A Model for Hepatitis B with Chronological and Infection Ages

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Abstract

We construct and study a differential infectivity model with chronological and infection age. The application is done on hepatitis B in Cameroon. We prove the global stability of the disease free equilibrium when the basic reproduction ratio \( R_0 \) is less than one and the existence and uniqueness of an endemic equilibrium when \( R_0 > 1 \).

Keywords: Nonlinear dynamical systems; Global stability; Lyapunov methods; Differential infectivity models; HBV; Cameroon

A Simplified Model without Vertical Transmission and With Control Measure

Hepatitis B is endemic in Africa [1-3] (see also the references cited therein) and some models have been constructed in order to understand its dynamic with ODE deterministic (with delay or not) or stochastic processes [2,4-8] or partial differential equations [9,10]. Studies like [9-13] recognized the importance of the age factor in the dynamics of infectious diseases like hepatitis B [10]. Moreover some studies like [4,14] obtained results with ODEs with discrete age(s) that could be generalized with continuous age assumption more realistic and relevant.

We introduce then a model with differential infectivity and chronological or infection ages. We denote by \( s(t,a) \) the density of susceptible at time \( t \) with chronological aged \( a \). We denote by \( i(t,a) \) the density of infective that will develop acute disease at time \( t \) contaminated since a time and \( e(t,a) \) the density of infective that will develop acute disease at time \( t \) contaminated since a time \( t \). The model we shall consider reads as follows:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) s(t,a) &= -\mu s(t,a) - \lambda(t) s(t,a), \quad t > 0, a > 0, \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i(t,a) &= -\gamma i(t,a) - \lambda(t) s(t,a) + \beta(t) e(t,a), \quad t > 0, a > 0, \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) e(t,a) &= -\gamma e(t,a), \quad t > 0, a > 0,
\end{align*}
\]

Here \( \lambda(t) = \int_0^1 \gamma(t,a) p(a) s(t,a) da \), \( \beta(t) = \int_0^1 \beta(t,a) e(t,a) da \).

The above model takes into account the chronological age of susceptible. This parameter has strong implication in the dynamics of infection. Indeed, depending on the age at which susceptible enters the infective’s classes, the disease will develop indifferent way. For hepatitis B virus (HBV), young infections lead to chronic infection while older infection leads to acute disease.

In the above model, we do not take into account possible vertical transmission and we do not consider any control strategy such vaccination campaign. It seems to be relevant together the assumption of WHO [3] that consider that vertical transmission of the disease does occur in sub-saharian Africa, but its influence of the dynamics of the disease is rather small because the proportion of chronic infections acquired prenatally is low. Under the above assumption, we assume that the chronological age for the infective classes do not play an important role. But the time since infection is a relevant biological variable because of the possibility to have a long latent period (especially for the asymptomatic carrier class, until several years).

The work is organized as follows. In Section 2, we prove the wellposedness of the PDE (1.1-1.2), derive preliminary results useful to study the long term behavior of the model. Sections 3, 4, and 5 is devoted to the uniqueness of endemic equilibrium when the biological basic reproduction rate \( R_0 \) is greater than 1 and study the global asymptotically stability of the disease free equilibrium if \( 1 > R_0 \). Finally Section 6 presents discussion.

Abstract Cauchy Problem Reformulation

Mathematical assumptions

We assume that:

\begin{enumerate}
\item \( \gamma(t,a) \geq 0, \gamma(t,a) = 0 \)
\item The function \( p \in L^1([0,\infty)) \) with \( p(a) \in [0,1] \) a.e. and not identically 0 and 1.
\item Function \( \beta(t,a) \in L^1([0,\infty)) \).
\end{enumerate}

