i+1} \\ & b_i \geq c_i \geq b_{i+1} \\ & \vdots \end{aligned}$$

We denote the set of all Gelfand-Zetlin patterns based on Λ by Γ_Λ. The set Γ_Λ has an important role in the representation theory of general (equivalently, special) and orthogonal linear Lie algebras. For example, let Λ be a dominant weight for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and suppose $L(\Lambda)$ is the corresponding irreducible representation with the highest weight Λ. In [5], Gelfand and Zetlin have proved that the set Γ_Λ can be viewed as a basis for $L(\Lambda)$. For the matrix representations of the elements of the Chevalley basis of $\mathfrak{sl}_n(\mathbb{C})$ with respect to Γ_Λ, see [1] or [3]. The set Γ_Λ is also important from the point of view of *branching rules.* Branching rules are descriptions of the reduction of irreducible representations upon restriction to a subalgebra (subgroup). The first branching rule discovered is possibly the well-known branching rule of representations of the symmetric group S_m in the early twenties.

Since then, it was an exciting job to discover other kinds of branching rules for finite groups, Lie groups, and Lie algebras. We can employ the set Γ_Λ to describe the branching rule for type $\mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{sl}_r(\mathbb{C})$. Suppose we like to restrict the representation $L(\Lambda)$ to $\mathfrak{sl}_r(\mathbb{C})$. For any Gelfand-Zetlin pattern $M \in \Gamma_\Lambda$, let M^i be the i-th row of M. Then, the restriction of $L(\Lambda)$ to $\mathfrak{sl}_r(\mathbb{C})$ is equal to the following direct sum decomposition:

$$\bigoplus_{M\in\Gamma_\Lambda} \frac{1}{|\Gamma_{M^{n-r+1}}|} L\big(M^{n-r+1}\big).$$

The aim of this article is to compute the number of elements of Γ_Λ. Although, one can use the well-known dimension formula of Weyl, but our formula is an alternative one, which uses the irreducible characters of the symmetric group. To give a survey of our main result, suppose

$$m = a_1 + 2a_2 + 3a_3 + \cdots + (n-1)a_{n-1}.$$

We consider a partition π of m with the parts:

$$\pi_i = a_i + a_{i+1} + \cdots + a_{n-1}.$$

Let χ_π be the irreducible character of the symmetric group S_m corresponding to π, (for standard terms about partitions and characters of S_m, see [11]). Also, for any permutation $\sigma \in S_m$, let $c(\sigma)$ be the number of disjoint cycles in the cycle decomposition of σ. It is clear that the function $\xi_n(\sigma) = n^{c(\sigma)}$ is a character of S_m, (for its irreducible constituents, see [12]). Our main result will be

$$\big|\Gamma_\Lambda\big| = \big[\chi_\pi, \xi_n\big],$$

where [,] is the inner product of characters in S_m. In the other words, we will see that

$$\big|\Gamma_\Lambda\big| = \frac{1}{m!} \sum_{\sigma\in S_m} \chi_\pi(\sigma) n^{c(\sigma)}.$$

2 Symmetry classes of tensors

In this section, we are going to review the notion of a symmetry class of tensors. The reader interested in the subject can find a detailed introduction in [9] or [10].

Let V be an n-dimensional complex inner product space and let G be a subgroup of the full symmetric group S_m. Let $V^{\otimes m}$ denote the tensor product of m copies of V and for any $\sigma \in G$, define the *permutation operator*:

$$P_\sigma : V^{\otimes m} \longrightarrow V^{\otimes m}$$

by

$$P_\sigma\big(v_1 \otimes v_2 \otimes \cdots \otimes v_m\big) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

Suppose that χ is a complex irreducible character of G and define the *symmetrizer*:

$$S_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma\in G} \chi(\sigma) P_\sigma.$$

The *symmetry class of tensors* associated with G and χ is the image of S_χ and it is denoted by $V_\chi(G)$. So

$$V_\chi(G) = S_\chi\big(V^{\otimes m}\big).$$

For example, if we let $G = S_m$ and $\chi = \varepsilon$, the alternating character, then we get $\wedge^m V$, the m-th Grassman space over V and if $G = S_m$ and $\chi = 1$, the principal character, then we obtain $V^{(m)}$, the m-th symmetric power of V, as symmetry classes of tensors.

Several monographs and articles have been published on symmetry classes of tensors during the last decades, see for example [9, 10].

