

# A General Stability Result for Viscoelastic Equations with Singular Kernels

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## Abstract

Viscoelasticity with regular relaxation functions has attracted the attention of many researchers over the last half century or so. Several results concerning existence and long-time behavior of solutions have been established. In particular the exponential, polynomial decay and recently what so called the general decay have been proved.

For viscoelasticity, with singular kernels, less attention has been given and few results of existence and exponential decay have been established. In this paper we extend the general decay result, established for regular-kernel viscoelasticity, to that with singular kernels. We also present some numerical test to illustrate our theoretical result.

**Keywords:** General decay; Viscoelastic damping; Singular relaxation function

**AMS Classification:** 35B37, 35L55, 74D05, 93D15.

## Introduction

Since the pioneer work of Dafermos [1,2] in 1970, the viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad (1.1)$$

with smooth kernel, has attracted a great deal of researchers and several existence and stability results have been established. See, for instance, the works of [3-19] where the relaxation function was assumed to be either of polynomial or of exponential decay. Messaoudi [20] studied (1.1) for more general decaying kernels and established some general decay results, from which the usual exponential and polynomial rates are only special cases. After that a series of papers have appeared, using similar techniques, and obtaining similar general decay results. See, among others, [21-25].

Following the work by Mustafa and Messaoudi [26], Lasiecka et al. discussed (1.1) with a relaxation function satisfying

$$g'(s) + H(g(s)) \leq 0, \forall s \geq 0,$$

Where  $H$  is a given continuous positive increasing convex function such that  $H(0) = 0$ , and developed an intrinsic method for determining optimal decay rates.

In all the above mentioned works, kernels are assumed to be regular on  $[0, +\infty)$ .

However, Kinetic theories for chain molecules as mentioned in [27] and some experimental data [28] suggests that this a realistic possibility, at least for some viscoelastic materials like dilute solutions of coiling polymers. Contrary to the regular kernel case, only very few results related to singular (at the origin) kernels have been established. For instance, Hrusa and Renardy [29] studied a model equation in non-linear viscoelasticity and proved local and global existence theorems, allowing the memory function to have a singularity. To achieve their result, they approximated the equation by equations with regular kernels and then used the energy estimates to prove convergence of the approximate solutions. In [30], the authors showed that a singular kernel may yield smoothing effects for the solution of an evolution problem, though the gain in regularity cannot be derived without

specifying the kind of singularity [31]. Gentili considered a linear viscoelastic material with a relaxation function which may exhibit an initial singularity. He used the Laplace transform to study existence, uniqueness and asymptotic behavior of the solution to the dynamic problem. To provide these results, the author required the relaxation function to satisfy only restrictions deriving from Thermodynamics. He also used the energy method to establish a stability theorem and obtained a regularity result for a class of singular kernels which ensures the asymptotic stability of the solution. Tatar considered

$$u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds = 0,$$

together with initial and Dirichlet-boundary conditions and for a relaxation function

$$g(t) = t^{-\alpha} e^{-\beta t} / \Gamma(1-\alpha), \quad 0 < \alpha < 1, \quad \beta > 0$$

and proved an exponential decay result. Notice that the kernel here exhibits an initial singularity, summable, and decays exponentially at infinity. This type of kernels appears mostly in fractional calculus [32]. Wu [33] extended Tatar's result to the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u|^p u = 0,$$

with  $\rho, p > 0$  and  $g$  in [34]. We refer the reader to Carillo et al. [35-37] for more recent results regarding viscoelastic problems with singular Kernels.

In this paper we are concerned with the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times (0, \infty) \\ u = 0, & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (1.2)$$

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Where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  ( $n \geq 1$ ) with a smooth boundary  $\partial\Omega$  and the relaxation function  $g$  is a positive non increasing function which can exhibit a singularity at 0. Our aim in this work is to show that, with a slight modification in the arguments of [20], we extend the general decay result, established for regular relation functions, to singular kernels. As we show later, the exponential decay results of Tatar [31] and Wu [27], among others, are only special cases. We also give some examples and present some numeric to illustrate our decay results. The paper is organized as follows. In section 2, we present some hypotheses and technical lemmas and our main decay result. In section 3, the proof of our main result, as well as some illustrating examples will be given. Section 4 is devoted to the numerical setting and tests of our decay results. We finish our paper with some concluding remarks [11].

