Journal of Physical Mathematics

Moffat et al., J Phys Math 2017, 8:2
DOI: 10.4172/2090-0902.1000232

Research Article Open Access

A New Approach to the Quantisation of Paths in Space-Time

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Abstract

A discrete path in space-time can be considered as a series of applications of the translation subgroup of the Poincare group. If there is a local mapping from this translation group into a neighbourhood of the identity of a quantum Weyl algebra fibre bundle, then the whole classical path can be lifted into the fibre bundle to form a unique quantum field as a section through the fibres. Under the further assumptions of scale relativity, we also show that a discrete closed loop in space time, corresponding to two classical paths sharing the same end points, is renormalisable and the finite limit has anomalous dimension equal to the fractal dimension. We end by introducing the possibility of a 'push forward' connection on the bundle $O(D)$ of Ehresmann type.

Keywords: Quantum Weyl algebra; Fibre bundle; Quantum Paths; Entropy

Introduction

Einstein's field equations for gravity are an 'effective' theory at the classical level, relating mass-energy flux to the changes in local space-time. At the underlying quantum level, we assume space-time is non-commutative due to the existence of additional non-commutative algebraic structure at each point x of space-time, forming a quantum operator 'fibre algebra' A(x). This structure then corresponds to the single fibre of a fibre bundle. A gauge group acts on each fibre algebra locally, while a 'section' through this bundle is then a quantum field of the form

$\{A(x);x\in M\}$ with M the underlying space-time manifold. In addition, we assume a local algebra $O(D)$ corresponding to the algebra of sections of such a principal fibre bundle with base space a finite and bounded subset of space-time, D. The algebraic operations of addition and multiplication are assumed defined fibrewise for this algebra of sections. The region D corresponds to the space-time constraints within which only that subset of $\{A(x);x\in M\}$ with $x\in D$ is of local physical interest or is capable of measurement. Alternatively, it can represent the set of sections $\{A(x);x\in M\}$ with compact support in D.

We relax the requirement of that each fibre algebra is norm separable and hence finite dimensional [1]. Here we assume that each fibre algebra has a faithful representation as a von Neumann algebra with trivial centre (a 'factor') acting on a separable Hilbert space via the Gelfand, Naimark, Segal (GNS) construction. This further implies that each fibre algebra is countably decomposable; every set of orthogonal projections is countable; and thus the fibre algebra has a faithful normal quantum state.

We assume, for now, that background space-time is globally flat. The union of all the local algebras generates algebra of all observables defined on the subset of space-time $\bigcup_\alpha D_\alpha$. The closure in the ultraweak operator topology of this set of local algebras generates the 'quasi-local' von Neumann algebra R of all observables. Choosing the ultraweak topology on R ensures that it contains an identity element. A key benefit of reformulating quantum field theory in this way as a'local algebra' formalism is the ability to consider coherently the many inequivalent irreducible representations, each corresponding to the Gelfand-Naimark-Segal (GNS) construction. These essentially represent different 'projections' of the same underlying algebraic structure.

Quantum Paths in Space-Time

Wald places representation of the Weyl form of the CCR at the centre of his approach to quantum field theory, generating the 'fundamental observables' and corresponding states [2]. We can add to that approach by interpreting it from the fibre bundle perspective. Thus we start by considering classical phase space. Given a dynamical system, entropy is defined through considering the phase space of the system. The emergent behaviour of this classical system gives rise to regions of phase space, each corresponding to similar macro-level behaviour. The number and variation in size of these regions reflects the overall complexity of the system. This identification is known as 'coarse graining'. The entropy of such a coarse grained region is essentially a count of all of the different micro-configurations constituting that region. A system starting in a low entropy state will tend to wander into larger coarse grained volumes; hence thermodynamic entropy tends to increase over time if the system is isolated, giving rise to the second law of thermodynamics. The structure of classical phase space is such that each set of initial conditions (x_μ, P_μ) generates a unique solution $S(x, P_\mu)$. For a Hamiltonian system it is possible to reformulate classical mechanics as a symplectic vector space of solutions, or equivalent initial conditions of location and momentum, equipped with a bilinear form Ω which ultimately derives from Hamilton's equations of motion;

With $\xi=(x, p)$ a 6-dimensional vector we have $\frac{\partial \xi}{\partial t}=\Omega\frac{\partial H}{\partial \xi}$ where $\Omega=\begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ -I_{3\times 3} & 0_{3\times 3}\end{bmatrix}$.