Let $v_1, \dots, v_m$ be arbitrary vectors in V and define the *decomposable symmetrized tensor*:

$$v_1 * v_2 * \cdots * v_m = S_\chi\big(v_1 \otimes v_2 \otimes \cdots \otimes v_m\big).$$

Let $\{e_1, \dots, e_n\}$ be a basis of V, and suppose that Γ_n^m is the set of all m-tuples of integers $\alpha = (\alpha_1, \dots, \alpha_m)$ with $1 \le \alpha_i \le n$. For $\alpha = (\alpha_1, \dots, \alpha_m) \in \Gamma_n^m$, we use the notation e_α^* for decomposable symmetrized tensor $e_{\alpha_1} * \cdots * e_{\alpha_m}$. It is clear that $V_\chi(G)$ is generated by all e_α^*; $\alpha \in \Gamma_n^m$. We define an action of G on Γ_n^m by

$$\alpha^\sigma = \big(\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(m)}\big)$$

for any $\sigma \in G$ and $\alpha \in \Gamma_n^m$. Given two elements $\alpha, \beta \in \Gamma_n^m$, we say that $\alpha \sim \beta$ if and only if α and β lie in the same orbit. Suppose that Δ is a set of representatives of orbits of this action and let G_α denote the stabilizer subgroup of α. Define

$$\Omega = \big\{\alpha \in \Gamma_n^m : \big[\chi, 1_{G_\alpha}\big] \neq 0\big\},$$

where [,] denotes the inner product of characters (see [7]). It is well known that $e_\alpha^* \neq 0$, if and only if $\alpha \in \Omega$, see for example [10]. Suppose $\bar{\Delta} = \Delta \cap \Omega$. For any $\alpha \in \bar{\Delta}$, we have the cyclic subspace:

$$V_\alpha^* = \big\langle e_{\alpha^\sigma}^* : \sigma \in G\big\rangle.$$

It is proved that we have the direct sum decomposition:

$$V_\chi(G) = \dot{\sum_{\alpha\in\bar{\Delta}}} V_\alpha^*,$$

see [10] for a proof. It is also proved that

$$s_\alpha := \dim V_\alpha^* = \chi(1)\big[\chi, 1_{G_\alpha}\big],$$

and in particular, if χ is linear then $s_\alpha = 1$ and so the set:

$$\big\{e_\alpha^* : \alpha \in \bar{\Delta}\big\}$$

is an orthogonal basis of $V_\chi(G)$. Also in the case of linear character χ, we have $e_{\alpha^\sigma}^* = \chi(\sigma^{-1})e_\alpha^*$. In the general case, let $\alpha \in \bar{\Delta}$ and suppose

$$e_{\alpha^{\sigma_1}}^*,\ e_{\alpha^{\sigma_2}}^*,\ \dots,\ e_{\alpha^{\sigma_t}}^*$$

is a basis of V_α^* with $\sigma_1 = 1$. Let

$$A_\alpha = \{\alpha^{\sigma_1}, \alpha^{\sigma_2}, \dots, \alpha^{\sigma_t}\}.$$

Then, we define $\hat{\Delta} = \bigcup_{\alpha \in \bar{\Delta}} A_\alpha$. It is clear that

$$\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega,$$

and the set:

$$\{e_\alpha^* : \alpha \in \hat{\Delta}\}$$

is a basis of $V_\chi(G)$. Finally, we remember a formula for dimension of symmetry classes. We have

$$\dim V_\chi(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)},$$

where $c(\sigma)$ denotes the number of disjoint cycles (including cycles of length one) in cycle decomposition of σ.

3 Symmetry classes as $\mathfrak{sl}_n(\mathbb{C})$-modules

In this section, we define a Lie module structure on $V_\chi(G)$, so let L be a complex Lie algebra and suppose that V is an L-module. For any $x \in L$, define

$$D(x) : V^{\otimes m} \longrightarrow V^{\otimes m}$$

by

$$D(x)\big(v_1 \otimes v_2 \otimes \cdots \otimes v_m\big) = \sum_{i=1}^{m} v_1 \otimes \cdots \otimes x v_i \otimes \cdots \otimes v_m.$$

We know that $D(x)S_\chi = S_\chi D(x)$ and so $V_\chi(G)$ is invariant under $D(x)$. Suppose

$$D^*(x) = D(x) \downarrow_{V_\chi(G)},$$

where the down arrow denotes restriction.

