A General System of Regularized Non-convex Variational Inequalities

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Abstract

In this communication, we introduced a general system of regularized non-convex variational inequalities (GSRNVI) and established an equivalence between this system and fixed point problems. By using this equivalence we define a projection iterative algorithm for solving GSRNVI, we also proved existence and uniqueness of GSRNVI. The convergence analysis of the suggested iterative algorithm is studied.

Keywords: General system of regularized non-convex variational inequalities; Uniformly t-prox regular sets; Iterative schemes; Convergence analysis

Introduction

The originally variational inequality problem introduced by Stampacchia [1] in the early sixties has a great impact and in influence in the development of almost all branches of pure and applied sciences and has witnessed an explosive growth in theoretical advances, algorithmic development. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various algorithms for solving variational inequalities and related optimizations [2-6]. Verma [7-10] studied some systems of variational inequality with single valued mappings and suggest some iterative algorithms to compute approximate solutions of these systems in Hilbert spaces. Agarwal et al. [11] studied sensitivity analysis for a system of generalized nonlinear mixed quasi variational inclusions with single valued mappings. Several authors studied different kinds of systems of variational inequalities and suggested iterative algorithms to find the approximate solutions of the systems [12-15]. We remark that the results regarding the existence of solutions and iterative schemes for solving the system of variational inequalities and related problems are being considered in the setting of convex sets and the technique defined on the characteristics of the projection operator over convex a set which does not hold in general when the sets are non-convex. It is well known that the uniform prox regular sets are convex and include the convex set as special cases. Wen [16] considered a system of non-convex variational inequalities with different nonlinear operator and asserted that this system is equivalent to the fixed point problem and suggested an iterative algorithm for the system of non-convex variational inequalities. The convergence analysis of the proposed iterative algorithm under some certain assumption is also studied. In [17] point out the equivalence formulation used by Wen [16] is not correct. Inspired and motivated by the works of [18-26], we introduced and studied a general system of regularized non-convex variational inequalities. By using the equivalence, we defined a projection iteration algorithm for solving GSRNVI. Further, we proved the existence and uniqueness of solutions of general system of regularized non-convex variational inequalities. The convergence analysis of the proposed iterative algorithm is also studied.

Basic Foundation

Let H be a real Hilbert space endowed with norm \( \| \cdot \| \) and an inner product \( \langle \cdot, \cdot \rangle \) respectively. Let \( \Omega \) be nonempty closed subsets of H. We represent \( d_{\Omega}(\cdot) \) or \( d_{\Omega}(\cdot;\Omega) \) the distance function from a point to a set \( \Omega \) that is

\[
d_{\Omega}(u) = \inf_{v \in \Omega} \| u - v \|
\]

**Definition 2.1**: Let \( u \in H \) be a point not lying in \( \Omega \). A point \( v \in \Omega \) is called a closed point or a projection of \( u \) onto \( \Omega \) if \( d_{\Omega}(u) = \| u - v \| \); The set of all such closed points is denoted by \( P_{\Omega}(u) \), that is

\[
P_{\Omega}(u) = \{ v \in \Omega : d_{\Omega}(u) = \| u - v \| \}
\]

**Definition 2.2**: The proximal normal cone of \( \Omega \) at a point \( u \in \Omega \) is given by

\[
N_{\Omega}(u) = \{ \zeta \in H : v \in N_{\Omega}(u+\alpha \zeta) \}
\]

where \( \alpha > 0 \) is a constant.

**Lemma 2.3** [26] Let \( \Omega \) be a nonempty closed subset of H. Then \( \zeta \in N_{\Omega}(u) \) if and only if there exists a constant \( \alpha = a(\zeta, u) > 0 \) such that

\[
\langle \zeta, v - u \rangle \leq \alpha \| v - u \| \quad \forall v \in \Omega
\]

**Lemma 2.4** [27] Let \( \Omega \) be a nonempty closed and convex subset of H. Then \( \zeta \in N_{\Omega}(u) \) if and only if

\[
\langle \zeta, v - u \rangle \leq 0 \quad \forall v \in \Omega
\]

**Definition 2.5** [4] Let \( f : H \rightarrow R \) be a locally Lipschitz near a point \( x \). The Clark’s directional derivative of \( f \) at \( x \) in the direction \( v \), denoted by \( f^\circ (x; v) \) is define by

\[
f^\circ (x; v) = \limsup_{y \rightarrow x; v} \frac{f(y + \tau v) - f(y)}{\tau}
\]

where \( y \) is a vector in H and \( \tau \) is a positive scalar.

