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A Note on the Pricing of American Capped Power Put Option

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Abstract

We give an explicit solution to the perpetual American capped power put option pricing problem in the Black-Scholes-Merton Model. The approach is mainly based on free-boundary formulation and verification. For completeness we also give an explicit solution to the perpetual American standard power (_ 1) option pricing problem.

Keywords: The perpetual American capped power put option; Geo-etric Brownian motion; Free-boundary

Introduction

A standard American power put option is a financial contract that allows the holder to sell an asset for a prescribed amount at any time. The price of this asset is raised to some power. The case of power being one corresponds to the usual American put option [1]. For the European power put option, the value of this option is given [2-4]. For the perpetual American power (≥ 1) put option, for completeness, we give the value of this option and the optimal stopping time. A capped power option is a power option whose maximum payoff is set to a prescribed level. For the European capped power put option, the value of this option is given [2-5]. For the perpetual American capped power put option, we give the value of this option and the optimal stopping time. Throughout this note, the approach is mainly based by free-boundary formulation and verification. Only one exception is Theorem 2.2. This note is organized as follows. In Section 2, we explicitly solve the perpetual American power put option pricing problem. In Section 3, we explicitly solve the perpetual American capped power put option pricing problem [6-8].

The Perpetual American Power Put Option

The arbitrage-free price of the perpetual American power put option is given by

$$V_*(x) = \sup E_x(e^{-rT}(K-(X_T)^i)^+) \tag{1}$$

where K is the strike price, T is a stopping time, i is a positive constant greater than or equal 1, and $x>0$ is the initial value of the stock price process $X=(Xt)t \geq 0$. In equation 1, the supremum is taken over all stopping times T of the process X started at x. The stock price process $X=(Xt)t \geq 0$ is assumed to be a geometric Brownian motion. That is,

$$dX_t = rX_t dt + \sigma X_t dB_t \tag{2}$$

Where r, $\sigma \geq 0$. The infinitesimal generator of X is given by

$$L_X = rx\frac{\partial}{\partial_x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2} \tag{3}$$

As in the case of standard perpetual American put option, we suppose that there exists a point $b \in (0,K)$ such that

$$Tb \quad Tb = \inf\{t \geq 0 : X_t \leq b\} \tag{4}$$

is optimal in equation 1 Then we solve the following free-boundary problem for unknown V and b. Here $0 < b^i < K$.

$$L_x V = rV \quad x \geq b \tag{5}$$

$$V(x) = (K - x^i)^+ \quad x = b \tag{6}$$

$$V'(x) = -ix^{i-1} \quad x = b \tag{7}$$

$$V(x) > (K - x^i)^+ \quad x > b \tag{8}$$

$$V(x) = (K - x^i)^+ \quad 0 < x < b \tag{9}$$

The steps to solve this free-boundary problem is same as in the case of the standard American put option (i=1) [5] so we only outline the steps. Since $V(x) \leq K$, the equation 5 implies

$$V(x) = cx^{-r/D} \tag{10}$$

Where $D = \frac{\sigma^2}{2}$ and c is an undetermined constant. Using equation 10, we solve two equations 6 and 7 to give

$$b = \left(\frac{K}{1+Di/r}\right)^{1/i} \tag{11}$$

$$c = \frac{Di}{r}\left(\frac{K}{1+Di/r}\right)^{1+\frac{r}{Di}} \tag{12}$$

Thus $V(x)$ is written as

$$V(x) = \begin{cases} \frac{Di}{r}\left(\frac{K}{1+Di/r}\right)^{1+\frac{r}{Di}_{x^{-r/D}}} & \text{if } x \in [b,\infty), \\ K - x^i & x \in (0,b] \end{cases} \tag{13}$$

Now we have the following theorem.

Theorem 1.1: $V(x)$ coincides with $V_*(x)$, and the optimal stopping time is given by Tb

The steps to prove this theorem is same as in the case of the standard American put option ($i = 1$) [5], so we omit.

It should be noted that for $i < 1$, $\lim_{x \to 0}|V'(x)| = \infty$, so we assume that $i \geq 1$ in this section.

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The Perpetual American Capped Power Put Option

The arbitrage-free price of the perpetual American capped power put option is given by^

$$V_*(x) = \sup_T E_x\left(e^{-TT}\min[K-(X_T)^i)\overline{C}]\right) \quad (14)$$

Where $\overline{C}(<K)$ and i are positive constants. First we suppose there exists a point $b \in (h, K^{\frac{1}{i}})$ such that

$$Tb = \inf\{t \geq 0 : X_t \leq b\} \quad (15)$$

is optimal in (14). Here h satisfies $K - h^i = \overline{C}$. Then we solve the following free-boundary problem for unknown V and b.

$$L_X V = rV \quad x > b \quad (16)$$

$$V(x) = (K - x^i)^+ \quad x = b \quad (17)$$

$$V'(x) = -ix^{i-1} \quad x = b \quad (18)$$

$$V(x) > (K - x^i)^+ \quad x > b \quad (19)$$

$$V(x) = (K - x^i)^+ \quad h < x \leq h \quad (20)$$

$$V(x) = \overline{C} \quad 0 < x \leq h \quad (21)$$

It is clear that for $0 < x \leq h$ the arbitrage-free price from equation14 is given by equation 21. Thus, $0 < x \leq h$ is the stopping region.