Definition 3.1. Define an action of Lie algebra L on $V_\chi(G)$ by

$$\begin{aligned} x\big(v_1 * \cdots * v_m\big) &= D^*(x)\big(v_1 * \cdots * v_m\big) \\ &= \sum_{i=1}^{m} v_1 * \cdots * x v_i * \cdots * v_m. \end{aligned}$$

Then, $V_\chi(G)$ becomes an L-module. In what follows, we will assume that $L = \mathfrak{sl}_n(\mathbb{C})$ and $V = \mathbb{C}^n$, the standard module for L. Then, $V_\chi(G)$ becomes an L-module. In [8], the irreducible constituents of $V_\chi(G)$ are determined. In this section, we give a summery of result of [8]. We also assume that $G = S_m$ and $\chi = \chi_\pi$, the irreducible character of S_m corresponding a partition π. For simplicity, we denote the symmetry class of tensors by

$V_\pi(S_m)$. To describe the irreducible constituents of $V_\pi(S_m)$, it is necessary to introduce some notations.

A Cartan subalgebra for L is

$$H = \big\{ \operatorname{diag}\big(h_1, h_2, \ldots, h_n\big) : h_1 + h_2 + \cdots + h_n = 0 \big\}.$$

For any $1 \leq i \leq n$, define a linear functional:

$$\mu_i : H \longrightarrow \mathbb{C}$$

by

$$\mu_i(h) = h_i,$$

where $h = \operatorname{diag}(h_1, h_2, \ldots, h_n)$, so we have

$$\mu_1 + \mu_2 + \cdots + \mu_n = 0,$$

and hence $\mu_1, \mu_2, \ldots, \mu_{n-1}$ is a basis for H^*.

Now let $\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}$ be the fundamental weights corresponding to H. It is easy to see that for any k:

$$\Lambda_k = \mu_1 + \mu_2 + \cdots + \mu_k.$$

Let $\alpha \in \Gamma_n^m$. We define a composition of m by $m(\alpha) = (m_1, m_2, \ldots, m_n)$, where m_i is the multiplicity of i in α. Suppose

$$\mu_\alpha = \mu_{\alpha_1} + \mu_{\alpha_2} + \cdots + \mu_{\alpha_m}.$$

So we have

$$\mu_\alpha = m_1\mu_1 + m_2\mu_2 + \cdots + m_n\mu_n.$$

Also we can see that

$$\mu_\alpha = \big(m_1 - m_2\big)\Lambda_1 + \big(m_2 - m_3\big)\Lambda_2 + \cdots + \big(m_{n-1} - m_n\big)\Lambda_{n-1}.$$

It is easy to prove that $\mu_\alpha = \mu_\beta$, if and only if $m(\alpha) = m(\beta)$. So, for any $\alpha \in \Gamma_n^m$, we introduce a partition $M(\alpha)$, which is just the multiplicity composition $m(\alpha)$ with a descending arrangement of entries. In fact, any partition of m, with height at most n, is of the form $M(\alpha)$, where $\alpha \in \bar{\Delta}$. For any $h \in H$ and $\alpha \in \hat{\Delta}$, we have

$$h \cdot e_\alpha^* = \left(\sum_{i=1}^{m} \mu_{\alpha_i} \right) (h) e_\alpha^*.$$

In other words, we have

$$h \cdot e_\alpha^* = \mu_\alpha(h) e_\alpha^*.$$

So the set of weights of $V_\pi(S_m)$ is $\{\mu_\alpha : \alpha \in \hat{\Delta}\}$. We also can see by an easy argument that the weight μ_α is dominant, if and only if $M(\alpha) = m(\alpha)$, i.e. $m(\alpha)$ is a partition. For two dominant weights μ_α and μ_β, it is routine to check that μ_β appears in $L(\mu_\alpha)$ as a weight, if and only if $m(\alpha)$ majorizes $m(\beta)$, (for definition of the majorization, see [11]). We are now ready to compute irreducible constituents of $V_\pi(S_m)$. Although, the following theorem is proved in [8] in a more general framework, we prove it here again because what we need is only this special case.

Theorem 3.2. *We have*

$$V_\pi(S_m) = L(\mu_\alpha)^{\chi_\pi(1)},$$

where $\alpha \in \bar{\Delta}$ *is any element with the property* $M(\alpha) = \pi$.

