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A Further Research on the Convergence of Wu-Schaback's Multi-quadric Quasi-Interpolation

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Abstract

The paper discusses the error estimate of Wu-Schaback's quasi-interpolant for a wider class of approximated functions (the functions with lower smoothness order). Three cases are considered: a function with a Lipschitz continuous first-order derivative, a continuous function and a Lipschitz continuous function, respectively.

Keywords: Quasi-Interpolation; Multi-quadric; Convergence; Error estimate

Introduction

Quasi-interpolation methods have been used widely in data analysis, and have great values not only in theory but also in many application areas such as medicine, geology, economy and computer science. Multiquadric functions were first proposed by Hardy [1] in 1968, and Franke [2] showed they performed well in many calculations including the numerical experiments. Powell [3], Beatson and Powell [4], and Beatson and Dyn [5] successively proposed a number of quasi-interpolation schemes and discussed the convergence of the schemes. In 1994, Wu and Schaback [6] proposed a useful quasi-interpolation operator $L_D f$ and discussed the convergence and shape preserving properties of this operator. In their convergence theorem (theorem A in our paper), they claimed interpolated functions $f(x)\in C^2$. Based on these papers, Zhang and Wu [7], and Ma and Wu [8] did further researches. In this paper, we discuss the convergence of operator $L_D f$ for a wider range of approximated functions (namely functions with lower smoothness). To prove the convergence, we use two theorems showed by Beatson and Powell [4], and our method differs from that in [6].

Preparation

We assume that there are finite scattered points $\{x_j\}_{j=0}^N$ in the bounded interval $[a,b]$ as follows:

$$a=x_0<x_1<...<x_{N-1}<x_N=b,$$

and the maximum spacing is defined as

$$h=\max_{1\le j\le N}(x_j-x_{j-1}).$$

For $f\in C[a,b]$, we define its norm as

$$\|f\|_\infty=\max_{a\le x\le b}|f(x)|,$$

and its modulus of continuity as

$$\omega(f,\delta)=\max_{\substack{a\le x,x+h\le b\\ |h|\le\delta}}|f(x+h)-f(x)|.$$

The basis functions used in this paper are

$$\phi_j(x)=\sqrt{c^2+(x-x_j)^2},\ j=0,\dots,N,$$

$$\psi_j(x)=\frac{\phi_{j+1}(x)-\phi_j(x)}{2(x_{j+1}-x_j)}-\frac{\phi_j(x)-\phi_{j-1}(x)}{2(x_j-x_{j-1})},\ j=1,\dots N-1,$$

where $c>0$ is a positive shape parameter

In 1994, Wu and Schaback proposed the quasi-interpolation operator L_D:

$$(L_D f)(x)=f_0\alpha_0(x)+f_1\alpha_1(x)+\sum_{j=2}^{N-2}f_j\psi_j(x)+f_{N-1}\alpha_{N-1}(x)+f_N\alpha_N(x),$$

where

$$\alpha_0(x)=\frac{1}{2}+\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},$$

$$\alpha_1(x)=\frac{\phi_2(x)-\phi_1(x)}{2(x_2-x_1)}-\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},$$

$$\alpha_{N-1}(x)=\frac{(x_N-x)-\phi_{N-1}(x)}{2(x_N-x_{N-1})}-\frac{\phi_{N-1}(x)-\phi_{N-2}(x)}{2(x_{N-1}-x_{N-2})},$$

$$\alpha_N(x)=\frac{1}{2}+\frac{\phi_{N-1}(x)-(x_N-x)}{2(x_N-x_{N-1})}.$$

They got the error estimate of this operator as follows:

Theorem A: For $f\in C^2[a,b]$ the quasi-interpolant $L_D f$ satisfies an error estimate of type

$$\|f-L_D f\|_\infty\le K_1h^2+K_2ch+K_3c^2\log h,$$

where positive constants K_1,K_2,K_3 are independent of h and c.