## Preliminaries

In this section we state our hypotheses, give, without proof a standard existence theorem, and state our main decay result. So, we assume

(H1)  $g: (0, +\infty) \rightarrow (0, +\infty)$  is a differentiable integrable function satisfying

$$1 - \int_0^{+\infty} g(s)ds = l > 0 \quad (2.1)$$

(H2) There exists a differentiable non increasing function  $\xi: (0, +\infty) \rightarrow (0, +\infty)$  such that

$$g'(t) \leq -\xi(t)g(t), \quad \forall t > 0.$$

**Remark 2.1:** These conditions allow a larger class of functions than that considered in [14,20] and others. In particular it allows singular integrable functions such as

$$g(t) = t^{-a}e^{-bt^\nu}, \quad 0 < a < 1, \quad 0 < \nu \leq 1, \quad b > 0.$$

Notice, also, that  $|\xi'(t)/\xi(t)| \leq M$  imposed in [14,20] is no longer required. Now, we introduce the energy functional

$$E(t) := \frac{1}{2} \int_{\Omega} \left( u_t^2 + (1 - \int_0^t g(s)ds) |\nabla u|^2 \right) dx + \frac{1}{2} (g \circ \nabla u)(t), \quad (2.2)$$

Where

$$(g \circ \nabla u)(t) = \int_{\Omega} \int_0^t g(t-s) |\nabla u(t) - \nabla u(s)|^2 ds dx.$$

For completeness, we state the following existence result, which can be proved using the Galerkin method [7]. For more about existence, see [38] and [29].

**Proposition 2.1:** Let  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$  be given. Assume that  $g$  satisfies (G1). Then problem (1.2) has a unique global solution

$$u \in C([0, +\infty); H_0^1(\Omega)), \quad u_t \in C([0, +\infty); L^2(\Omega)). \quad (2.3)$$

Our main stability result is

**Theorem 2.2:** Assume that (H1) and (H2) hold. Then, for any  $t_0 > 0$ , there exist two positive constants  $k$  and  $K$  such that the solution of (1.2) satisfies

$$E(t) \leq K e^{-k \int_{t_0}^t \xi(s)ds} \quad \forall t \geq t_0. \quad (2.4)$$

## Main Result

In this section we prove our main decay result. We will use  $c$ , throughout this paper, to denote a generic positive constant. We start with the following lemmas.

**Lemma 2.2:** [20] Let  $u$  be the solution of (1.2). Then the energy

functional satisfies

$$E'(t) = \frac{1}{2} (g' \circ \nabla u) - \frac{1}{2} g(t) \int_{\Omega} |\nabla u|^2 dx \leq 0. \quad (3.1)$$

**Lemma 2.3:** [20] For any  $u \in H_0^1(\Omega)$ , we have

$$\int_{\Omega} \left( \int_0^t g(t-\tau) (u(t) - u(\tau)) d\tau \right)^2 dx \leq (1-l) C_p^2 (g \circ \nabla u)(t),$$

$C_p$  is the Poincaré constant.

**Lemma 2.4:** Under the assumption (H1) and (H2), the functional

$$\Phi(t) := \int_{\Omega} u_t dx$$

Satisfies, along the solution, the estimate

$$\Phi'(t) \leq -\frac{l}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} u_t^2 + c(g \circ \nabla u). \quad (3.2)$$

**Proof:** By using equation (1.2), we easily see that

$$\Phi'(t) = \int_{\Omega} u_t^2 dx - \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \nabla u(t) \cdot \int_0^t g(t-\tau) \nabla u(\tau) d\tau dx. \quad (3.3)$$

We then estimate the third term in the right-hand side of (3.3), as in Lemma 3.4 [20], to obtain

$$\begin{aligned} \int_{\Omega} \nabla u(t) \cdot \int_0^t g(t-\tau) \nabla u(\tau) d\tau dx &\leq \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx \\ &+ \frac{1}{2} \left[ \left(1 + \frac{1}{\eta}\right) (1-l) (g \circ \nabla u)(t) + (1+\eta) (1-l)^2 \int_{\Omega} |\nabla u(t)|^2 dx \right], \quad \forall \eta > 0. \end{aligned} \quad (3.4)$$