The tangent cone to \( \Omega \) at a point \( x \in \Omega \), denoted by \( T_{\Omega}(x) \) is defined by

\[
T_{\Omega}(x) = \{ v \in H : d_{\Omega}^\circ(x; v) = 0 \}
\]

The normal cone to \( \Omega \) at a point \( x \in \Omega \), denoted by \( N_{\Omega}(x) \) is defined by

\[
N_{\Omega}(x) = \{ v \in H : v \in N_{\Omega}(x) \}
\]

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The Clarke normal cone denoted by \( Q^n_C(x) \) is defined by

\[
Q^n_C(x) = \overline{co}(Q^c_C(x))
\]

where \( co(S) \) denotes the closure of the convex hull of \( S \) and \( Q^c_C(x) \subset Q^c_C(x) \); where \( Q^c_C(x) \) is a closed convex cone and \( Q^n_C(x) \) may not be closed.

**Definition 2.6**: [4] For a given \( t \in (0, +\infty) \); a subset \( H \) of \( C \) is called the normalized uniformly prox-regular (or uniformly \( t \)-prox-regular) if every nonzero proximal normal to \( C \) can be realized by an \( t \)-ball.

That is for all \( x \in H \), and \( 0 \neq \zeta \in Q^c_C(x) \)

\[
\zeta, x - x, \leq \frac{1}{2} \|x - x\|, \quad x \in H,
\]

Therefore, for all \( x \in H \), and \( 0 \neq \zeta \in Q^c_C(x) \) with \( \zeta, x - x \leq \frac{1}{2} \|x - x\|, \quad x \in H \),

**Lemma 2.7**: [23] A closed set \( H \subset C \) is convex if and only if it is uniformly \( t \)-prox-regular for every \( t > 0 \). Proposition 2.8 [25] Let \( t > 0 \) and \( \Omega \) be a nonempty closed and uniformly \( t \)-prox-regular subset of \( H \). Let \( U(t) = \{u \in H : 0 \leq d_u (u) < t\} \) then the following statements are hold:

(a) for all \( x \in U(t) ; \) \( P_{in} (x) \neq \emptyset \)

(b) for all \( t \geq 0 \); \( Pt \) is Lipschitz continuous mapping with constant \( \frac{t}{t - t'} \) on \( U(t') = \{u \in H : 0 \leq d_u (u) < t'\} \).

If \( \Omega^o (x) \) is a closed set valued mapping, hence \( \Omega^o (x) = Q^o_C(x) \) and \( \Omega^o (x) = \overline{co}(Q^c_C(x)) \)

The union of two disjoint intervals \([a, b]\) and \([c, d]\) is uniformly \( t \)-prox-regular with \( t = \frac{d - b}{c - a} \). The infinite union of disjoint intervals is also uniformly \( t \)-prox-regular and \( t \) depends on the distance between the intervals.

**Basic Remarks and Formulations**

Let \( t \) be an uniformly \( t \) -prox-regular (nonconvex) set and \( g : \Omega \rightarrow \Omega \) be a given mapping for \( i = 1; 2; 3 \). For given mappings \( T_i, T_2, T_3 : \Omega \rightarrow \Omega \), we consider the following problems of finding \((x', y', z') \in \Omega \times \Omega \times \Omega \) such that

\[
< r_{T_i} (x', y', z') + g(x'), g(y') - g(y'), g(z') - g(z') > \geq 0, \quad g(x') \in \Omega, \quad r_i > 0
\]

\[
< r_{T_2}(y', y') + g(y'), g(y') - g(y'), g(z') - g(z') > \geq 0, \quad g(y') \in \Omega, \quad r_0 > 0
\]

\[
r_{T_3}(x', x') + g(x'), g(x') - g(x'), g(y') - g(y') > \geq 0, \quad g(x') \in \Omega, \quad r_3 > 0
\]

(3.1)

The problem (3.1) is called a general system of regularized non convex variational inequalities. We note that if \( T_1 = T_2 = T_3 = T : \Omega \rightarrow \Omega \) is an unvariant nonlinear operator, \( g = I_i \); \( i = 1; 2; 3 \) (the identity operator) and \( x = y = z = x_{in} \), then the system (3.1) reduces to the following classical variational inequalities defined on the nonconvex set \( \Omega \), find \( u \in \Omega \), such that

\[
\langle Tu, v - u \rangle \geq 0, \quad \forall v \in \Omega,
\]

and (3.2) is equivalent to find \( u \in \Omega \) such that

\[
0 \in Tu + Q^C_{\Omega}(u)
\]

Where \( Q^C_{\Omega}(u) \) denotes the normal cone of \( \Omega \) at \( u \) over the non convex set.