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Abstract Cauchy problem

The aim of this section is to deal with (1.1). To do so, we consider the Banach spaces

\[ X = \mathbb{R}^3 \times L^1(0,\infty)^3, \quad X_0 = [0,\infty) \times L^1(0,\infty)^3, \]

and endowed the norm

\[ \left\| \begin{bmatrix} \alpha_i \\ \alpha_2 \\ \alpha_3 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \right\|_{L^1(0,\infty)^3} = \sum_{i=1}^3 (|\alpha_i| + |\psi_i|_{L^1(0,\infty)}), \]

as well as the non-densely defined linear operator \( A : D(A) \subset X \rightarrow X \)

defined by

\[ D(A) = \{ u \in W^{1,1}(0,\infty)^3 \} \]

and the map \( \phi \) defined by

\[ \phi(x) = \begin{cases} -\phi_1(0) \\ -\phi_2(0) \\ -\phi_3(0) \\ -\phi_1(\tau) - \mu \phi_3(\tau) \\ -\phi_2(\tau) - (\mu + \gamma) \phi_3(\tau) \\ -\phi_3(\tau) - (\mu + \gamma) \phi_3(\tau) \end{cases} \]

as well as the nonlinear map \( F : X_0 \rightarrow X \)

defined by

\[ F = \begin{cases} 0 \\ 0 \\ -\phi_1(t) \\ -\phi_2(t) \\ -\phi_3(t) \end{cases} \]

rewrites as the following system

\[ \begin{align*}
\phi_1(a) &= -\phi_1(0) + \int_0^a (\beta_1(\tau) \phi_1(\tau) + \beta_2(\tau) \phi_2(\tau) + \beta_3(\tau) \phi_3(\tau)) d\tau \\
\phi_2(a) &= -\phi_2(0) + \int_0^a (\beta_1(\tau) \phi_1(\tau) + \beta_2(\tau) \phi_2(\tau) + \beta_3(\tau) \phi_3(\tau)) d\tau \\
\phi_3(a) &= -\phi_3(0) + \int_0^a (\beta_1(\tau) \phi_1(\tau) + \beta_2(\tau) \phi_2(\tau) + \beta_3(\tau) \phi_3(\tau)) d\tau.
\end{align*} \]

Let us notice that \( D(A) = X_0 \).

Now by identifying \((\alpha_1(0), \alpha_2(0), \alpha_3(0), \psi_1(0), \psi_2(0), \psi_3(0))\) in (1.1) together with \( u(t) = (0,0,0,\alpha(t),\psi(t),\psi(t)) \) one obtains that \( u(t) \) satisfies the following abstract Cauchy problem

\[ \frac{du(t)}{dt} = Au(t) + F(u(t)), \quad t > 0, \]

with initial data \( u(0) = x \).

We also consider the positive cones \( X = \mathbb{R}^3 \times L^1(0,\infty)^3, \quad X_0 = X \cap X \).

**Lemma 2.1:** (Hille-Yosida property) Operator \( A : D(A) \subset X \rightarrow X \)
is a Hille-Yosida operator. More precisely we have \((\mu, \infty) \subset p(A)\) the resolvent set of \( A \) and for each \( \lambda > \mu \) each \((\alpha_1, \alpha_2, \alpha_3, \psi_1, \psi_2, \psi_3)\) \( \in X \) we have

\[ (\lambda - A)^{-1} = \begin{bmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \]

Moreover we have for each \( \lambda > -\mu \)

\[ (\lambda - A)^{-1} X_0 \subset X_0 \]

**Proof:** Let \( x = (\alpha_1, \alpha_2, \alpha_3, \psi_1, \psi_2, \psi_3)^T \in X \) and \( \lambda > -\mu \) be given. Then the equation

\[ \begin{cases} \phi_1'(a) &= -\phi_1(0) + \int_0^a (\beta_1(\tau) \phi_1(\tau) + \beta_2(\tau) \phi_2(\tau) + \beta_3(\tau) \phi_3(\tau)) d\tau \\
\phi_2'(a) &= -\phi_2(0) + \int_0^a (\beta_1(\tau) \phi_1(\tau) + \beta_2(\tau) \phi_2(\tau) + \beta_3(\tau) \phi_3(\tau)) d\tau \\
\phi_3'(a) &= -\phi_3(0) + \int_0^a (\beta_1(\tau) \phi_1(\tau) + \beta_2(\tau) \phi_2(\tau) + \beta_3(\tau) \phi_3(\tau)) d\tau.
\end{cases} \]

that is

\[ \begin{align*}
\phi_1(a) &= e^{-(\lambda + \mu) a} \phi_1(0) + \int_0^a e^{-(\lambda + \mu)(t-a)} \psi_1(t) ds \\
\phi_2(a) &= e^{-(\lambda + \mu) a} \phi_2(0) + \int_0^a e^{-(\lambda + \mu)(t-a)} \psi_2(t) ds \\
\phi_3(a) &= e^{-(\lambda + \mu) a} \phi_3(0) + \int_0^a e^{-(\lambda + \mu)(t-a)} \psi_3(t) ds.
\end{align*} \]