This is of the form of a symplectic vector space $V\oplus V^*$ with V a real finite vector space and with dual V^*. The skew-symmetric rank 2 tensor Ω then takes the general form;

$$\Omega(x\oplus\eta \ , \ x'\oplus\eta' \ ) \ = \ \eta'\bullet x \ - \ \eta\bullet x' \ .$$

In our case V is the configuration space, V^* the (dual) momentum space and $V\oplus V^*$ the phase space, a product vector bundle over V with

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Received May 10, 2017; **Accepted** May 31, 2017; **Published** June 06, 2017

Citation: Moffat J, Oniga T, Wang CHT (2017) A New Approach to the Quantisation of Paths in Space-Time. J Phys Math 8: 232. doi: 10.4172/2090-0902.1000232

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fibre V^*. By choosing particular values such as (1,0,0,0,0,0) we can pull out particular elements; $\Omega(x \oplus \eta, 0 \oplus \eta') = \eta' \bullet x = \eta'_1$.

From this point of view the Dirac canonical quantisation of elements of phase space such as η'_1 is equivalent to the canonical quantization; $\Omega \to \hat{\Omega}$ as a (not necessarily bounded) linear operator, and this form of canonical quantisation extends smoothly to countably infinite phase space. The approach is particularly transparent in flat space-time [2]. Given the canonical quantization; $\Omega \to \hat{\Omega}$ we can form the Weyl unitaries $\hat{W} = \exp i\hat{\Omega}$. Then closure of linear combinations of these unitaries and their adjoints in the normed operator topology is then a C*-algebra called the Weyl algebra.

In classical mechanics, given a particular dynamical relationship, we can select out the subset of phase space consisting of initial values. Each initial value vector *(x(0), p(0))* generates a unique solution $S=\{(x(t), p(t)); t>0\}$ propagating through phase space as a function of time t. In fibre bundle terms the solution space is also a product bundle with bundle projection $\pi(S)=S(0)=(x(0), p(0))$ onto phase space. We can then extend the definition of Ω to the space of solutions S as Ω(S(1), S(2))=Ω(π(S(1), π(S(2)). This allows us to define an inner product on S as $\langle S(1), S(2)\rangle = -i\Omega\left(S(1)^*, S(2)\right)$ where $S(1)^*$ is the complex conjugate solution. It turns out (Wald, 1994) that this defines an inner product on S relative to which a one particle Hilbert space can be defined. For a quantum system of bosonic harmonic oscillators we can then assemble a symmetric tensor product Fock space in the usual way, using creation and annihilation operators.

An example of the Weyl form in a two dimensional locally flat space-time is now given for a local algebra *O(D)*, having a representation as observables acting on the Hilbert space $L^2(x,t)$ with Lebesgue measure.

For a small increment of space-time $(\delta x, \delta t)$ we consider the Poincare Translation subgroup element $T(\delta x, \delta t):(x,t) \to (x+\delta x, t+\delta t)$ and define; $U_{T(\delta x,\delta t)} f(x,t) = f(x-\delta x, t-\delta t)$ for $f(x,t) \in L^2(x,t)$

Then U is a local group homomorphism of the translation group T as observables acting on $L^2(x,t)$. Define also;

$$
\begin{aligned}
& V_{\delta t} = U_{T(0,\delta t)}; V_{\delta x} = U_{T(\delta x,0)} \\
& \Rightarrow V_{\delta t} V_{\delta t'} f(x,t) = U_{T(0,\delta t)} U_{T(0,\delta t')} f(x,t) = f(x, t-\delta t' - \delta t) \\
& = U_{T(0,\delta t'+\delta t)} f(x,t) = V_{\delta t+\delta t'} f(x,t) \Rightarrow \; V_{\delta t} V_{\delta t'} = V_{\delta t+\delta t'}
\end{aligned}
$$

A similar result applies for $V_{\delta x}$, by symmetry.