In 1992, Beatson and Powell [4] proposed the quasi-interpolation operator L_B:

$$(L_B f)(x)=f(x_0)\beta_0(x)+\sum_{j=1}^{N-1}f(x_j)\psi_j(x)+f(x_N)\beta_N(x),\quad x\in R,$$

where

$$\beta_0(x)=\frac{1}{2}+\frac{[(x-x_1)^2+c^2]^{1/2}-[(x-x_0)^2+c^2]^{1/2}}{2(x_1-x_0)},\quad x\in R,$$

$$\beta_N(x)=\frac{1}{2}-\frac{[(x-x_N)^2+c^2]^{1/2}-[(x-x_{N-1})^2+c^2]^{1/2}}{2(x_N-x_{N-1})},\quad x\in R.$$

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They proved the following result:

Theorem B: In interval $[a,b]$, the error function $\{f(x)-(L_B f)(x)\}$ satisfies the bound

$$\| f - L_B f \|_\infty \le (1+c/h)\omega(f,h).$$

Meanwhile, in [4] the quasi-interpolation operator L_C was defined as follows:

$$(L_C f)(x) = (L_B f)(x) + f'(x_0)\gamma_0(x) + f'(x_N)\gamma_N(x), \quad x \in R,$$

where

$$\gamma_0(x) = \frac{1}{2}(x-x_0) - \frac{1}{2}[(x-x_0)^2 + c^2]^{1/2}, \qquad x \in R,$$

$$\gamma_N(x) = \frac{1}{2}[(x_N-x)^2 + c^2]^{1/2} - \frac{1}{2}(x_N - x), \quad x \in R.$$

They got the following theorem:

Theorem C: If f has a Lipschitz continuous first-order derivative, then the maximum error of the quasi-interpolant $L_C f$ satisfies the bound

$$\| f - L_C f \|_\infty \le \frac{1}{4}c^2\Omega\left[1 + 2\log\left(1 + \frac{b-a}{c}\right)\right] + \frac{1}{8}h^2\Omega,$$

where $\Omega = \operatorname*{ess\,sup}_{a \le x \le b} | f''(x) |$.

Main Result

It should be noticed that in Theorem A, Wu and Schaback demanded the approximated function $f \in C^2[a,b]$. In this paper, we weaken this condition step by step. Using Theorem B and Theorem C proposed by Beatson and Powell, we get three theorems about convergence estimate for the approximated functions with lower smoothness.

Theorem 1: If f has a Lipschitz continuous first-order derivative, then we can draw the conclusion:

$$\| f - L_D f \|_\infty \le Mch + \frac{1}{4}c^2\Omega\left[1 + 2\log\left(1 + \frac{b-a}{c}\right)\right] + \frac{1}{8}h^2\Omega,$$

where $M = \operatorname*{ess\,sup}_{a \le x \le b} | f^{(2)}(x) |$.

Proof: We notice that quasi-interpolant $L_D f$ and $L_B f$ have the following relationship:

$$L_D f = L_B f + \frac{f(x_1) - f(x_0)}{x_1 - x_0}\gamma_0(x) + \frac{f(x_N) - f(x_{N-1})}{x_N - x_{N-1}}\gamma_N(x). \qquad (1)$$

In [4], Beatson and Powell have showed the relationship between $L_B f$ and $L_C f$:

$$L_C f = L_B f + f'(x_0)\gamma_0(x) + f'(x_N)\gamma_N(x), \qquad (2)$$

where

$$\gamma_0(x) = \frac{1}{2}(x - x_0) - \frac{1}{2}\phi_0(x),$$

$$\gamma_N(x) = \frac{1}{2}\phi_N(x) - \frac{1}{2}(x_N - x).$$

For $x \in [a,b]$, we can easily get the two inequalities:

$$|\gamma_0| \le \frac{1}{2}c, |\gamma_N| \le \frac{1}{2}c \ . \qquad (3)$$

Using (1), (2),(3), we can get

$$| L_D f - L_C f | = \left\| \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} - f'(x_0) \right] \right\| \gamma_0(x) | + \left\| \left[\frac{f(x_N) - f(x_{N-1})}{x_N - x_{N-1}} - f'(x_N) \right] \right\| \gamma_N(x) |$$
$$\le \frac{1}{2}c\, | f'(x_0 + \theta\Delta x_0) - f'(x_0) | + \frac{1}{2}c\, | f'(x_N - \theta\Delta x_{N-1}) - f'(x_N) |$$
$$\le Mch$$