On the other hand one has

\[ \begin{align*}
\|\phi_1\|_{L^1(0,\infty)} &\leq \frac{1}{\lambda + \mu} \|\phi_1\|_{L^1(0,\infty)} \\
\|\phi_2\|_{L^1(0,\infty)} &\leq \frac{1}{\lambda + \mu} \|\phi_2\|_{L^1(0,\infty)} \\
\|\phi_3\|_{L^1(0,\infty)} &\leq \frac{1}{\lambda + \mu} \|\phi_3\|_{L^1(0,\infty)}.
\end{align*} \]

As a consequence, \((\mu, \infty) \subset p(A)\) and \( \|\lambda - A\| L(X) \leq \frac{1}{\lambda + \mu}, \forall \lambda > -\mu \)

This completes the proof of the Hille-Yosida property. Finally the explicit formula of the resolvent operator implies that (2.4) holds true.

**Theorem 2.2:** There exists a continuous semi flow \( \{U(t)\}_{t \geq 0} \) on \( X \) into itself such that for each \( x \in X_0 \), the map \( t \mapsto U(t)x \) is the unique integrated solution of (2.3) with initial data \( x \), namely \( t \mapsto U(t)x \) satisfies

\[ \begin{align*}
(i) \quad & U(t)x = x + \int_0^t U(s)x ds + \int_0^t F(U(s)x) ds \\
(ii) \quad & U(t)x = x + \int_0^t U(s)x ds + \int_0^t F(U(s)x) ds \quad \forall t \geq 0
\end{align*} \]
Moreover we have for each \( x \in X_0 \):
\[
\frac{\Lambda}{\mu + \gamma_x + \gamma_x} \leq \lim \inf_{t \to \infty} \| U(t) x \| \leq \lim \sup_{t \to \infty} \| U(t) x \| \leq \frac{\Lambda}{\mu}
\]

Proof: Let us first notice that for each \( M > 0 \) there exists \( \lambda > 0 \) such that
\[
F(x) + \lambda x \in X_+ \quad \forall x \in B_0(M) \cap X_0.
\]
With \( B_0(M) \) denote the ball of radius \( M \) centered at 0. One obtains the existence of a maximal positive semi flow for (2.3) on \( X_0 \) itself. It remains to prove that this semi flow is globally defined. To do so, let \( x \in X_0 \), be given and recall that
\[
U(t)x = (0, 0, 0, s(t), i(t), e(t))^{T}
\]
Consider the quantity
\[
P(t) = \|U(t)x\| = \int_0^\infty s(t, a) da + \int_0^\infty i(t, \tau) d\tau + \int_0^\infty e(t, \tau) d\tau,
\]
the total population at time \( t \). Then it satisfies the differential inequality
\[
\frac{dP(t)}{dt} \leq \Lambda - \mu P(t)
\]
Thus the map \( t \to P(t) \) cannot blow up in finite and the global existence result follows.

Let us, in addition, notice that, from this inequality one gets (by density)
\[
\lim \sup_{t \to \infty} \|U(t)x\| \leq \frac{\Lambda}{\mu} \quad \forall x \in X_0,
\]
One the other hand one has
\[
\frac{dP(t)}{dt} = \Lambda - \mu P(t) - (\gamma_x + \gamma_x) \int_0^\infty s(t, a) d\tau \geq \Lambda - (\mu + \gamma_x + \gamma_x) P(t),
\]
so that
\[
\lim \sup_{t \to \infty} \|U(t)x\| \geq \frac{\Lambda}{\mu + \gamma_x + \gamma_x} \quad \forall x \in X_0,
\]
This completes the proof of the result.