Since we suppose that $b \in (h, K^{\frac{1}{i}})$, b is same as in the case of the American power put option. Thus the free-boundary b is given by

$$b = \left(\frac{K}{1 + Di/r}\right)^{1/i} \quad (22)$$

$V(x)$ is written as

$$V(x) = \begin{cases} & x \in [b, \infty) \\ \frac{Di}{r}\left(\frac{K}{1+Di/r} - \right)^{1 + \frac{r}{Di}_{x^{-r/D}}} & \textbf{if } x \in (h, b] \\ \frac{K - x^i}{C} & x \in (0, h] \end{cases} \quad (23)$$

Note that $b > h$ is equivalent to $\overline{C} > K.\frac{Di/r}{1+Di/r}$

Theorem 2.1: Suppose that $\overline{C} > K.\frac{Di/r}{1+Di/}$ then $V(x)$ coincides with $V_*(x)$ and the optimal stopping time is given by $x \in (h, \infty)$.

Proof: From our earlier consideration we can suppose that $x \in (h, \infty)$. Since $P(Xs = h) = 0$ and $P(X_s = b) = 0$, the change-of-variable formula (see Remark 2.3 in [5]) with the smooth-fit condition (18) gives

$$V(X_t) = V(X_0) + \int_0^t e^{-rs}(L_x V - rV)(X_s) I(X_s \neq h, X_s \neq b) ds \quad (24)$$

$$+ \int_0^t V_x(X_s)\sigma X_s I(X_s \neq h, X_s \neq b) dB_s$$

$$+ \frac{1}{2}\int_0^t V_x(X_s +) - V_x(X_s -)) I(X_s = h) dl_s^c(X)$$

S When $V(x) = (K - x^i)$, we see that

$(L_x V - rV)(x) = x^i(1-i)(r + \frac{\sigma^2}{2} i) - rk < 0$ for $h < x \leq b$. For $i \geq 1$, it clearly holds. For $i < 1$, from $x^i < h$ it is easily seen to hold. Thus

$$e^{-rt}(K - X_t^i)^+ \leq e^{-rt} V(X_t) \leq V(x) + M_t$$

Where $M = (M_t)_{t \geq 0}$ is defined by

$$M_t = \int_0^t V_x(X_s)\sigma X_s I(X_s \neq h, X_s \neq b) dB_s$$

is a continuous martingale (because $|V'(x)|$ is bounded for all $x > 0$. Thus it is easily verified by standard means (using the localization of M and Fatou's lemma) that we get that

$$V_*(x) \leq V(x)$$

For all $x \in (h, \infty)$

Next we set $t = T_b \wedge T_n$ in equation 24. Here $(T_n)_{n \geq 1}$ is a localization sequence of bounded stopping times for M. For $s \leq T_{b,} X_s \geq b > h$. Hence the fourth term in the right-hand side of this equation is zero. Moreover using equation16, we find the second term in the right-hand side is also zero. Finally using the optional sampling theorem we get

$$V(x) = \frac{\overline{C}}{C}(K - \overline{C})^{\frac{r}{Di}}.x^{-\frac{r}{D}}$$

$$E_x(e^{-r(T_b \wedge T_n)} V(X_{Tb \wedge Tn})) = V(x) \quad (25)$$

Letting n go to infinity and using the dominated convergence and, we get that

$$V_*(x) \geq V(x)$$

for all $x \in (h, \infty)$. The proof is completed.

Theorem 2.2: Suppose that $\overline{C} \leq K.\frac{Di/r}{1+Di/r}$ then $V(x)$ is written as

$$V(x) = \frac{\overline{C}}{C}(K - \overline{C})^{\frac{r}{Di}}.x^{-\frac{r}{D}} \quad \text{if } x \in (0, h] \quad (26)$$

$x \in (h, \infty)$

The optimal stopping time is given by T_h.

Proof: We set $T_h = \inf\{t \geq 0 : X_t \leq h\}$. Then

$$E_x(e^{-rTh}(K - (X_{Th})^i)^+) \quad (27)$$

$$= E_x(e^{-rTh}(K - h^i))$$

$$= \overline{C}(K - \overline{C})^{\frac{r}{Di}.x^{-\frac{r}{D}}}$$

$$= \overline{C}(K - \overline{C})^{\frac{r}{Di}.x^{-\frac{r}{D}}} \quad (28)$$

Where the final equality follows by the formula for the expected first hitting time for a geometric Brownian motion

Now we show for $K^{\frac{1}{i}} > x > h$,

$$\overline{C}\wedge(K - \overline{C})^{\frac{r}{Di}}.x^{-\frac{r}{D}} > K - x^i \quad (29)$$

We set $f(x)$ to be equal to the left-hand side minus the right-hand side in equation 29. Clearly $f(h) = 0$. To show that $f'(x) > 0$, it suffices to show

$$\overline{C}x^{\frac{r}{D}}\left(-\frac{r}{D}\right)x^{-\frac{r}{D}-1} + i(K - \overline{C})x^{-1} \geq 0 \quad (30)$$

Because $x > h$. Since $\overline{C} \leq K.\frac{Di/r}{1+Di/r}$ (equation30) holds. Since $V(x)$ $V(x) > 0$ equation 29 and 30 imply that (h,∞) is a continuous region. On the other hand, $(0,h]$ is a stopping region. Thus T_h is the optimal stopping time and $V(x)$ is given by equation 26. The proof is completed.

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