By combining (3.3) and (3.4), we arrive at

$$\Phi'(t) \leq \int_{\Omega} u_t^2 dx + \frac{1}{2} \left(1 + \frac{1}{\eta}\right) (1-l) (g \circ \nabla u)(t) - \frac{1}{2} [1 - (1+\eta)(1-l)^2] \int_{\Omega} |\nabla u(t)|^2 dx, \quad \forall \eta > 0 \quad (3.5)$$

By choosing  $\eta = l/(1-l)$ , (3.2) is established.

**Lemma 2.5:** Under the assumption (H1) and (H2), the functional

$$\Psi(t) := -\int_{\Omega} u_t \int_0^t g(t-s) (u(t) - u(s)) ds dx$$

With  $t_0 > 0$  satisfies, along the solution and for any  $\delta > 0$ , the estimate

$$\Psi'(t) \leq -\left( \int_0^t g(s) ds - \delta \right) \int_{\Omega} u_t^2 + \delta \int_{\Omega} |\nabla u|^2 + \frac{c}{\delta} (g \circ \nabla u) - \frac{c}{\delta} (g' \circ \nabla u). \quad (3.6)$$

**Proof:** Direct computations, using (1.2), yield

$$\begin{aligned} \Psi'(t) &= \int_{\Omega} \nabla u(t) \cdot \left( \int_0^t g(t-\tau) (\nabla u(t) - \nabla u(\tau)) d\tau \right) dx \\ &- \int_{\Omega} \left( \int_0^t g(t-\tau) \nabla u(\tau) d\tau \right) \cdot \left( \int_0^t g(t-\tau) (\nabla u(t) - \nabla u(\tau)) d\tau \right) dx \\ &- \int_{\Omega} u_t \int_0^t g'(t-\tau) (u(t) - u(\tau)) d\tau dx - \left( \int_0^t g(s) ds \right) \int_{\Omega} u_t^2 dx \end{aligned} \quad (3.7)$$

Similarly to (3.3), we estimate the right-hand side terms of (3.7). So, by Lemma 2.3, Young's inequality and the fact that

$$\int_0^t g(\tau) d\tau \leq \int_0^{+\infty} g(\tau) d\tau = 1-l,$$

the first term of (3.7) gives

$$\begin{aligned} &-\int_{\Omega} \nabla u(t) \cdot \left( \int_0^t g(t-\tau) (\nabla u(t) - \nabla u(\tau)) d\tau \right) dx \\ &\leq \delta \int_{\Omega} |\nabla u|^2 dx + \frac{1-l}{4\delta} (g \circ \nabla u)(t), \quad \forall \delta > 0. \end{aligned} \quad (3.8)$$

Similarly to (3.4), the second term of (3.7) can be estimated as follows

$$\int_{\Omega} \left( \int_0^t g(t-s) \nabla u(s) ds \right) \left( \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds \right) dx \quad (3.9)$$

$$\leq \left( 2\delta + \frac{1}{4\delta} \right) (1-l)(g \circ \nabla u)(t) + 2\delta(1-l)^2 \int_{\Omega} |\nabla u|^2 dx.$$

As for the third and the fourth terms of (3.7) we have

$$-\int_{\Omega} u_t \int_0^t g'(t-\tau) (u(t) - u(\tau)) d\tau dx$$

$$\leq \delta \int_{\Omega} |u_t|^2 dx - \frac{g(t_0)}{4\delta} C_p^2 \int_0^t g'(t-\tau) |\nabla u(t) - \nabla u(\tau)|^2 d\tau dx \quad (3.10)$$

$$\leq \delta \int_{\Omega} |u_t|^2 dx - \frac{g(t_0)}{4\delta} C_p^2 (g' \circ \nabla u)(t)$$

By combining (3.7)-(3.10), the assertion of the lemma is established.