Stationary States

The disease free equilibrium

The disease free equilibrium corresponds to a stationary (that is time independent solution)
\[
(s_0, i_0, e_0) = (0, 0, 0),
\]
of (1.1)-(1.2). As a consequence we have the following lemma

Lemma 3.1: The dynamical system provided by (1.1)-(1.2) has a unique disease free equilibrium where \( s \) is given by \( s(\cdot) = e^{\mu t} \)

From now one we set
\[
S_0 = \int_0^\infty p(a) s(a) da,
\]
and we consider the biological basic reproduction rate
\[
R_0 = \Lambda \int_0^\infty (\beta_x(\tau) e^{-\gamma_x \tau} + \beta_{e}(\tau) e^{-\gamma_e \tau}) d\tau,
\]
where we have set

Endemic equilibrium

We look for stationary solutions \( s; i; e \) such that \( i; e \) not identical zero satisfying
\[
s'(a) = -\mu s(t, a) - \lambda s(a), a > 0,
\]
\[
s(0) = \Lambda,
\]
\[
i'(\tau) = (\mu + \gamma_x) i(\tau), \tau > 0,
\]
\[
e'(\tau) = (\mu + \gamma_e) e(\tau), \tau > 0,
\]
\[
i(0) = \lambda \int_0^\infty p(a) s(a) da, e(0) = \lambda \int_0^\infty (1 - p(a)) s(a) da
\]

Theorem 3.2: Recalling definition (3.5), if \( R_0 > 1 \), system (1.1)-(1.2) has a unique endemic equilibrium point denoted by \( (s^*, i^*, e^*) \).

Proof: we have
\[
s(a) = \Lambda e^{-\mu t},
\]
\[
i(\tau) = e^{-\mu \tau} \Lambda \int_0^\infty p(a) e^{-\mu t} a da
\]
\[
e(\tau) = e^{-\mu \tau} \Lambda \int_0^\infty (1 - p(a)) e^{-\mu t} a da
\]
Thus
\[
\lambda = \int_0^\infty \beta_x(\tau) e^{-\mu \tau} \tau d\tau \Lambda \int_0^\infty p(a) e^{-\mu t} a da + \int_0^\infty \beta_e(\tau) e^{-\mu \tau} \tau d\tau \Lambda \int_0^\infty (1 - p(a)) e^{-\mu t} a da
\]
We are looking for endemic stationary state, that is \( \lambda > 0 \), so that
\[
1 = \int_0^\infty \beta_x(\tau) e^{-\mu \tau} \tau d\tau \Lambda \int_0^\infty p(a) e^{-\mu t} a da + \int_0^\infty \beta_e(\tau) e^{-\mu \tau} \tau d\tau \Lambda \int_0^\infty (1 - p(a)) e^{-\mu t} a da
\]
Now the map \( \lambda = f(\lambda) \) is non increasing with \( f(\lambda) = \int_0^\infty \beta_x(\tau) e^{-\mu \tau} \tau d\tau \Lambda \int_0^\infty p(a) e^{-\mu t} a da + \int_0^\infty \beta_e(\tau) e^{-\mu \tau} \tau d\tau \Lambda \int_0^\infty (1 - p(a)) e^{-\mu t} a da
\]
and \( f(\lambda) = 0 \) when \( \lambda = \infty \). As a consequence, since \( R_0 = f(0) > 1 \), there exists a unique \( \lambda > 0 \) such that
\[
f(\lambda^*) = 0
\]
Finally, the functions
\[
s_x(\tau) = \Lambda e^{-\mu t} \beta_x(\tau),
\]
\[
i_x(\tau) = e^{-\mu \tau} \Lambda \int_0^\infty p(a) e^{-\mu t} a da
\]
\[
e_x(\tau) = e^{-\mu \tau} \Lambda \int_0^\infty (1 - p(a)) e^{-\mu t} a da
\]
provides the unique endemic stationary state of system (1.1)-(1.2).