Now introduce a deformation of the form; $T(\delta x, 0) \to Z_{\delta x} f(x,t) = \exp(it\delta x) f(x-\delta x, t)$. The mappings V and Z are unitary representations on $L^2(x,t)$ and so also is their product $V \times W : (\delta x, \delta t) \to T(\delta x, \delta t) \to V_{\delta t} Z_{\delta x}$

Then we have;

$$
\begin{aligned}
V_{\delta t} Z_{\delta x} f(x,t) &= V_{\delta t} \exp(it\delta x) f(x-\delta x, t) \\
&= \exp(it\delta x) f(x-\delta x, t-\delta t) \\
Z_{\delta x} V_{\delta t} f(x,t) &= Z_{\delta x} f(x, t-\delta t) \\
&= \exp(i(t-\delta t)\delta x) f(x-\delta x, t-\delta t) \\
&= \exp(-i\delta t \delta x) V_{\delta t} Z_{\delta x} f(x,t) \\
Z_{\delta x} V_{\delta t} &= \exp(-i\delta t \delta x) V_{\delta t} Z_{\delta x}
\end{aligned}
$$

Then $T(\delta x,\delta t) \to (Z_{\delta x}, V_{\delta t})$ is a local Weyl representation of the CCR on $L^2(x,t)$. By the Stone-von Neumann theorem, the resulting C*-algebra and its weak closure as a von Neumann algebra must be unitarily isomorphic to Wald's equivalent 'algebraic approach' to quantum field construction and his Weyl Algebra, since we can assume all relevant Hilbert spaces are separable in application to observed real systems [3].

In summary, this example indicates that a continuous local group homomorphism from a neighbourhood of the identity of T to a neighbourhood of the identity of the set of observables in *O(D)* exists as a Weyl representation of the CCR. It can be easily extended to 4 dimensions by replacing x by the 3-vector $\boldsymbol{x}=(x_1, x_2, x_3)$.

Definition: A discrete path in space-time is one which consists of linked causally directed intervals in space-time, each of the same Euclidean 4-dimensional length. More formally, we define the path as a series of n linked increments $\mathbf{a}(j)$ each of the same Euclidean interval length $|\mathbf{a}(j)|$ but varying direction such that the path begins at $\mathbf{x(0)}$ and ends at $\mathbf{x(1)}$, with $T(\mathrm{a}(j))$:x→x+a(j) elements of the translation subgroup T. The total path is then generated by the product $\prod_{j=1}^{j=n} T(\mathbf{a}(j))\mathbf{x}(\mathbf{0})$ with the final end point $\mathbf{x}(\mathbf{1}) = \mathbf{x}(\mathbf{0}) + \sum_{j=1}^{j=n} \mathbf{a}(j)$. For a fixed initial point $\mathbf{x(0)}$ we can identify this path with the finite group product $\prod_{j=1}^{j=n} T(\mathbf{a}(j)) \in T$.

Theorem 1: Let there be given a continuous local group homomorphism from a neighbourhood V of the identity of T to the neighbourhood W of the identity of the set of observables in *O(D)* as a Weyl representation of the CCR. Then a discrete classical path *CP* in space-time can be lifted to a quantum section *QP* through *O(D)*.

Proof: We construct the section iteratively, following a method suggested by Pontryagin [4]. Let CP be a discrete classical path in space-time. From the definition, the path is a series of n linked increments a(j) each of the same Euclidean interval length |a(j)| but varying direction such that the path begins at x(0) and ends at x(1), with $T(\mathrm{a}(j))$:x → x+a(j) elements of the translation subgroup T. We can thus, as noted earlier, characterise CP as the product mapping $T^n(\mathrm{a}(j))$ in the translation group T for a given x(0). We have, by assumption, a continuous local group homomorphism $\phi: T(a) \to U_{T(a)}$ from V, a neighbourhood of the identity of T to W, a neighbourhood of the identity of *(O(D))*. Clearly, by redefining the number of links in CP if necessary, we can assume that $|\mathbf{a}(j)|$ is sufficiently small so that $T(\mathrm{a}(j)) \in V$ for all j and by choosing appropriate units we can assume that CP consist of n links each of length *1/n*.