**Proof of Theorem 2.2:** We set

$$F(t) := NE(t) + \varepsilon \Phi(t) + \Psi(t), \quad (3.11)$$

where  $N$  and  $\varepsilon$  are positive constants. Direction substitution leads to

$$F'(t) \leq - \left[ \left( \int_0^t g(s) ds - \delta \right) - \varepsilon \right] \int_{\Omega} u_t^2 dx - \left[ \frac{l\varepsilon}{2} - \delta \right] \int_{\Omega} |\nabla u|^2 + \left[ \frac{N}{2} - \frac{c}{\delta} \right] (g' \circ \nabla u) + c(g \circ \nabla u).$$

Since  $g(s) > 0$ , we can choose  $t_1 > t_0$  such that  $\int_0^{t_1} g(s) ds = g_0 > 0$ . Hence, we obtain

$$F'(t) \leq - \left[ (g_0 - \delta) - \varepsilon \right] \int_{\Omega} u_t^2 dx - \left[ \frac{l\varepsilon}{2} - \delta \right] \int_{\Omega} |\nabla u|^2 + \left[ \frac{N}{2} - \frac{c}{\delta} \right] (g' \circ \nabla u) + c(g \circ \nabla u), \quad \forall t \geq t_1.$$

At this point, we choose  $\delta$  so small that

$$g_0 - \delta > \frac{1}{2} g_0, \quad \delta < \frac{l}{8} g_0$$

And

$$\frac{g_0}{4} < \varepsilon < \frac{g_0}{2}$$

So that

$$(g_0 - \delta) - \varepsilon > 0, \quad \frac{l\varepsilon}{2} - \delta > 0.$$

Finally, we then pick  $N$  so large that  $\frac{N}{2} - \frac{c}{\delta} \geq 0$  and  $F(t) \propto E(t)$ . Therefore, we have, for some  $\gamma > 0$ ,

$$F'(t) \leq -\gamma E(t) + c(g \circ \nabla u), \quad \forall t \geq t_1. \quad (3.12)$$

By using (H2) and (3.12), we easily deduce

$$(\xi(t)F(t))' = \xi(t)F'(t) + \xi'(t)F(t)$$

$$\leq -\gamma \xi(t)E(t) + c(g' \circ \nabla u), \quad \forall t \geq t_1,$$

Since  $\xi' \leq 0$ . By recalling  $F(t) \propto E(t)$ , we get

$$L'(t) \leq -\gamma \xi(t)E(t) \leq -k \xi(t)L(t), \quad \forall t \geq t_1$$

Where

$$L(t) = \xi(t)F(t) + cE(t) \propto E(t). \quad (3.13)$$

$$L(t) \leq K e^{-k \int_{t_1}^t \xi(s) ds} \quad \forall t \geq t_1. \quad (3.14)$$

Then, (2.4) follows by (3.14) and continuity of  $\xi$  and  $E$ .

Decay examples. To illustrate our result, we give four examples of singular relaxation functions.

1. Let

$$g(t) = c_0 t^{-a} e^{-bt}, \quad 0 < a < 1, \quad b > 0,$$

And  $c_0$  small enough so that  $\int_0^{+\infty} g(t) < 1$ . Direct calculations show that  $\xi(t) = b + at^{-1}$ . So, the decay estimate (2.4) gives

$$E(t) \leq K t^{-ak} e^{-bkt},$$

for two positives constants  $k$  and  $K$ .

2. Let

$$g(t) = c_0 [\ln(1+t)]^{-a-bt}, \quad 0 < a < 1, \quad b > 0,$$

and  $c_0$  small enough so that  $\int_0^{+\infty} g(t) < 1$ . Direct calculations show that

$$\xi(t) = b + \frac{a}{(1+t)\ln(1+t)}. \quad \text{So, the decay estimate (2.4) gives}$$

$$E(t) \leq K [\ln(1+t)]^{-ak} e^{-bkt},$$

for two positives constants  $k$  and  $K$ .

3. Let  $g(t) = c_0 \frac{e^{-\sqrt{t}}}{\sqrt{t}}$ , for  $c_0$  small enough so that  $\int_0^{+\infty} g(t) < 1$ . Direction

calculation show that  $\xi(t) = \frac{1}{2t} + \frac{1}{2\sqrt{t}}$ . So, the decay estimate (2.4) gives

$$E(t) \leq K t^{-k/2} e^{-k\sqrt{t}},$$

for two positives constants  $k$  and  $K$ .