Dynamical Properties

Assumption 4.1

Assume that the maps \( a \to \beta_x(\tau) \) and \( a \to \beta_e(\tau) \) are bounded and uniformly continuous from \([0, \infty)\) into itself.

Volterra Integral Formulation:

The solutions of (1.1)-(1.2) can be reformulated as follows
\[
\frac{d}{dt} s(t) + \frac{d}{dt} i(t) = -\mu s(t) - \lambda i(t) s(t)
\]
\[
s(t, 0) = \Lambda, \quad i(0, 0, 0) = 0
\]
With
\[ \lambda(t) = \int_{0}^{\infty} \beta(t) \alpha(t, \tau) \phi(\tau) d\tau, \]
and
\[ i(t, \tau) = \begin{cases} e^{\mu \tau} b_{0}(a - \tau) & \text{if } a - \tau \geq 0, \\ e^{\mu \tau} b_{a}(a - \tau) & \text{if } a - \tau < 0, \end{cases} \]
where \( t \to b(t) = i(t, 0) \) and \( t \to b(t) = e(t, 0) \) are the unique continuous functions satisfying for \( k = 1, \ldots, \)

\[ b_{k}(t) = p_{k}[s(t, \tau)] \int_{0}^{\infty} \beta(t) e^{\lambda t} b_{k}(t, \tau) + \beta(t) e^{\lambda t} b_{k}(t, \tau) d\tau \]

where we have set
\[ p_{k}[\phi] = \int_{0}^{\infty} p(a) \phi(a) da, \quad p_{k}[\phi] = \int_{0}^{\infty} (1 - p(a)) \phi(a) da. \]

By using results in Sell and You [15], one can find suitable conditions to prove that \( \{U(t)_{\mid_{t=0}} \) is asymptotically smooth and derived the following proposition.

**Proposition 4.2** Let Assumption 4.1 be satisfied. Then there exists a compact set \( A \subset X_{0} \) such that

(i) \( A \) is invariant under the semi flow \( \{U(t)_{\mid_{t=0}} \)

(ii) \( A \) attracts the bounded sets of \( X_{0} \) under \( \{U(t)_{\mid_{t=0}} \). This means that for each bounded set \( B \subset X_{0} \), we have

\[ \lim_{t \to \infty} \delta(U(t), B, A) = 0, \]

where \( \delta \) is defined as
\[ \delta(A, B) = \sup_{t \to \infty} \inf_{y \in B} \| x - y \| \]
Moreover \( A \) is locally asymptotically stable. Next one considers the following quantities \( \tau_{k} = \sup \{ t \geq 0; \beta(t) > 0, \} \) \( k \in [1, e] \) and the following set:
\[ M = \{ \phi, \psi \in L^{1}_{(0, \infty)} : \int_{0}^{\infty} \phi(t) d\tau + \int_{0}^{\infty} \psi(t) d\tau > 0 \} \]
We set also
\[ M_{a} = \{ [0, \infty) \times L^{1}_{(0, \infty)} \times M \subset X_{0}, \}
\[ \partial M_{a} = M_{a} \setminus M \]
From this Volterra integral formulation one obtains the following lemma:

**Lemma 4.3** The sets \( M_{a} \) and \( \partial M_{a} \) are positively invariant under the semi flow \( \{U(t)_{\mid_{t=0}} \). Moreover if \( x \in \partial M_{a} \) then

\[ \lim_{t \to \infty} U(t)x - x_{x} = 0 \]
with \( x_{x} = (0, 0, \sum_{k=1}^{e} \beta_{k}(a) \Lambda e^{-\mu \tau}, 0) \)

Global Stability of the Disease free Equilibrium when \( R_{0} < 1 \) and Simulations

**Stability with a Lyapunov like function**

**Lemma 5.1** Let \( t \in \mathbb{R} \) and \( a > 0 \) be given. For a globally in time solution \( s \), we get the following inequality:

\[ s(t, a) \leq s_{0}(a), \ \forall a > 0. \]
We have on characteristics:

\[ s(t, a) = e^{-\mu(t-a)} \]

As a consequence we obtain for each \( s \in \mathbb{R}, t>0 \) and \( a>0 \) that

\[ s(t+a, a) = e^{-\mu a} S_{a}(a) \]

If \( t = a \) then for each \( s \in \mathbb{R}: \)

\[ s(s+a, a) = e^{-\mu a} S_{a}(a) \leq S_{a}(a) \]

Let \( t \in \mathbb{R} \) and \( a > 0 \) be given. Choose \( s \in \mathbb{R} \) such that \( s=t-a \). The above equality re-writes as

\[ s(t, a) \leq s_{0}(a) \]
and the result follows.