We also assume that for n large; $|\mathbf{x}(t_2)-\mathbf{x}(t_1)| < \frac{1}{n} \Rightarrow T(\mathbf{x}(t_1))^{-1} T(\mathbf{x}(t_2)) = T(\mathbf{a}(t_1)) \in V$.

Let m be a positive integer strictly less than n, and suppose that the path *QP* has been defined such that its initial value is *QP(0)*=A(x(0)). We proceed by induction. Assume that *QP* has been defined for all values of x(t) with $0 \le t \le \frac{m}{n}$, and satisfies, for all such t with $0 \le t \le \frac{m}{n}$;

(a). Fixed endpoint; *QP(0)*=A(x(0));

(b). Local lifting to the Weyl algebra near the identity; If $\mathbf{x}(s)$, $\mathbf{x}(t)$ in *CP* satisfy $|\mathbf{x}(s)-\mathbf{x}(t)| \le \frac{1}{n}$ then $T(\mathbf{x}(s))^{-1} T(\mathbf{x}(t)) \in V$ and $\varphi\left(T(\mathbf{x}(s))^{-1} T(\mathbf{x}(t))\right) = QP(\mathbf{x}(s))^{-1} QP(\mathbf{x}(t)) \in W$

We now extend the path $QP(\mathbf{x}(t))$ with $0 \le t \le \frac{m}{n}$, stepping forward one additional link on CP so that $t = \frac{m}{n} + \frac{1}{n}$, by the following construction;

$$
QP\left(\mathbf{x}\left(\frac{m+1}{n}\right)\right) = QP\left(\mathbf{x}\left(\frac{m}{n}\right)\right) \varphi\left(T\mathbf{x}\left(\frac{m}{n}\right)^{-1} T\mathbf{x}\left(\frac{m+1}{n}\right)\right) \quad (1)
$$

From eqn. (1) the extension of the path QP still satisfies (a): QP(0)=A(x(0)) since φ acting on the identity of the group local translations T is the identity operator in O(D). We need to show that the extension under induction still satisfies (b).

Let h be a real number with $|h| \le \frac{1}{n}$. If h is positive then by induction h satisfies the extension shown in equation (1). Thus we have;

$$QP\left(\mathbf{x}\left(\frac{m}{n}+h\right)\right) = QP\left(\mathbf{x}\left(\frac{m}{n}\right)\right)\varphi\left(T\mathbf{x}\left(\frac{m}{n}\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right)\right)$$

If on the other hand h is negative then setting $\mathbf{x}(s) = \mathbf{x}\left(\frac{m}{n}+h\right)$ and $\mathbf{x}(t) = \mathbf{x}\left(\frac{m}{n}\right)$; since *s, t* are both equal to or less than $\frac{m}{n}$ by the inductive hypothesis they therefore satisfy;

$$\varphi\left((T\mathbf{x}(s))^{-1} T\mathbf{x}(t)\right) = (QP\mathbf{x}(s))^{-1} QP\mathbf{x}(t) \text{ with } t = \frac{m}{n}+h \text{ and } s = \frac{m}{n}$$

$$\Rightarrow \varphi\left(\left(T\mathbf{x}\left(\frac{m}{n}\right)\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right)\right) = \left(QP\mathbf{x}\left(\frac{m}{n}\right)\right)^{-1} QP\mathbf{x}\left(\frac{m}{n}+h\right)$$

$$\Rightarrow QP\mathbf{x}\left(\frac{m}{n}+h\right) = QP\mathbf{x}\left(\frac{m}{n}\right)\varphi\left(T\mathbf{x}\left(\frac{m}{n}\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right)\right)$$

Thus equation (1) holds for both positive and negative values of h.

