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A Multivariate Weight Enumerator for Tail-biting Trellis Pseudocodewords

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Abstract

Tail-biting-trellis representations of codes allow for iterative decoding algorithms, which are limited in effectiveness by the presence of pseudocodewords. We introduce a multivariate weight enumerator that keeps track of these pseudocodewords. This enumerator is invariant under many linear transformations, often enabling us to compute it exactly. The extended binary Golay code has a particularly nice tail-biting-trellis and a famous unsolved question is to determine its minimal AWGN pseudodistance. The new enumerator provides an inroad to this problem.

Keywords: Tail-biting trellis; Binary Golay code; Pseudocodewords; Weight enumerators; Invariant theory

Introduction

This paper makes headway into a longstanding problem, namely that of whether the Golay tail-biting trellis found by Calderbank, Forney, and Vardy [1] has any pseudocodewords of AWGN pseudoweight less than 8. This problem was addressed by many experts in the field before being abandoned as being computationally too hard. We circumvent this issue by introducing a new kind of multivariate weight enumerator that is invariant under the action of a large group of matrices. This enumerator is not covered by the encyclopedic work of Nebe, Rains, and Sloane [2], but the method, dating back to Gleason [3], of using invariant theory so as to constrain drastically the form of the enumerator polynomial, works well. In particular, we show that there are no Golay pseudocodewords of period less than 5 and AWGN pseudoweight less than 8. Deeper explorations of these techniques should permit us to extend this result to periods of 5 and higher.

Tail-biting Trellises

Definition: A *tail-biting trellis* $T = (V, E, A)$ of depth n is a directed graph with vertex set V and edge set E such that each edge has a label taken from alphabet A and with the following property: the set V can be partitioned into n subsets $V = V_0 \cup V_1 \cup \ldots \cup V_{n-1}$ such that every edge in T either begins at a vertex of V_i and ends at a vertex of V_{i+1} for some i with $0 \le i \le n-2$ or begins at a vertex of V_{n-1} and ends at a vertex of V_0.

The set of edge labels along a cycle in T starting at a vertex in V_0 is an n-tuple in A^n. We say that T represents a linear block code $\mathcal{C}$ over A if $\mathcal{C}$ is precisely the set of all edge-label sequences in T.

Conventional trellises, corresponding to the case where $|V_0| = 1$, have an older history dating back to the 60's. Trellis representations of linear block codes provide efficient decoding algorithms such as the two-way, or BCJR, algorithm [4,5] and the Viterbi algorithm [6]. Their complexity depends on the state complexity $\sigma := max_i|V_i|$ and associated to each code is a minimal conventional trellis, unique up to isomorphism.

Tail-biting trellises date back to 1979 [7]. Their advantage is that they can have much lower state complexity (in fact $\sqrt{\ }$) than that, σ, of the minimal conventional trellis associated to the code. Also, they are the simplest kind of factor graph with cycles. On the other hand, there are different notions of minimality [8] and so no uniqueness for minimal tail-biting trellises associated to a given code. Additionally, their construction can be hard, as in one famous case given next.

The Extended Binary Golay Code

One of the most extraordinary binary linear codes is the [24, 12, 8] extended binary Golay code. Muder [9] showed that the state complexity of its minimal conventional trellis is at least 256 and Forney [10] that this bound is met. It follows that any associated tail-biting trellis has state complexity at least 16 and in 1996 Forney issued the challenge of finding a tail-biting trellis meeting this bound. In 1999, Calderbank, Forney, and Vardy successfully answered this challenge [1].

The corresponding trellis-oriented generator matrix is

$$\begin{pmatrix}
1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&1&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&1&1&1\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&1\\
0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0\\
1&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&1&1&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{pmatrix}$$

Pseudocodewords

Because of the local nature of iterative decoding algorithms they can decode to vectors that are not codewords of the original code but come from some 'covering' code instead. This is most commonly studied in the case of belief propagation algorithms on Tanner graphs of low density parity check codes, where since finite covers of Tanner

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graphs are locally identical to the original graph, the algorithm can be misled by codewords associated to the cover. This leads to a theory of pseudocodewords for which advanced algebraic tools have already been developed by, for instance, Koetter, Li, Vontobel, and Walker [11]. Much work has also been carried out to define suitable pseudoweights for various channels and the minimum pseudoweight of a nonzero pseudocodeword becomes a better performance measure for the code than its minimum distance.