We will use the lemma above in the proof of the following theorem.

**Theorem 5.2** For the model (1.1)-(1.2), if \( R_{0} < 1 \), then the disease free equilibrium is globally asymptotically stable in \( \partial M_{a} \).

**Proof:** Recall that

\[ R_{0} = \int_{0}^{\infty} \beta(t) e^{-(\mu + \gamma) t} d\tau \int_{0}^{\infty} p(a) e^{-\mu a} da \]

Choose and such that

\[ \Gamma_{1}(t) = (\mu + \gamma) \Gamma_{1}(t) + \beta(t) \]
\[ \Gamma_{1}(0) = \int_{0}^{\infty} e^{-\mu \tau} \beta(\tau) s(\tau) d\tau \]

And

\[ \Gamma_{2}(t) = (\mu + \gamma) \Gamma_{2}(t) + \beta(t) \]
\[ \Gamma_{2}(0) = \int_{0}^{\infty} e^{-\mu \tau} \beta(\tau) s(\tau) d\tau \]
Note that
\[ R_{0} = \int_{0}^{\infty} p(a) s_{0}(a) da + \Gamma_{1}(0) \int_{0}^{\infty} (1 - p(a)) s_{0}(a) da \]
Then one gets (by density):

\[ \frac{d}{dt} \int_{0}^{\infty} \Gamma_{1}(t)(t, a) d\tau = -\int_{0}^{\infty} \Gamma_{1}(t)(t, a) d\tau - (\mu + \gamma) \int_{0}^{\infty} \Gamma_{1}(t)(t, a) d\tau \]

\[ = \lambda(t) \int_{0}^{\infty} p(a) s(t, a) da + \Gamma_{1}(0) \int_{0}^{\infty} (1 - p(a)) s(t, a) da \]

\[ = \lambda(t) \int_{0}^{\infty} (1 - p(a)) s(t, a) da + \Gamma_{1}(0) \int_{0}^{\infty} (1 - p(a)) s(t, a) da \]

\[ = \lambda(t) \int_{0}^{\infty} (1 - p(a)) s(t, a) da + \Gamma_{1}(0) \int_{0}^{\infty} (1 - p(a)) s(t, a) da \]

so that

\[ \frac{d}{dt} \int_{0}^{\infty} \Gamma_{1}(t)(t, a) d\tau = -\int_{0}^{\infty} \Gamma_{1}(t)(t, a) d\tau - (\mu + \gamma) \int_{0}^{\infty} \Gamma_{1}(t)(t, a) d\tau \]

\[ = \lambda(t) \int_{0}^{\infty} (1 - p(a)) s(t, a) da + \Gamma_{1}(0) \int_{0}^{\infty} (1 - p(a)) s(t, a) da \]

\[ = \lambda(t) \int_{0}^{\infty} (1 - p(a)) s(t, a) da + \Gamma_{1}(0) \int_{0}^{\infty} (1 - p(a)) s(t, a) da \]
\[
\frac{d}{dt}\int_0^\infty \left[ \Gamma_i(\tau)i(\tau,\tau) + \Gamma_e(\tau)e(\tau,\tau) \right] d\tau
\]

so that
\[
\frac{d}{dt}\int_0^\infty \left[ \Gamma_i(\tau)i(\tau,\tau) + \Gamma_e(\tau)e(\tau,\tau) \right] d\tau
\leq \left[ \Gamma_i(0)\int_0^\infty p(a)s(t,a)da + \Gamma_e(0)\int_0^\infty (1-p(a))s_e(a)da \right] \times
\int_0^\infty \beta_i(\tau)i(\tau,\tau) + \beta_e(\tau)e(\tau,\tau) d\tau
\]