It follows that;

$$\left(QP\mathbf{x}\left(\frac{m+1}{n}\right)\right)^{-1} QP\mathbf{x}\left(\frac{m}{n}+h\right) = \left(QP\mathbf{x}\left(\frac{m}{n}\right)\varphi\left(T\mathbf{x}\left(\frac{m}{n}\right)^{-1} T\mathbf{x}\left(\frac{m+1}{n}\right)\right)\right)^{-1}\left(QP\mathbf{x}\left(\frac{m}{n}\right)\varphi\left(T\mathbf{x}\left(\frac{m}{n}\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right)\right)\right)$$
$$= \varphi\left(T\mathbf{x}\left(\frac{m+1}{n}\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right)\right)$$

Thus the path extension satisfies both requirements (a) and (b) completing the inductive step $\frac{m}{n} \to \frac{m+1}{n}$, provided we satisfy the local topological constraints, namely;

We have that if n is sufficiently large then there is a neighbourhood U such that;

$$|x(t_2) - x(t_1)| < \frac{1}{n} \Rightarrow \text{by continuity } T(x(t_1))^{-1} T(x(t_2)) = T(a(t_1)) \in U^{-1}U \subset V$$

and $\varphi\left(T(x(t_1))^{-1} T(x(t_2))\right) = \varphi(T(a(t_1)) \in W$

Setting, for small h>0;

$$t_1 = \frac{m+1}{n}; t_2 = \frac{m}{n}+h \Rightarrow |\mathbf{x}(t_1) - \mathbf{x}(t_2)| \text{ small}$$

$$\Rightarrow T\mathbf{x}\left(\frac{m+1}{n}\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right) = T(\mathbf{x}(t_1))^{-1} T(\mathbf{x}(t_2)) = T(\mathbf{x}(t_2)) T(\mathbf{x}(t_1))^{-1} = T(\mathbf{x}(t_2) - \mathbf{x}(t_1)) \in V$$

$$\varphi\left(T\left(\mathbf{x}\left(\frac{m+1}{n}\right)\right)^{-1} T\mathbf{x}\left(\frac{m}{n}+h\right)\right) \in W \text{ as required.}$$

Since ϕ(identity of T)=identity of $O(D)$ the induction hypothesis is true for $\frac{m}{n} = 0;\ h = \frac{1}{n}$. By induction the path QP(t) can be extended in O(D) for all discrete steps m less than or equal to n.

Theorem 2: The constructed quantum path QP is unique.

Proof: The initial point of QP is unique by condition (a). If $QP(\mathbf{x}(t))$ is unique for all

$t \le t_0$ then let $t_0 < t \le t_0 + \varepsilon$. Then;

$$T\mathbf{x}(t_0)^{-1} T\mathbf{x}(t) \in V \Rightarrow \varphi\left(T\mathbf{x}(t_0)^{-1} T\mathbf{x}(t)\right) = QP\mathbf{x}(t_0)^{-1} QP\mathbf{x}(t) \Rightarrow QP\mathbf{x}(t) = QP\mathbf{x}(t_0)\varphi\left(T\mathbf{x}(t_0)^{-1} T\mathbf{x}(t)\right)$$

Thus the path QP is uniquely determined for all points $t<t_0+\varepsilon$. The result follows by induction.

Theorem 3: There is a projection π from the fibre bundle $O(D)$ mapping the quantum path back to the translation subgroup T

Proof: From condition (a) this is clear for the initial point. We again use induction to prove the general case. If the projection π maps the observable QP $\mathbf{x}(t)$ back to $T\mathbf{x}(t)$ for $t \le t_0$ then let $t_0 < t \le t_0 + \varepsilon$. Then as before we have;

$$QP\mathbf{x}(t) = QP\mathbf{x}(t_0)\varphi\left(T(\mathbf{x}(t_0))^{-1} T\mathbf{x}(t)\right)$$

$$\text{thus } \pi QP\mathbf{x}(t) = \pi QP\mathbf{x}(t_0)\pi \circ \varphi\left(T(\mathbf{x}(t_0))^{-1} T\mathbf{x}(t)\right) = T\mathbf{x}(t_0)\left(T(\mathbf{x}(t_0))^{-1} T\mathbf{x}(t)\right) = T\mathbf{x}(t)$$

The result follows by induction.