**Lemma 3.1**: \( (x', y', z') \in \Omega \times \Omega \times \Omega \) is a solution set of problem (3.1) if and only if

\[
g(x') = P_{\Omega}(g(x') - r_{T_1}(y', x'))
\]

\[
g(y') = P_{\Omega}(g(y') - r_{T_2}(y', y'))
\]

\[
g(z') = P_{\Omega}(g(z') - r_{T_3}(x', z'))
\]

(3.4)

where \( P_{\Omega} \) is the projection of \( H \) on to the uniformly \( t \)-prox-regular set \( \Omega \). In the proof of Lemma 3.1, there occur three fatal errors. First in view of Proposition 2.8, for any \( t \in (0, 1) \) the projection of points in the tube \( U(t) = \{u \in H : 0 < d_u (u) < t'\} \) onto the set \( \Omega \) exists and unique, that is for any \( x \in U(t) \), the set \( P_{in} (x) \) is nonempty and singleton. From the Lemma 3.1 and Proposition 2.6, the points \( g_1(y') - r_{T_1}(y', x') \) and \( g_2(x') - r_{T_2}(x', y') \) and \( g_3(z') - r_{T_3}(x', z') \) should be in \( U(t) \) for some \( t' \in (0, t) \) it is not necessary true, hence (3.4) are not necessarily well defined. If \( t' < \frac{1}{2} \|T(x', y')\| \) and \( r_i < \frac{1}{2} \|T(x', y')\| \) for \( t' \in (0, t) \), then we have

\[
d_{in}(g_1(y') - r_{T_1}(y', x')) \leq d_{in}(g_1(y')) + r_1 \|T_1(y', x')\|
\]

Therefore, \( (g_1(y') - r_{T_1}(y', x')) \in U(t') \)

Hence, \( (g_1(y') - r_{T_1}(y', x')) \in \Omega \) Similarly we have \( (g_2(z') - r_{T_2}(z', y')) \in U(t') \) and \( (g_3(z') - r_{T_3}(x', z')) \in U(t') \)

If \( t' < \frac{1}{2} \|T_1(y', x')\| \) and \( r_i < \frac{1}{2} \|T_1(y', x')\| \) for \( t' \in (0, t) \), then the equation (3.4) are well defined.

Secondly the following general system of regularized non convex variational inclussions is equivalence to the system (3.1):

\[
0 \in r_{T_1}(y', x') + g(x') - g(y') + r_{Q^C_{\Omega}}(g(x'))
\]

\[
0 \in r_{T_2}(y', y') + g(y') - g(y') + r_{Q^C_{\Omega}}(g(y'))
\]

\[
0 \in r_{T_3}(x', z') + g(z') - g(z') + r_{Q^C_{\Omega}}(g(z'))
\]

(3.5)

Since \( Q^C_{\Omega}(g(x')) \) and \( Q^C_{\Omega}(g(y')) \) and \( Q^C_{\Omega}(g(z')) \) are cone, the system (3.1) is equivalent the following system:

\[
0 \in r_{T_1}(y', x') + g(x') - g(y') + Q^C_{\Omega}(g(x'))
\]

\[
0 \in r_{T_2}(y', y') + g(y') - g(y') + Q^C_{\Omega}(g(y'))
\]

\[
0 \in r_{T_3}(x', z') + g(z') - g(z') + Q^C_{\Omega}(g(z'))
\]

(3.6)

The system (3.1) is equivalent to the system (3.5) which is not true in general.

**Example 3.2** Let \( H = \mathbb{R} \) and \( t \in [0, b] \cap [c, d] \) be the union of two disjoint intervals \([0, b] \cap [c, d] \) where \( 0 < c < d \). Then \( t \) is an uniformly \( t \)-prox-regular set with \( \Omega \), and the system (3.6) is not true.
where for $i=1, 2, 3$, $s, i, m \in \mathbb{R}, \theta_i < 0$ and $b^{-}\infty \leq k < \frac{c}{b^\infty}$ are arbitrary but fixed.