Further, due to Theorem C, we can get

$$\| L_D f - f \|_\infty \le \| L_D f - L_C f \|_\infty + \| L_C f - f \|_\infty$$
$$\le Mch + \frac{1}{4}c^2\Omega\left[1 + 2\log\left(1 + \frac{b-a}{c}\right)\right] + \frac{1}{8}h^2\Omega. \qquad \#$$

Remark 1: Usually we choose $c = O(h)$, then Theorem 1 is basically in accordance with Theorem A.

Further, for the approximated function $f(x)$ with lower smoothness, we can get the following results:

Theorem 2: If is f(x) Lipschitz continuous in [a,b], then

$$\| f - L_D f \|_\infty \le Mc + (1 + c/h)\omega(f,h),$$

where $M = \operatorname*{ess\,sup}_{a \le x \le b} | f'(x) |$

Proof: Due to (3), it is obvious that

$$| L_D f - L_B f | = \left\| \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] \right\| \gamma_0(x) | + \left\| \left[\frac{f(x_N) - f(x_{N-1})}{x_N - x_{N-1}} \right] \right\| \gamma_N(x) |$$
$$\le Mc,$$

Finally, using Theorem B, we have

$$\| L_D f - f \|_\infty \le \| L_D f - L_B f \|_\infty + \| L_B f - f \|_\infty$$
$$\le Mc + (1 + c/h)\omega(f,h). \qquad \#$$

Remark 2: Since $f(x)$ is Lipschitz continuous in $[a,b]$ and $M = \operatorname*{ess\,sup}_{a \le x \le b} | f'(x) |$ we have $\omega(f,h) = \max_{\substack{a \le x_1, x_2 \le b \\ |x_1 - x_2| \le h}} | f(x_1) - f(x_2) | \le Mh.$

Then Theorem 2 can be rewrote as:

If $f(x)$ is Lipschitz continuous in interval $[a,b]$, then

$$\| f - L_D f \|_\infty \le Mc + (1 + c/h)Mh = M(h + 2c),$$

where $M = \operatorname*{ess\,sup}_{a \le x \le b} | f'(x) |$

At last, for the general continuous approximated function $f(x)$, the following theorem of convergence is valid:

Theorem 3: If $f(x)$ is continuous in $[a,b]$, and the interpolation knots are $\{x_j = x_0 + jh\}_{j=0}^N$ (namely equally distributed), then we have the estimation:

$$\| f - L_D f \|_\infty \le (1 + 2c/h)\omega(f,h).$$

Proof: Due to

$$| L_B f - L_D f | \le \left| \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right| | \gamma_0(x) | + \left| \frac{f(x_N) - f(x_{N-1})}{x_N - x_{N-1}} \right| | \gamma_N(x) |$$
$$\le \frac{\omega(f,h)}{h}c,$$

using Theorem B, we have

$$\| f - L_D f \|_\infty \le \| f - L_B f \|_\infty + \| L_B f - L_D f \|_\infty$$
$$\le (1 + c/h)\omega(f,h) + (c/h)\omega(f,h)$$
$$= (1 + 2c/h)\omega(f,h). \qquad \#$$

Remark 3: Assuming $c=O(h)$ in Theorem 3, we can conclude the convergence of Wu-Schaback's quasi-interpolation operator dealing with continuous approximated functions when the interpolated knots are equally distributed.

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