Renormalisation of Discrete Fractal Paths in Space-Time

The principle of relativity is captured within the assumptions of the Riemannian geometry of 4-manifolds, where formulae equating a tensor expression to zero remain invariant under covariant and contravariant coordinate transformations. It is a natural extension of these ideas to additionally postulate that the scales of measurement inscribed on the clocks or measuring rods used by an observer should also not be absolute. Mathematically this can be captured by the additional requirement that the tensor formulae should be invariant under transformations of scale [5]. From this perspective a relativistic quantum system is a *scale free system* as first defined by James [6].

The derivation of a particular quantum relationship has to be inferred, in a rather *ad hoc* way, from the context and can be captured in the abstract by a function Φ linking system inputs and outputs.

Definition: A system is scale free if observers using different scales observe the same functional relationship Φ [6].

Definition: A system input variable is dimensionally independent if it cannot be described dimensionally by a combination of other inputs; otherwise it is described as dimensionally dependent [7].

Assume we have a scale free system with output value a, functional relationship Φ; k dimensionally independent input variables a_1, a_2,……., a_k and 2 dimensionally dependent input variables b_1,b_2. Given the mathematical relationship linking inputs to output; $a=\Phi(a_1,\ldots a_k,b_1,b_2)$ it is possible to vary the arguments $a_1,\ldots a_k$ using arbitrary positive numbers so that:

$$a_1' = A_1 a_1, \ldots, a_k' = A_k a_k$$

By definition, the dimensions of a,b_1,b_2 may be represented as power monomials in the dimensions $a_1,\ldots a_k$ for example:

$$[b_1] = [a_1]^{p_1} \ldots [a_k]^{r_1}$$
$$[b_2] = [a_1]^{p_2} \ldots [a_k]^{r_2}$$
$$[a] = [a_1]^{p} \ldots [a_k]^{r}$$

We therefore obtain the transformations:

$$b_1' = A_1^{p_1} \ldots A_k^{r_1} b_1$$
$$b_2' = A_1^{p_2} \ldots A_k^{r_2} b_2$$
$$a' = A_1^{p} \ldots A_k^{r} a$$

The above transformations form a group of continuous gauge transformations with $A_1 \dots A_k$ as the parameters. For a scale free system, our physical relationship can then be represented as a relationship between gauge transformation group invariants:

$$\Pi = \Phi(\Pi_1, \Pi_2)$$

These invariants are given by:

$$\Pi_1 = \frac{b_1}{a_1^{p_1} \dots a_k^{r_1}}$$

$$\Pi_2 = \frac{b_2}{a_1^{p_2} \dots a_k^{r_2}}$$

$$\Pi = \frac{a}{a_1^{p} \dots a_k^{r}}$$

The invariants Π_1 and Π_2 are similarity parameters and the functional relationship Φ has the equivalent form;

$$\Phi(a_1, \dots a_k, b_1, b_2) = a_1^p \dots a_k^r \Phi'\left(\frac{b_1}{a_1^{p_1} \dots a_k^{r_1}}, \frac{b_2}{a_1^{p_2} \dots a_k^{r_2}}\right)$$

Three possibilities are available for this system under renormalisation of one of the similarity parameters; [7]

a) Φ tends to a non-zero finite limit as $\Pi_2 \to 0$ This means that Φ can be replaced by its limiting expression, with complete separation of variables and the functional relationship is a product of powers whose values can be determined by dimensional analysis.

b) Φ has power law asymptotics of the form $\Phi = \Pi_2^{\alpha_1} \Phi'\left(\frac{\Pi_1}{\Pi_2^{\alpha_2}}\right)$, as $\Pi_2 \to 0$. The power law form of the limiting expression still leads to separation of variables, but with characteristic exponents equal to the 'anomalous' fractional dimensions of a form of renormalisation [8,9].

c). neither a). nor b) holds; Φ has no finite limit different from zero and no power-law asymptotics.

In summary, for scale relativity, as a scale, free system, application of a renormalisation group is mathematically equivalent to the intermediate asymptotics approach. We can exploit this equivalence to prove the following result.