4. Let

$$g(t) = c_0 (1+t)^{-b} (\ln(1+t))^{-a}, \quad 0 < a < 1, \quad b > 1,$$

and  $c_0$  small enough so that  $\int_0^{+\infty} g(t) < 1$ . Direction calculations show

$$\text{that } \xi(t) = \frac{b}{1+t} + \frac{a}{(1+t)\ln(1+t)}. \quad \text{So, the decay estimate (2.4) gives}$$

$$E(t) \leq K (1+t)^{-kb} (\ln(1+t))^{-ka},$$

for two positives constants  $k$  and  $K$ .

## Numerical Test

In this section, we present a two dimensional case of system (1.1) in order to illustrate our theoretical decay result.

## Numerical scheme

For computational purposes, we rewrite (1.1) as follows

$$u_{tt} - \left( 1 - \int_0^t g(s) ds \right) \Delta u - \int_0^t g(t-s) (\Delta u(s) - \Delta u(t)) ds = 0. \quad (4.1)$$

We consider a square domain  $[0, 1] \times [0, 1]$  meshed uniformly in  $N_x \times N_y$  grids with the space steps  $\Delta x = \Delta y = 1/\max(N_x, N_y)$ . We chose a time interval  $[0, T]$  subdivided uniformly into  $N_t = T/\Delta t$  sub-intervals with a time step  $\Delta t = \alpha \Delta x$ . The solution  $u(x, y, t)$  approximated at each point of the mesh  $(x_i, y_j) = (i\Delta x, j\Delta y)$  and at any time  $t^n = n\Delta t$  is given by  $U_{i,j}^n \approx u(x_i, y_j, t^n) = u_{i,j}^n$ . At each time  $t^n$  the interval  $[0, t^n]$  is subdivided uniformly in  $n$  sub-intervals using the same time step  $\Delta t$  where the function  $g$  at each time  $s^k = k\Delta t$  is given by  $g(s^k)$ .

The full discretization of (1.1) in time and space is given by

$$\Delta_t U_{i,j}^n - l \Delta U_{i,j}^n + \int_0^{t^n} g(t^n-s) (\Delta U_{i,j}(s) - \Delta U_{i,j}^n) ds = 0, \quad (4.2)$$

With the boundary condition

$$U_{0,j}^n = U_{N_x,j}^n = U_{i,0}^n = U_{i,N_y}^n = 0, \quad \forall 0 \leq i, j \leq N_x, N_y. \quad (4.3)$$

And the initial conditions

$$U_{i,j}^0 = \sin(\pi x_i) \sin(\pi y_j), \frac{\%U_{i,j}^1 - U_{i,j}^0}{\Delta t} = 1, \forall 0 \leq i, j \leq N_x, N_y, \quad (4.4)$$

Where

$$\Delta_l U_{i,j}^n = \frac{U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}}{\%(\Delta t)^2}, l = 1 - \int_0^t g(s) ds$$

$$\Delta U_{i,j}^n = \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{(\Delta x)^2} - \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{(\Delta y)^2}$$

And

$$\Delta U_{i,j}(s) = \frac{U_{i+1,j}(s) - 2U_{i,j}(s) + U_{i-1,j}(s)}{(\Delta x)^2 \%} + \frac{U_{i,j+1}(s) - 2U_{i,j}(s) + U_{i,j-1}(s)}{(\Delta y)^2}.$$

Then, the proposed scheme to solve problem (1.1) is as follows:

$$\forall i = 0, \dots, N_x, \forall j = 0, \dots, N_y, \forall n = 1, \dots, N_t - 1 :$$

$$U_{i,j}^{n+1} = 2U_{i,j}^n - U_{i,j}^{n-1} + (\Delta t)^2 \left( l \Delta U_{i,j}^n + \int_0^{t^n} g(t^n - s) (\Delta U_{i,j}^n - \Delta U_{i,j}(s)) ds \right). \quad (4.5)$$