Assume $x' = y' = z = b$ and $r > 0$, $i = 1, 2, 3$

$\alpha > \max \left\{ -r_i \theta_i e^{\beta_i} - r_i \theta_i e^{\beta_i}; i = 1, 2, 3 \right\}$

be the fixed arbitrary. Hence for all $w \in \Omega_i$

$\left\{ r_i T_i(y, x') + g_i(x') \geq g_i(y') \right\} + \alpha \| w - g_i(x') \|^2 = r_i \theta_i e^{\beta_i} (w - k_b^n) + \alpha (w - k_b^n)^2$

$=(w - k_b^n) + \alpha (w - k_b^n) + r_i \theta_i e^{\beta_i}$

If $w \in [0, b]$, then

$k_b^n \leq w - k_b^n \leq b - k_b^n = b(1 - k_b^{-1})$

and

$-k_b^n \alpha \| w - k_b^n \|^2 + r_i \theta_i e^{\beta_i} \leq \alpha (w - k_b^n) + \alpha (w - k_b^n)^2$

For $w \in (c, d)$, we have

$c - k_b^n \leq w - k_b^n \leq d - k_b^n$

and

$\alpha (c - k_b^n) + r_i \theta_i e^{\beta_i} \leq \alpha (w - k_b^n) + r_i \theta_i e^{\beta_i} \leq (d - k_b^n) + r_i \theta_i e^{\beta_i}$

$(w - k_b^n) + \alpha (w - k_b^n)^2 \geq 0 \; \forall \; w \in \Omega_i, \; (\beta_i)$

From (3.7)-(3.8), we have

$r_i T_i(y, x') + g_i(x') - g_i(y') + \alpha \| w - g_i(x') \|^2 \geq 0 \; \forall \; y \in \Omega_i$

Since $r_i \theta_i e^{\beta_i} (w - k_b^n) < 0$ for all $w \in [c, d]$, i.e.,

$r_i T_i(y, x') + g_i(x') - g_i(y') - \alpha \| w - g_i(x') \|^2 \leq 0 \; \forall \; y \in \Omega_i$

Hence

$r_i T_i(y, x') + g_i(x') - g_i(y') \geq 0, \; \forall \; w \in \Omega_i$

cannot hold. Similarly we have

$r_i T_i z, x') + g_i(y') - g_i(z') \geq 0, \; \forall \; y \in \Omega_i$

while the inequality

$r_i T_i z, x') + g_i(z') - g_i(y') \geq 0, \; \forall \; z \in \Omega_i$

cannot hold. Again in similar way we have

$r_i T_i z, x') + g_i(z') - g_i(y') \geq 0, \; \forall \; y \in \Omega_i$

the inequality

$r_i T_i z, x') + g_i(y') - g_i(z') \geq 0, \; \forall \; z \in \Omega_i$

cannot hold. Therefore we can see that every solution of (3.3) is a solution of (3.3) but converse need not be true in general. On the basis of example we define as the general system of regularized non-convex variational inequality. For given nonlinear mappings $T_i : H \times H \rightarrow H$ and $g_i : H \times H \rightarrow H$ $i = 1, 2, 3$ we consider the general system of regularized non-convex variational inequality for finding $(x', y', z') \in H \times H \times H$ such that $(g_i(x'), g_i(y'), g_i(z')) \in \Omega_i \times \Omega_i \times \Omega_i$, and

$\left\{ r_i T_i(y, x') + g_i(x') - g_i(y') \right\} + \alpha \| w - g_i(x') \|^2 = 2 \times \frac{r_i}{r_i T_i(y, x') + g_i(x') - g_i(y') \| g_i(x') - g_i(y') \|}^2 \geq 0, \; \forall \; x \in \Omega_i$

Proposition 3.3: Let $\Omega_i$ be an uniformly $t$-prox regular set. The system (3.9) is equivalent to the system (3.6).