Tail-biting trellises also yield pseudocodewords. Namely, given $T = (V, E, A)$ and any positive integer m, define tail-biting trellis $T^m := (V^m, E^m, A)$ of depth mn by letting V_i^m be a copy of V_j where $j=i \pmod n$ and letting the edges from V_i^m to V_{i+1}^m be those from $V_{i \pmod n}$ to $V_{(i+1) \pmod n}$. The edge-labels on cycles in T^m starting at V_0^m yield a code C_m such that C_1 is the original code C represented by T.

Assume that C is binary. If $(c_0, \ldots, c_{mn-1})$ is in C_m, its associated pseudocodeword is defined to be $\mathbf{p} := (p_0, \ldots, p_{n-1})$ where p_j counts the number of nonzero c_i for $i \pmod n = j$ (note that sometimes this is normalized by dividing each entry by m). We say that $\mathbf{p}$ is of period m. There are different kinds of pseudoweight-for example, the AWGN pseudoweight of $\mathbf{p}$ is defined to be $(\sum p_i)^2 / \sum p_i^2$ [12].

In the case where C is the extended Golay code, a trellis-oriented generator matrix for C_m is given as follows. Let M be the generator matrix given in section 2. Let A be the matrix obtained by zeroing out the ones in the bottom left hand corner of M and $B = M - A$, i.e. the matrix obtained by zeroing out everything but the bottom left hand corner. Let Z be the zero matrix of the same dimensions as M. Define M_m to be the $12m$-by-$24m$ block matrix

$$\begin{pmatrix} A & B & Z & Z & \ldots Z & Z \\ Z & A & B & Z & \ldots Z & Z \\ Z & Z & A & B & \ldots Z & Z \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ Z & Z & Z & Z & \ldots A & B \\ B & Z & Z & Z & \ldots Z & A \end{pmatrix}$$

Then C_m is the $[24m, 12m, 8]$ binary linear code with generator matrix M_m. Note that in the limit as $m \to \infty$, M_m yields the infinite recurring generator matrix for the Golay convolutional code.

Much work has been done on pseudoweights of pseudocodewords in this case, notably in Aji et al. [13] and Forney et al. [14]. The question of whether there exist nonzero pseudocodewords of AWGN pseudoweight less than 8 apparently became a major challenge but remains open to this day. There are many nontrivial near-misses - for example, there are pseudocodewords of period 2 with 2 in 2 positions, 1 in 8 positions, and 0 elsewhere, which therefore have AWGN pseudoweight $12^2/16=9$, and pseudocodewords of period 3 with 3 in 2 positions, 2 in 2 positions, 1 in 6 positions, and 0 elsewhere, which therefore have AWGN pseudo weight $16^2/32=8$.

This question motivated the current work. It is feasible to find all 2^{24} codewords of C_2 and hence all corresponding pseudocodewords but to do the same with C_3 is already computationally intense. Resolving the question by brute force would require doing the same for all C_m for $1 \le m \le 16$ since cycles can go around up to 16 times before necessarily returning to the same vertex of V_0.

Pseudocodeword Weight Enumerators

Let $\mathbf{p} := (p_0, \ldots, p_{n-1})$ be a pseudocodeword of period m. Attach to $\mathbf{p}$ the monomial $x_0^{r_0} x_1^{r_1} \ldots x_m^{r_m}$ where r_i is the number of occurrences of i in $\mathbf{p}$. Note that $r_0 + r_1 + \ldots + r_m = n$. For example, the two pseudocodewords referred to in the penultimate paragraph of section 3 yield monomials $x_0^{14} x_1^8 x_2^2$ and $x_0^{14} x_1^6 x_2^2 x_3^2$ respectively. Note that the AWGN pseudoweight can be calculated from the corresponding monomial.