Since \( R_e \leq 1 \) then,
\[
\frac{d}{dt}\int_0^\infty \left[ \Gamma_i(\tau)i(\tau,\tau) + \Gamma_e(\tau)e(\tau,\tau) \right] d\tau
\leq \left[ \Gamma_i(0)\int_0^\infty p(a)s(t,a)da + \Gamma_e(0)\int_0^\infty (1-p(a))s_e(a)da \right] \times
\int_0^\infty \beta_i(\tau)i(\tau,\tau) + \beta_e(\tau)e(\tau,\tau) d\tau
\]

\[
\leq \left[ \Gamma_i(0)\int_0^\infty p(a)s(t,a)da - s_e(a)da + \Gamma_e(0)\int_0^\infty (1-p(a))(s(t,a) - s_e(a))da \right] \times
\int_0^\infty \beta_i(\tau)i(\tau,\tau) + \beta_e(\tau)e(\tau,\tau) d\tau
\]

On the attractor \( A \) we check by lemma 5.1 that
\[
s(t,a) \leq s_e(a), \quad \forall t \in \mathbb{R}, \quad a > 0,
\]
so that the functional
\[
V[i,e](t) := \int_0^\infty \left[ \Gamma_i(\tau)i(\tau,\tau) + \Gamma_e(\tau)e(\tau,\tau) \right] d\tau,
\]
is non-increasing along the complete orbits.

\( V[i,e] \) is a strict Lyapunov function for DFE on \( A \subseteq \partial M_0 \) and global stability of DFE when \( R_e < 1 \) follows.

**Simulations**

We first simplify the model by assuming that \( \beta_i \) and \( \beta_e \) are both constant parameters. Then introducing
\[
I(t) = \int_0^\infty i(t,\tau)d\tau \quad \text{and} \quad E(t) = \int_0^\infty e(t,\tau)d\tau.
\]

We will use data in Tables 1-3 for the case of Cameroon.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>( p(a) )</th>
<th>( \beta_i )</th>
<th>( B_i )</th>
<th>( V_i )</th>
<th>( V_e )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, A=60 yrs]</td>
<td>0.811.e(0.189a)</td>
<td>10-6</td>
<td>10-20</td>
<td>0.036</td>
<td>0.02</td>
<td>15.10^{-5}</td>
</tr>
</tbody>
</table>

**Table 2:** Values for \( R_e < 1 \).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>( p(a) )</th>
<th>( \beta_i )</th>
<th>( B_i )</th>
<th>( V_i )</th>
<th>( V_e )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, A=60 yrs]</td>
<td>0.811.e(0.189a)</td>
<td>0.000006</td>
<td>10-20</td>
<td>0.09</td>
<td>0.05</td>
<td>15.10^{-5}</td>
</tr>
</tbody>
</table>

**Table 3:** Values for \( R_e > 1 \).

**Discussion**

Simulations illustrate the asymptotic stability of DFE in section 5. The model described by equations (1.1-1.2) exhibit a rich dynamic. We observe that the biological basic reproduction rate \( R_e \) is fundamental for the study of the basic dynamical properties. Applied to hepatitis B, the model suggests that infection rates play a great role in the description of the disease (see expression of \( R_e \)). Simulations conducted follow our results and suggest the fact that the endemic equilibrium is asymptotically stable if \( R_e > 1 \) (Figures 1-9).
Figure 3: Function $E(t)$ from Data-“C”, $R_0<1$.

Figure 4: Function $I(t)$ from data of Data-“C”, $R_0<1$.

Figure 5: Function $t \rightarrow \int_{0}^{t} s(r)dr$ from Data-“C”, $R_0>1$.

Figure 6: Function $E(t)$ from Data-“C”, $R_0>1$.

Figure 7: Function $I(t)$ from data of Data-“C”, $R_0>1$.

Figure 8: Prevalence of asymptomatic carriers $E(t)$ from Data-“C”, $R_0>1$. 
References