### Consistency-Global truncation error

The global truncation error  $e_{i,j}^n$  is obtained by replacing the approximate solution  $U_{i,j}^n$  by the value of the exact solution  $u(x, y, t)$  at the point  $(x_i, y_j, t^n)$  ( $u(x_i, y_j, t^n) = u_{i,j}^n$ ) in the scheme (4.2) as follows:

$$e_{i,j}^n = \Delta_l u_{i,j}^n - \left( 1 - \int_0^{t^n} g(s) ds \right) \Delta u_{i,j}^n + \int_0^{t^n} g(t^n - s) (\Delta u_{i,j}(s) - \Delta u_{i,j}^n) ds \neq 0. \quad (4.6)$$

Suppose that  $u$  is at least  $C^2$  and compute the Taylor approximation of the second derivative of  $u$  with respect to  $t, x$  and  $y$  respectively:

$$\Delta_l u_{i,j}^n = \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = \frac{\partial^2 u_{i,j}^n}{\partial t^2} + \frac{2}{4!} \Delta t^2 \frac{\partial^4 u_{i,j}^n}{\partial t^4} + O(\Delta t^4), \quad (4.7)$$

$$\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} = \frac{\partial^2 u_{i,j}^n}{\partial x^2} + \frac{2}{4!} \Delta x^2 \frac{\partial^4 u_{i,j}^n}{\partial x^4} + O(\Delta x^4),$$

$$\frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} = \frac{\partial^2 u_{i,j}^n}{\partial y^2} + \frac{2}{4!} \Delta y^2 \frac{\partial^4 u_{i,j}^n}{\partial y^4} + O(\Delta y^4),$$

$$\Delta u_{i,j}^n = \frac{\partial^2 u_{i,j}^n}{\partial x^2} + \frac{\partial^2 u_{i,j}^n}{\partial y^2} + \frac{2}{4!} \left( \Delta x^2 \frac{\partial^4 u_{i,j}^n}{\partial x^4} + \Delta y^2 \frac{\partial^4 u_{i,j}^n}{\partial y^4} \right) + O(\Delta x^4 + \Delta y^4), \quad (4.8)$$

$$\Delta_l u_{i,j}(s) = \frac{\partial^2 u_{i,j}(s)}{\partial x^2} + \frac{\partial^2 u_{i,j}(s)}{\partial y^2} + \frac{2}{4!} \left( \Delta x^2 \frac{\partial^4 u_{i,j}(s)}{\partial x^4} + \Delta y^2 \frac{\partial^4 u_{i,j}(s)}{\partial y^4} \right) + O(\Delta x^4 + \Delta y^4). \quad (4.9)$$

By substituting (4.7)-(4.8)-(4.9) into (4.6), we find

$$e_{i,j}^n = \frac{\partial^2 u_{i,j}^n}{\partial t^2} - \left( 1 - \int_0^{t^n} g(s) ds \right) \Delta u_{i,j}^n + \int_0^{t^n} g(t^n - s) (\Delta u_{i,j}(s) - \Delta u_{i,j}^n) ds$$

$$+ \frac{1}{12} \left[ \Delta t^2 \frac{\partial^4 u_{i,j}^n}{\partial t^4} + \Delta x^2 \left( \frac{\partial^4 u_{i,j}^n}{\partial x^4} + \int_0^{t^n} g(t^n - s) \frac{\partial^4 u_{i,j}(s)}{\partial x^4} ds \right) \right]$$

$$+ \Delta y^2 \left( \frac{\partial^4 u_{i,j}^n}{\partial y^4} + \int_0^{t^n} g(t^n - s) \frac{\partial^4 u_{i,j}(s)}{\partial y^4} ds \right) + O(\Delta t^4 + \Delta x^4 + \Delta y^4).$$

But the first term in the right hand side vanishes, using problem (1.1). Hence, we obtain

$$e_{i,j}^n = \frac{1}{12} \left[ \Delta t^2 \frac{\partial^4 u_{i,j}^n}{\partial t^4} + \Delta x^2 \left( \frac{\partial^4 u_{i,j}^n}{\partial x^4} + \int_0^{t^n} g(t^n - s) \frac{\partial^4 u_{i,j}(s)}{\partial x^4} ds \right) \right]$$

$$+ \Delta y^2 \left( \frac{\partial^4 u_{i,j}^n}{\partial y^4} + \int_0^{t^n} g(t^n - s) \frac{\partial^4 u_{i,j}(s)}{\partial y^4} ds \right) + O(\Delta t^4 + \Delta x^4 + \Delta y^4).$$

$$O(\Delta t^2 + \Delta x^2 + \Delta y^2) \quad (4.10)$$

We conclude that scheme (4.5) is consistent of order 2 in time and in space.