Proof. First of all, note that

$$\bar{\Delta} = \{\alpha \in G_n^m : M(\alpha) \trianglelefteq \pi\},$$

where G_n^m denotes the set of all increasing sequences in Γ_n^m and $\trianglelefteq$ denotes majorization. Suppose that $\eta_1, \ldots, \eta_s$ is the set of all dominant weights of $V_\pi(S_m)$, ordered in such a way that $\eta_j \preceq \eta_i$ implies $i \leq j$ (we say $\eta_j \preceq \eta_i$ iff $\eta_i - \eta_j$ is a sum of positive roots). As we saw above, for any i, there is $\alpha \in \bar{\Delta}$ such that $\eta_i = \mu_\alpha$. Suppose that r_i equals number of $\beta \in \hat{\Delta}$ such that $m(\alpha) = m(\beta)$. Let m_{ij} be the multiplicity of η_i in η_j. Define a sequence of integers as follows:

$$c_1 = r_1,$$
$$c_i = r_i - \sum_{j=1}^{i-1} m_{ij} c_j.$$

Now we have

$$V_\pi(S_m) = \dot{\sum_i} L(\eta_i)^{c_i},$$

so we must show that $c_1 = \chi_\pi(1)$ and $c_i = 0$ for $i \neq 1$. First, note that if $\alpha \in \bar{\Delta}$ has the property $\mu_\alpha = \eta_1$, then $m(\alpha) = \pi$ and hence we have

$$\begin{aligned} c_1 &= r_1 \\ &= \left|\{\beta \in \hat{\Delta} : m(\alpha) = m(\beta)\}\right| \\ &= \left|\{\beta \in \hat{\Delta} : \alpha \sim \beta\}\right| \\ &= s_\alpha \\ &= \chi_\pi(1)\left[1_{(S_m)_\alpha}, \chi_\pi\right] \\ &= \chi_\pi K_{\pi,m(\alpha)} \\ &= \chi_\pi K_{\pi,\pi} \\ &= \chi_\pi(1). \end{aligned}$$

Note that $K_{*,*}$ denote the well-known Kostka numbers, see [11] or [4] for definition. Now, we compute c_2; we have $c_2 = r_2 - m_{2,1}c_1$. As in the above case, we easily see that $r_2 = \chi_\pi(1)K_{\pi,m(\alpha')}$, where α' corresponds to η_2. It is proved that (see [4])

$$K_{\pi,m(\alpha')} = m_{2,1}.$$

Hence,

$$\begin{aligned} c_2 &= \chi_\pi K_{\pi,m(\alpha')} - K_{\pi,m(\alpha')}\chi_\pi(1) \\ &= 0. \end{aligned}$$

By a similar argument, we see that $c_i = 0$ for other "i"s and the result follows. □

Note that this theorem affords a new method of constructing of all irreducible $\mathfrak{sl}_n(\mathbb{C})$-modules, namely, let $\Lambda = \mu_\pi$ be any integral dominant weight of $\mathfrak{sl}_n(\mathbb{C})$. As in [10], we have

$$V_\pi(S_m) = \sum_{i=1}^{\chi_\pi(1)} V_\pi^i(S_m),$$

where $V_\pi^i(S_m)$ is defined as follows: let

$$F : S_m \longrightarrow GL_{\chi_\pi(1)}(\mathbb{C})$$

be the corresponding representation of χ_π with $F(\sigma) = [a_{ij}(\sigma)]$. We introduce the partial symmetrizer S_π^i by

$$S_\pi^i = \frac{\chi(1)}{m!} \sum_{\sigma \in S_m} a_{ii}(\sigma) P_\sigma.$$

Now, $V_\pi^i(S_m)$ is precisely the image of S_π^i. So we have

$$L(\Lambda) = V_\pi^i(S_m)$$

for all $1 \leq i \leq \chi_\pi(1)$. One of the most important consequences of this construction is the following dimension formula, which is different from Weyl's one.