Proof: Let $(x', y', z') \in H \times H \times H$ with $(g_i(x'), g_i(y'), g_i(z')) \in \Omega_i \times \Omega_i \times \Omega_i$, i = 1; 2; 3 be a solution set of the system (3.9). If $r_i T_i(y, x') + g_i(x') - g_i(y') = 0$ then

$(x', y', z') \in Q_{\alpha_i}^{t_0}(g_i(x'))$

From Lemma 2.3, we have

$0 \in r_i T_i(y, x') + g_i(x') - g_i(y') + Q_{\alpha_i}^{t_0}(g_i(x'))$

Hence

$0 \in r_i T_i(y, x') + g_i(x') - g_i(y') + Q_{\alpha_i}^{t_0}(g_i(x'))$

Similarly we have

$0 \in r_i T_i(y, x') + g_i(y') - g_i(z') + Q_{\alpha_i}^{t_0}(g_i(y'))$

Conversely, $(x', y', z') \in H \times H \times H$ with $(g_i(x'), g_i(y'), g_i(z')) \in \Omega_i \times \Omega_i \times \Omega_i$ and $r_i T_i(y, x') + g_i(x') - g_i(y') \neq 0$ then

$(x', y', z') \in Q_{\alpha_i}^{t_0}(g_i(x'))$

Lemma 3.4: For $i = 1, 2, 3$, let $r_i, t_i, t_i$ be the same as in the system (3.9), then $(x', y', z') \in H \times H \times H$ with $(g_i(x'), g_i(y'), g_i(z')) \in \Omega_i \times \Omega_i \times \Omega_i$ is a solution of the system (3.9) if and only if $(x', y', z')$ satisfies the system (3.4) with $t_i = 1 + \frac{1}{1 + ||T_i z, y' ||}, \; r_i = 1 + \frac{1}{1 + ||T_i z, y' ||}$ and $t_i = 1 + \frac{1}{1 + ||T_i z, y' ||}$ for $t_i \neq 0$.

Proof: Let $(x', y', z') \in H \times H \times H$ with $(g_i(y')) \in \Omega_i \times \Omega_i \times \Omega_i$ and $r_i T_i(y, x') + g_i(x') - g_i(y') + Q_{\alpha_i}^{t_0}(g_i(x'))$

$\Rightarrow g_i(x') - r_i T_i(y, x') \in g_i(x') + Q_{\alpha_i}^{t_0}(g_i(x'))$
\( g_1(y') - r T_1(y', x') \in (I + Q^{\alpha}_{\Omega}) (g_1(x')) \)
\( g_1(x') = P_{\Omega} \left[ g_1(y') - r T_1(y', x') \right], \)
where I is an identity mapping. Similarly we have
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Similarly we obtain

\[ x = 0, \quad y = 0, \quad z = 0. \]

Then the iterative sequences \( \{x_n, y_n, z_n\} \) of (4.14) such that

\[ y_n - g_i(y_n) \leq p_i y_n - y_n \]

Since \( T_i \) is relaxed \((\eta_i, v_i)\)-cocoercive mapping and \( \lambda_i \)-Lipschitz continuous with first variable, we obtain

\[ \|y_n - g_i(y_n)\| \leq p_i \|y_n - y_n\| \]

Lemma 3.4 guarantees that \((\hat{x}, \hat{y}, \hat{z}) \in H \times H \times H \) with \((g_i(\hat{x}), g_i(\hat{y}), g_i(\hat{z})) \in \Omega_1 \times \Omega_2 \times \Omega_3 \) for \( i = 1, 2, 3 \) is a solution set of the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in \Omega \), compute the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( \Omega \) in the following way:

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [x_n - g_i(x_n) + P_{\Omega_i} (P_{\Omega_i} (x_n) - r_i T_i (x_n, y_n))] \]

\[ y_{n+1} = (1 - \alpha_n) y_n + \alpha_n [y_n - g_i(y_n) + P_{\Omega_i} (P_{\Omega_i} (y_n) - r_i T_i (x_n, y_n))] \]

\[ z_{n+1} = (1 - \alpha_n) z_n + \alpha_n [z_n - g_i(z_n) + P_{\Omega_i} (P_{\Omega_i} (z_n) - r_i T_i (x_n, z_n))] \]

where \( \alpha_n \) is a sequence in \([0, 1]\).

Again assume that \( g = I \), \( i = 1, 2, 3 \) is an identity mappings, then we have the following algorithm.