Theorem 4: Under the assumptions of scale relativity, a discrete closed loop in space-time; corresponding to two discrete non-oriented paths sharing the same end points, is renormalisable and has a finite limit as the step size of the curve tends to zero.

Proof: Assume that we have a fractal closed loop L in space-time with Euclidean diameter d. We approximate L by a discrete closed path $L(\eta)$ where η is the Euclidean length of each segment of $L(\eta)$. Standard dimensional analysis shows that $N(\eta)$, the number of segments in the path $L(\eta)$, is a function of the form $f\left(\frac{d}{\eta}\right)$. We will establish the nature of this function and its renormalisation limit, following a suggestion of Barenblatt and Isaakovich [7].

The fractal, self-similar nature of the discrete path implies that if we consider a finer segmentation of segment length ξ, then $N(\xi) \propto N(\eta) N(\xi \mid \eta) = \frac{f\left(\frac{d}{\eta}\right) f\left(\frac{\eta}{\xi}\right)}{f(1)^2}$ where $N(\xi|)$ is the number of segments of length ξ in a segment of length η, and $N(d) = N(\eta \mid \eta) = f(1)$.

It follows that $\frac{f\left(\frac{d}{\xi}\right)}{f(1)} = \frac{f\left(\frac{d}{\eta}\right) f\left(\frac{\eta}{\xi}\right)}{f(1)^2}$ thus $f\left(\frac{d}{\xi}\right) = \frac{f\left(\frac{d}{\eta}\right) f\left(\frac{\eta}{\xi}\right)}{f(1)}$

This implies that f, for the limiting case, must be of the form $f\left(\frac{x}{y}\right) = C\left(\frac{x}{y}\right)^D$ with C and D constants; $C=f(1)$. Thus we have;

$$f\left(\frac{d}{\eta}\right) = f(1)\left(\frac{d}{\eta}\right)^D \Rightarrow N(\eta) = \left(\frac{d}{\eta}\right)^D \text{ and } L(\eta) = \eta\left(\frac{d}{\eta}\right)^D$$

Locally along the limiting smooth form this implies $L(\xi \mid \eta) = \xi\left(\frac{\eta}{\xi}\right)^D = \eta^D \xi^{1-D}$.

The renormalisation limit is thus finite and we identify D with the path fractal dimension. We end by introducing the possibility of a 'push forward' connection on the bundle $O(D)$ of Ehresmann type.

Definition: If π is the projection map from $O(D) \to D$, let $\mathbf{x}$ be an element of D and p an element of the fibre $\pi^{-1}(x)$, so that $\pi(p)=\mathbf{x}$. The 'push forward' of π, denoted π_* is a connection we define as follows. Let $t \to A(x(t))$ be a section passing through the point p in the fibre $\pi^{-1}(x(t_0))$, so that $p=A(x(t_0))$ and $\pi(p)=x(t_0)$, a point on the curve $t \to x(t)$ defined on the base space D and passing through the point $x(t_0)$ with velocity $v_{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t}|_{t_0}$. If $v_p = \frac{\partial(Ax(t))}{\partial t}|_{t_0}$, with v_p the velocity of the curve $t \to A(x(t))$ at p, then $\pi_*(v_p)=v_x$

We now define the tangent space to $O(D)$ as the linear space $TO(D)$ generated by the set $\left\{\frac{\partial A(x(t))}{\partial x_\mu}; t \to A(x(t)) \text{ is a path in } O(D) \text{ and } x_\mu = t, x, y, z\right\}$

Similarly, we define the tangent space to D, denoted TD, as the linear space generated by the set;

$$\left\{\frac{\partial(\mathbf{x}(t))}{\partial x_\mu}; t \to \mathbf{x}(t) \text{is a path in } D \text{ and } x_\mu = t, x, y, z\right\}$$

With these definitions, we see that the push forward connection π_* maps $TO(D)$ to TD.

Conclusion

Through the paper we explained the series of applications of the translation subgroup of the Poincare group. In the end, we introduced the possibility of a 'push forward' connection on the bundle $O(D)$ of Ehresmann type.

Acknowledgments

CW is grateful to the Cruickshank Trust, Scotland for financial support.

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