Next, define the *pseudocodeword weight enumerator* W_m associated to pseudocodewords of period m to be the sum of all these monomials as we run through the codewords of C_m. Note that W_m is a polynomial in $m + 1$ variables $x_0, \ldots, x_m$ with non-negative integer coefficients. So, for example, W_1 is the usual weight enumerator. For the extended Golay code,

$$W_1 = x_0^{24} + 759 x_0^{16} x_1^8 + 2576 x_0^{12} x_1^{12} + 759 x_0^8 x_1^{16} + x_1^{24}$$

As noted above, a brute force calculation of W_2 for the extended Golay code is feasible. We thereby obtain:

$$\begin{aligned} W_2 = {} & x_0^{24} + 294 x_0^{16} x_1^8 + 759 x_0^{16} x_2^8 + 9792 x_0^{14} x_1^8 x_2^2 + 5152 x_0^{12} x_1^{12} + 178248 x_0^{12} x_1^8 x_2^4 \\ & + 2576 x_0^{12} x_2^{12} + 340032 x_0^{10} x_1^{12} x_2^2 + 748608 x_0^{10} x_1^8 x_2^6 + 24288 x_0^8 x_1^{16} + 2550240 x_0^8 x_1^{12} x_2^4 \\ & + 1234980 x_0^8 x_1^8 x_2^8 + 759 x_0^8 x_2^{16} + 680064 x_0^6 x_1^{16} x_2^2 + 4760448 x_0^6 x_1^{12} x_2^6 + 748608 x_0^6 x_1^8 x_2^{10} \\ & + 1700160 x_0^4 x_1^{16} x_2^4 + 2550240 x_0^4 x_1^{12} x_2^8 + 178248 x_0^4 x_1^8 x_2^{12} + 680064 x_0^2 x_1^{16} x_2^6 + 340032 x_0^2 x_1^{12} x_2^{10} \\ & + 9792 x_0^2 x_1^8 x_2^{14} + 4096 x_1^{24} + 24288 x_1^{16} x_2^8 + 5152 x_1^{12} x_2^{12} + 294 x_1^8 x_2^{16} + x_2^{24} \end{aligned}$$

Knowing W_2 is enough to establish that there are no nonzero pseudocodewords of period 2 with pseudoweight less than 8. Our strategy then will be to try to compute W_m for all m by using the fact that W_m has some very nice transformation properties.

Invariant Theory

In her 1962 Harvard PhD thesis [15], MacWilliams showed that the weight enumerator W of the dual of a binary linear code C is closely related to that of the code. In particular, W is invariant under the transformation

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

If the weights of all code words are divisible by 4 (C is then called doubly even), then W is also invariant under the transformation

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Thus, W is invariant under all possible compositions of these two transformations, of which there are 192, forming what is called the Clifford-Weil group G_1.

In 1970, Gleason [3] observed that this imposes a strong restriction on the structure of W , since every homogeneous polynomial in x_0, x_1 invariant under G_1 is a polynomial in the weight enumerators $W_H (= x_0^8 + 14 x_0^4 x_1^4 + x_1^8)$ of the extended [8, 4, 4] Hamming code and $W_G (= W_1$ given in the previous section) of the extended Golay code. This permits quick computation of weight enumerators of large self-dual doubly even codes.

In the last four decades, hundreds of papers have appeared generalizing and applying Gleason's results, culminating in the book [2] by Nebe, Rains, and Sloane unifying these theories. They define the

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Type of a self-dual code such that the weight enumerator of any code of that Type lies in the invariant ring of a certain Clifford-Weil group associated with that Type, and furthermore such that this invariant ring is spanned by weight enumerators of that Type. There are also fascinating algebra isomorphisms due to Broué and Enguehard between various rings of modular forms and rings of weight enumerators [16].

We are guided by a similar philosophy below, in seeking to compute the multivariate weight enumerators W_m for the Golay case. This theory is new, not covered by the above book.