Corollary 3.3. *Let $L(\Lambda)$ be an irreducible $\mathfrak{sl}_n(\mathbb{C})$-module with the highest weight Λ and define the corresponding number m and the partition π as in Section 1. Then,*

$$\dim L(\Lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} \chi_\pi(\sigma) n^{c(\sigma)}.$$

4 The number of the Gelfand-Zetlin patterns

We are ready now to give the interesting relation between dimension of the symmetry classes of tensors $V_\pi(S_m)$ and the number of Gelfand-Zetlin patterns based on Λ. Note that we have the following relations between the weight Λ and the partition π:

$$m = a_1 + 2a_2 + 3a_3 + \cdots + (n-1)a_{n-1},$$
$$\pi_i = a_i + a_{i+1} + \cdots + a_{n-1},$$
$$a_i = \pi_i - \pi_{i+1}.$$

Main theorem. *The number of Gelfand-Zetlin patterns based on Λ is equal to the inner product $[\chi_\pi, \xi_n]$. Equivalently, we have*

$$\dim V_\pi(S_m) = \chi_\pi(1)\left|\Gamma_\Lambda\right|.$$

Remark 4.1. In [12], the inner product $[\chi_\pi, \xi_n]$ is expressed in term of Kostka numbers. Suppose that $\rho = [\rho_1, \ldots, \rho_s]$ is a partition of m with distinct parts $b_1, \ldots, b_l$. Suppose that the multiplicity of b_i in ρ is r_i. If $s \leq n$, define

$$f(n, \rho) = \frac{n!}{(n-s)! r_1! r_2! \cdots r_l!},$$

and let $f(n,\rho)=0$ for $s>n$. Then the multiplicity of χ_π in ξ_n is equal to

$$\sum_{\rho} f(n,\rho)K_{\pi,\rho},$$

where $K_{\pi,\rho}$ is the Kostka number.

Remark 4.2. Note that we can *normalize* Λ in such a way that we have $a_{n-1}=1$. To do this, let $d=a_{n-1}-1$. Define

$$\Lambda^*=\bigl(a_1-d, a_2-d, \ldots, a_{n-2}-d, 1\bigr).$$

Although in general $L(\Lambda)$ and $L(\Lambda^*)$ are non-isomorphic representations, it is clear that $|\Gamma_\Lambda|=|\Gamma_{\Lambda^*}|$. For Λ^*, we have

$$m^*=m-\frac{n(n-1)}{2}\bigl(a_{n-1}-1\bigr),$$

and also the corresponding partition π^* has the parts:

$$\pi_i^*=\pi_i-(n-1)\bigl(a_{n-1}+1\bigr).$$

Hence, we have also the following normalized formula for the number of Gelfand-Zetlin patterns:

$$\bigl|\Gamma_\Lambda\bigr|=\bigl[\chi_{\pi^*}, \xi_n^*\bigr].$$

References

[1] A. O. Barut and R. Raczka. *Theory of Group Representations and Applications.* 2nd ed. PWN-Polish Scientific Publishers, Warszawa, 1980.

[2] R. Carter. *Lie Algebras of Finite and Affine Type.* Cambridge Studies in Advanced Mathematics, **96**, Cambridge University Press, Cambridge, 2005.

[3] L. Frappat, A. Sciarrino and P. Sorba. *Dictionary on Lie Algebras and Superalgebras.* Academic Press, San Diego, CA, 2000.

[4] W. Fulton and J. Harris. *Representation Theory. A First Course.* Graduate Texts in Mathematics, **129**, Springer-Verlag, New York, 1991.

[5] I. M. Gelfand and M. L. Zetlin. Finite dimensional representations of a group of unimodular matrices. *Dukl. Akad. Nauk. SSSR*, **71** (1950), 825–828.

[6] J. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Graduate Texts in Mathematics, **9**, Springer-Verlag, New York, 1972.

[7] M. Isaacs. *Character Theory of Finite Groups.* Academic Press, New York, 1976.

[8] A. R. Madadi and M. Shahryari. Symmetry classes of tensors as $\mathfrak{sl}_n(\mathbb{C})$-modules. *Linear and Multilinear Algebra*, **56** (2008), 517–541.

[9] M. Marcus. *Finite Dimensional Multilinear Algebra, Part I.* Pure and Applied Mathematics, **23**, Marcel Dekker, New York, 1973.

[10] R. Merris. *Multilinear Algebra.* Algebra, Logic and Applications, **8**, Gordon and Breach Science Publishers, Amsterdam, 1997.

[11] B. Sagan. *The Symmetric Group. Representation, Combinatorial Algorithms, and Symmetric Functions.* The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991.

[12] M. Shahryari and M. A. Shahabi. On a permutation character of S_m. *Linear and Multilinear Algebra*, **44** (1998), 45—52.

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