Algorithm 4.5: Let mapping \( T_i \) and constant \( r_i > 0 \) for \( i = 1, 2, 3 \) be the same as the in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in H \), compute the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( \Omega \) in the following way:

\[ x_{n+1} = P_{\Omega_i} (y_{n+1} - r_i T_i (x_{n+1}, y_{n+1})) \]

\[ y_{n+1} = P_{\Omega_i} (x_{n+1} - r_i T_i (x_{n+1}, y_{n+1})) \]

\[ z_{n+1} = P_{\Omega_i} (x_{n+1} - r_i T_i (x_{n+1}, z_{n+1})) \]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\).

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r_i > 0 \) for \( i = 1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in H \), compute the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( \Omega \) in the following way:

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [x_n - g_i(x_n) + P_{\Omega_i} (P_{\Omega_i} (x_n) - r_i T_i (x_n, y_n))] \]

\[ y_{n+1} = (1 - \alpha_n) y_n + \alpha_n [y_n - g_i(y_n) + P_{\Omega_i} (P_{\Omega_i} (y_n) - r_i T_i (x_n, y_n))] \]

\[ z_{n+1} = (1 - \alpha_n) z_n + \alpha_n [z_n - g_i(z_n) + P_{\Omega_i} (P_{\Omega_i} (z_n) - r_i T_i (x_n, z_n))] \]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r_i > 0 \) for \( i = 1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in H \), compute the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( \Omega \) in the following way:

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [x_n - g_i(x_n) + P_{\Omega_i} (P_{\Omega_i} (x_n) - r_i T_i (x_n, y_n))] \]

\[ y_{n+1} = (1 - \alpha_n) y_n + \alpha_n [y_n - g_i(y_n) + P_{\Omega_i} (P_{\Omega_i} (y_n) - r_i T_i (x_n, y_n))] \]

\[ z_{n+1} = (1 - \alpha_n) z_n + \alpha_n [z_n - g_i(z_n) + P_{\Omega_i} (P_{\Omega_i} (z_n) - r_i T_i (x_n, z_n))] \]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r_i > 0 \) for \( i = 1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in H \), compute the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( \Omega \) in the following way:

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [x_n - g_i(x_n) + P_{\Omega_i} (P_{\Omega_i} (x_n) - r_i T_i (x_n, y_n))] \]

\[ y_{n+1} = (1 - \alpha_n) y_n + \alpha_n [y_n - g_i(y_n) + P_{\Omega_i} (P_{\Omega_i} (y_n) - r_i T_i (x_n, y_n))] \]

\[ z_{n+1} = (1 - \alpha_n) z_n + \alpha_n [z_n - g_i(z_n) + P_{\Omega_i} (P_{\Omega_i} (z_n) - r_i T_i (x_n, z_n))] \]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.

Theorem 4.7 Let the mappings \( T_i \), \( g \) and \( r_i > 0 \) for \( i = 1, 2, 3 \) be the same as in the system (3.9). For arbitrary initial points \( x_0, y_0, z_0 \in H \), compute the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( \Omega \) in the following way:

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [x_n - g_i(x_n) + P_{\Omega_i} (P_{\Omega_i} (x_n) - r_i T_i (x_n, y_n))] \]

\[ y_{n+1} = (1 - \alpha_n) y_n + \alpha_n [y_n - g_i(y_n) + P_{\Omega_i} (P_{\Omega_i} (y_n) - r_i T_i (x_n, y_n))] \]

\[ z_{n+1} = (1 - \alpha_n) z_n + \alpha_n [z_n - g_i(z_n) + P_{\Omega_i} (P_{\Omega_i} (z_n) - r_i T_i (x_n, z_n))] \]

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\):