Symmetric Power Invariance

Let A be a 2-by-2 matrix. Then A acts on x_0, x_1 by linear transformations. Substituting these into $x_0^m, x_0^{m-1}x_1, x_0^{m-2}x_1^2, \ldots, x_1^m$, yields a linear transformation of those $m+1$ terms and hence produces an $m+1$-by-$m+1$ matrix, denoted $Sym^m(A)$. For example, if $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ then $A(x_0) = \frac{1}{\sqrt{2}}(x_0 - x_1)$ and so $A(x_0^2) = (\frac{1}{\sqrt{2}}(x_0 - x_1))^2 = \frac{1}{2}(x_0^2 - 2x_0x_1 + x_1^2)$. Similarly one computes that $A(x_0x_1) = \frac{1}{2}(x_0^2 - x_1^2)$ and $A(x_1^2) = \frac{1}{2}(x_0^2 + 2x_0x_1 + x_1^2)$. It follows that A sends

$$\begin{pmatrix} x_0^2 \\ x_0x_1 \\ x_1^2 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_0^2 \\ x_0x_1 \\ x_1^2 \end{pmatrix}$$

The above 3-by-3 matrix is then $Sym^2(A)$. Applying this to all 192 matrices in the Clifford-Weil group G_1 above yields 96 3-by-3 matrices (we say that the homomorphism Sym^2 has a kernel of order 2). Each such matrix defines a linear transformation in x_0, x_1, x_2. One can check using a computer algebra system such as Magma that W_2 is invariant under all 96 of these transformations.

In fact W_2 is also invariant under the transformation $x_0 \mapsto x_0, x_1 \mapsto x_1, x_2 \mapsto -x_2$. Compositions formed from this and the 96 transformations above yield a group G_2 of 384 3-by-3 matrices, all leaving W_2 invariant.

If we had known a priori that W_2 is invariant under G_2, then we could have used Magma to compute all homogeneous degree 24 polynomials invariant under G_2 (it turns out that they are spanned by 6 such polynomials I_1, ..., I_6). The correct linear combination of those 6 polynomials (in fact $I_1 + 294I_6$) could then be found by exploiting a computation of low weight codewords of C_2. This is the strategy we employ in section 7 below to calculate W_3 and W_4, which would otherwise have been out of reach. The main (nontrivial) point, proven in David Conti's upcoming University College Dublin PhD thesis, is that, for every m, W_m is invariant under every matrix $Sym^m(A)$ where A is in G_1 and under certain diagonal matrices. This produces a typically large group G_m of $(m+1)$-by-$(m+1)$ matrices leaving W_m invariant.

The group G_2 is the group of 3-by-3 quasipermutation matrices which have exactly one nonzero entry, a 4th root of unity, in every row and every column. This makes it isomorphic to the wreath product $C_4 \wr S_3$. It is also a complex reflection group, which makes its invariant theory particularly nice, leading to the following pretty formula. Let

$$f(x_0,x_1,x_2) := x_0^{24} + x_1^{24} + x_2^{24} + 759(x_0^{16}x_1^8 + x_0^{16}x_2^8 + x_0^8x_1^{16} + x_0^8x_2^{16} + x_1^{16}x_2^8 + x_1^8x_2^{16})$$
$$+2576(x_0^{12}x_1^{12} + x_0^{12}x_2^{12} + x_1^{12}x_2^{12}) + 3186x_0^8x_1^8x_2^8 .$$

Then $W_2(x_0,x_1,x_2) = 2^{12} f((x_0+x_2)/2, x_1, (x_0-x_2)/2)$.

Golay Pseudocodeword Enumerators

We move first to computing W_3. The above theory shows that W_3 is invariant under $Sym^3(A)$ for A in G_1. This yields 192 transformations. In addition, W_3 is invariant under the transformation

$x_0 \mapsto x_0, x_1 \mapsto x_1, x_2 \mapsto -x_2, x_3 \mapsto -x_3$. All possible compositions of the above transformations yield a group G_3 of 1152 4-by-4 matrices leaving W_3 invariant.