### Stability-Estimation of the discrete energy

The discrete energy of system (4.2)-(4.4) is given by

$$E^{n+1/2} = \frac{1}{2} \sum_{i,j} \left| \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\% \Delta t} \right|^2 \Delta x \Delta y + \frac{1}{2} \left( 1 - \int_0^{t^n} g(s) ds \right) \sum_{i,j} \Delta U_{i,j}^n U_{i,j}^{n+1} \Delta x \Delta y$$

$$+ \frac{1}{2} \sum_{k=0}^n g(t^n - s^k) \sum_{i,j} (-\Delta U_{i,j}^n + \Delta U_{i,j}^k) U_{i,j}^{n+1} \Delta t \Delta x \Delta y. \quad (4.11)$$

If we compute the variation of the discrete energy, we obtain

$$\Delta E = E^{n+1/2} - E^{n-1/2} = \frac{1}{2} \sum_{i,j} \sum_{k=0}^{n-1} F(k) \Delta t \Delta x \Delta y,$$

Where

$$F(k) = \begin{cases} g(t^1) U_{i,j}^{n-1} (\Delta U_{i,j}^{n-1} - \Delta U_{i,j}^n) & , k = n-1, \\ \left[ g(t^{n-k}) U_{i,j}^{n-1} - g(t^{n-k-1}) U_{i,j}^{n-1} \right] \Delta U_{i,j}^k & , k = 0, \dots, n-2. \end{cases}$$

The quantity  $\Delta E$  is numerically strictly negative as expected.

### Decay behavior of the discrete energy

In order to show the non-increasing behavior of  $g$  and the decay relation  $E(t^n) \leq Eps(t^n)$ , we consider the examples stated in Section 3 and choose the following parameters:

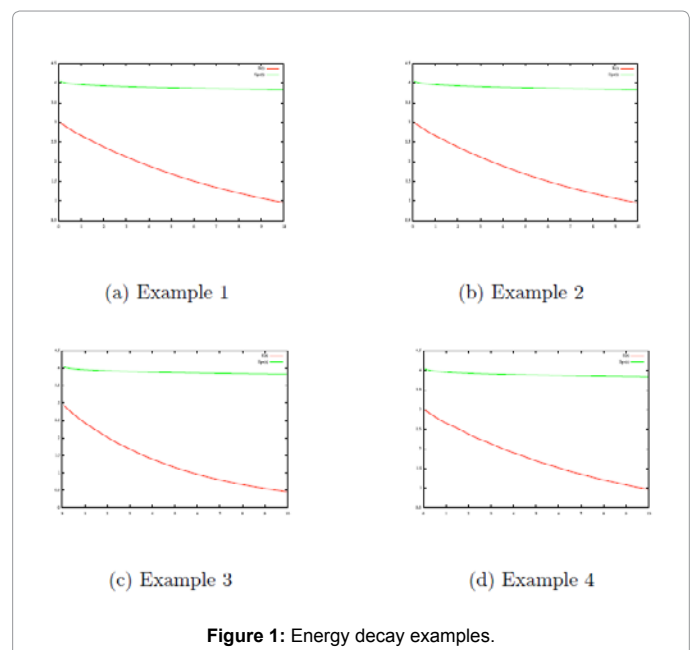
$$\alpha = 0.3, N_x = N_y = 50, T = 10, k = 0.01, K = 4, c_0 = 0.01, a = 0.1, b = 1.1.$$

The space step is  $\Delta x = \Delta y = 0.1$  and the time step  $\Delta t = 0.033$  (Figure 1).

### Conclusions

In this work, we have the following conclusions

- The general decay result, known for viscoelastic problems with regular kernels, has been extended successfully to problems with singular kernels.



- The decay result is established with weaker conditions on the function  $\xi(t)$ .
- The exponential decay results of [33,34] are only special cases.
- The numerical tests presented for the four types of relaxation functions are in accordance with our theoretical result.

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