Now we prove the strong convergence of the sequences generated by Algorithm 4.4 to a unique solution set of the system.
\[
x' = (1 - \alpha_n) x + \alpha_n \left[ x' - g_1 (y') + P_{x_1} (g_1 (y') - r_1 T_1 (y', x')) \right]
\]
\[
y' = (1 - \alpha_n) y' + \alpha_n \left[ y' - g_2 (z') + P_{x_1} (g_2 (z') - r_1 T_1 (y', x')) \right]
\]
\[
z' = (1 - \alpha_n) z' + \alpha_n \left[ z' - g_3 (z') + P_{x_1} (g_3 (z') - r_1 T_1 (y', x')) \right]
\]
\[
\text{where} \{\alpha_n\} \text{ is a sequence in } [0, 1]: \text{Since } n \in \mathbb{N}, (g(y^*), g(y_n)) \in \Omega_t; \\
\text{continuous mapping in the first variable and } g_2 \text{ is } \mu_2\text{-strong cocoercive mapping and } \lambda_1\text{-Lipschitz continuous mapping, we have}
\]
\[
\text{Combining (4.16)-(4.19) we get}
\]
\[
\left\| z_{n+1} - z' \right\| \leq (1 - \alpha_n) \left\| z_n - z' \right\| + \alpha_n (\left\| z_n - z' \right\| + \theta_n \left\| x_n - x' \right\|) 
\]
\[
\left\| y_{n+1} - y' \right\| \leq (1 - \alpha_n) \left\| y_n - y' \right\| + \alpha_n (\left\| y_n - y' \right\| + \theta_n \left\| y_n - y' \right\|) 
\]
\[
\left\| x_{n+1} - x' \right\| \leq (1 - \alpha_n) \left\| x_n - x' \right\| + \alpha_n (\left\| x_n - x' \right\| + \theta_n \left\| y_n - y' \right\|) 
\]
\[
\text{Therefore, for each } n \in \mathbb{N}, g_3 (z') - r_1 T_1 (z', y') + g_2 (z_n) - r_1 T_1 (z_n, y_n) = \mu (t').
\]
\[
\text{Since } T_1 \text{ is relaxed } (\eta_1, \nu_1)\text{-cocoercive mapping and } \lambda_1\text{-Lipschitz continuous mapping in the first variable and } g_3 \text{ is } \mu_3\text{-strong cocoercive mapping and } \xi_1\text{-Lipschitz continuous mapping, as same way of (4.16) - (4.20), we get}
\]
\[
\left\| z_{n+1} - z' \right\| \leq (1 - \alpha_n) \left\| z_n - z' \right\| + \alpha_n (p_3 \left\| z_n - z' \right\| + \theta_n \left\| x_n - x' \right\|) 
\]
\[
\left\| y_{n+1} - y' \right\| \leq (1 - \alpha_n) \left\| y_n - y' \right\| + \alpha_n (p_3 \left\| y_n - y' \right\| + \theta_n \left\| y_n - y' \right\|) 
\]
\[
\left\| x_{n+1} - x' \right\| \leq (1 - \alpha_n) \left\| x_n - x' \right\| + \alpha_n (p_3 \left\| x_n - x' \right\| + \theta_n \left\| y_n - y' \right\|) 
\]
\[
\text{where } p_3 \text{ and } \theta_n \text{ are same as in (4.12). Now}
\]
\[
\left\| (x_{n+1}, y_{n+1}, z_{n+1}) - (x', y', z') \right\| \leq \left\| (1 - \alpha_n) (x_n, y_n, z_n) - (x', y', z') \right\| + \alpha_n \left\| (p_3 + \theta_n) (x_n, y_n, z_n) - (x', y', z') \right\|
\]
\[
\text{and}
\]
\[
\left\| (x_{n+1}, y_{n+1}, z_{n+1}) - (x', y', z') \right\| \leq \left\| (1 - \alpha_n) (x_n, y_n, z_n) - (x', y', z') \right\| + \alpha_n \left\| (p_3 + \theta_n) (x_n, y_n, z_n) - (x', y', z') \right\|
\]
\[
\text{where } \lambda' \text{ is same as in (4.13). From condition (4.2) we get } \lambda' < 2 (0, 1). \text{ Since } \sum_{n=0}^{\infty} \alpha_n = \infty, \text{ we get}
\]
\[
\text{Therefore from (4.23)-(4.24)}
\]
\[
\left\| (x', y', z') - (x^*, y^*, z^*) \right\| \rightarrow 0 \text{ as } n \rightarrow \infty
\]
\[
\text{and the sequences } \{(x_n, y_n, z_n)\} \text{ suggested by algorithm 4.4, converges strongly to a unique solution set } (x^*, y^*, z^*) \text{ of the general system of regularized nonconvex variational inequalities. In a similar way, we can prove the convergence of iterative sequences generated by Algorithm 4.5 and Algorithm 4.6.}
\]

\[\text{References}\]


15. Inchan I, Petrot N (2011) System of general variational inequalities involving different nonlinear operators related to fixed point problems and its applications. Fixed point Theory.