Next, using Magma, we compute all homogeneous degree 24 polynomials in x_0, x_1, x_2, x_3 invariant under G_3. Magma produces 26 polynomials I_1, ..., I_{26} that span this space. Using low weight codewords in C_3 yields a simple linear combination of them that must equal W_3. Namely, $I_1 + 441I_{14} + 513I_{18} + 7560I_{22} + 288I_{23} + 11520I_{25} + 4608I_{26}$, which gives:

$$W_3 = x_0^{24} + 441x_0^{16}x_1^8 + 513x_0^{16}x_2^8 + 759x_0^{16}x_3^8 + 7560x_0^{14}x_1^8x_2^2 + 288x_0^{14}x_1^6x_2^2x_3^2$$
$$+14112x_0^{14}x_2^{10} + 11520x_0^{13}x_1^7x_2^3x_3 + 2304x_0^{13}x_1^5x_2^3x_3^3 + 4608x_0^{12}x_1^{12} + 792x_0^{12}x_1^{10}x_2^2 + \ldots$$
$$\ldots + 7560x_1^2x_2^8x_3^{14} + 33291x_2^{24} + 16371x_2^{16}x_3^8 + 4608x_2^{12}x_3^{12} + 441x_2^8x_3^{16} + x_3^{24} .$$

There are 212 terms in W_3, too many to list here, but note that each monomial that appears corresponds to pseudocodewords that have pseudoweight at least 8.

As for W_4, we similarly obtain a group G_4 of 384 5-by-5 matrices that leave W_4 invariant. The space of homogeneous degree 24 polynomials in x_0, ..., x_4 invariant under G_4 is spanned by 153 very lengthy polynomials which Magma gives explicitly. Of these 153 polynomials, 87 have the property that every monomial occurring in them corresponds to pseudocodewords of pseudoweight at least 8. We show that W_4 is a span of these 87 polynomials by excluding the other 66 polynomials as follows. Every such polynomial contains a monomial that occurs in it and none of the remaining 152 polynomials. Examining these special monomials, we see that, for those 66 polynomials, if the special monomial were present, it would come from a codeword of C_4 of weight at most 24. By analyzing low weight codewords of C_4, we can show that this does not happen. Thus, there are no nonzero pseudocodewords of period ≤ 4 of pseudoweight less than 8.

Likewise, for m ≥ 5, there is an explicitly given group G_m of $m + 1$-by-$m + 1$ matrices leaving W_m invariant. Unfortunately, both the computation of homogeneous degree 24 polynomials invariant under G_m and the analysis of low weight code words of C_m become computationally too expensive. It is clear that there are $147m$ code words of C_m of weight 8, but beyond that patterns are hard to spot. For example, the codewords of C_m of weight 12 correspond to pseudocodewords either consisting of 12 ones and 12 zeros or 2 twos, 8 ones, and 14 zeros, but there is no clear, even conjectural, formula for the number of either kind.

Conclusions and Further Work

We have introduced new and useful multivariate polynomials attached to a tail-biting trellis. These keep track of what kinds of pseudocodewords exist and indeed how many there are of each kind. This can in turn be used to measure how good the code is as regards iterative decoding, with various formulae for pseudoweight being used, depending on the channel. It has been a longstanding question to determine whether the AWGN pseudo weight of a nonzero pseudocodeword for the tail-biting trellis of the extended binary Golay code is ever less than 8. Our new invariant theory methods allow us to answer this question in the negative for all pseudocodewords of period ≤ 4.

Pseudocodeword weight enumerators W_m are defined for any tail-biting trellis. It is not true, however, that they are invariant under the same group G_m as for the Golay code above. For example, for tail-biting trellises attached to the extended [8, 4, 4] Hamming code, the polynomials W_m are invariant under slightly smaller groups than G_m. The author's PhD student, David Conti, is developing a theory that should hopefully clarify the notion of Type for a tail-biting trellis and allow one to define an analogue of the Clifford-Weil group for each Type.

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