

# A Peirce decomposition for (-1,-1)-Freudenthal-Kantor triple systems

Noriaki KAMIYA

University of Aizu, Aizu-Wakamatsu, 965-8580 Fukushima, Japan

E-mail: kamiya@u-aizu.ac.jp

## Abstract

In this paper, we study a Peirce decomposition for (-1,-1)-Freudenthal-Kantor triple systems and give several examples.

2000 MSC: 17A40

## 1 Introduction

Our aim is to give a characterization of many mathematical and physical fields by means of concept of triple systems (here, triple systems mean a vector space equipped with a triple product  $\langle xyz \rangle$ ). It seems that such concept is useful to an application of nonassociative algebras as well as the characterization of Yang-Baxter equations, and the construction of Lie (super)algebras and Jordan (super)algebras ([3]-[10], [12], [13], [15]).

In this article, first we will consider a Peirce decomposition of (-1,-1)-Freudenthal-Kantor triple systems, and give an example of a Peirce decomposition of simple (-1,-1)-Freudenthal-Kantor triple systems. Second, we will study the decomposition of simple Lie superalgebras associated with such triple systems.

Throughout this paper, we shall be concerned with algebras and triple systems over a field  $\Phi$  that is characteristic not 2 and do not assume that our algebras and triple systems are finite dimensional, unless otherwise specified.

Summarizing briefly this article we will mainly discuss the following result.

**Main Theorem.** *Let  $U$  be a (-1,-1)-Freudenthal-Kantor triple system with a tripotent element  $e$ , i.e.  $(eee)=e$ . Then we have the Peirce decomposition*

$$U = U_{00} \oplus U_{1,-1} \oplus U_{01} \oplus U_{11}$$

where  $U_{\lambda,\mu} = \{x \in U \mid (eex) = \lambda x \text{ and } (xee) = \mu x\}$ . In particular, for balanced cases, we have

$$U = U_{11} \oplus U_{1,-1}$$

which implies

$$x = \frac{x + R(x)}{2} + \frac{x - R(x)}{2} \quad \text{and} \quad R^2(x) = x$$

where  $R(x) = (xee)$ .

In the rest of this section, I shall give the definition and some results for a certain triple system in order to make this paper as self-contained as possible.

For  $\varepsilon = \pm 1$  and  $\delta = \pm 1$ , a vector space  $U(\varepsilon, \delta)$  over  $\Phi$  with the triple product  $\langle -, -, - \rangle$  is called a  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system if

$$[L(a, b), L(c, d)] = L(\langle abc \rangle, d) + \varepsilon L(c, \langle bad \rangle) \quad (1.1)$$

$$K(\langle abc \rangle, d) + K(c, \langle abd \rangle) + \delta K(a, K(c, d)b) = 0 \tag{1.2}$$

where

$$L(a, b)c = \langle abc \rangle, \quad K(a, b)c = \langle acb \rangle - \delta \langle bca \rangle, \quad [A, B] = AB - BA$$

A triple system is said to be a *generalized Jordan triple system*, if  $\varepsilon = -1$  and only the identity (1.1) is satisfied.

The triple products are generally denoted by

$$\langle xyz \rangle, \quad \{xyz\}, \quad (xyz), \quad [xyz]$$

**Example 1.1.** Let  $V$  be a vector space equipped with a bilinear form  $\langle x|y \rangle = \varepsilon \langle y|x \rangle$ . Then  $V$  is a  $(\varepsilon, \varepsilon)$ -Freudenthal-Kantor triple system with respect to the product

$$\langle xyz \rangle := \langle x|z \rangle y + \langle y|z \rangle x$$

**Example 1.2.** Let  $V$  be a Jordan triple system. Then this triple system is a special case of the  $(-1, 1)$ -Freudenthal-Kantor triple system, because the identity  $K(a, b)c \equiv 0$  (identically zero) implies that  $\langle acb \rangle = \langle bca \rangle$ , and the identity (1.1) implies that

$$\langle ab \langle cde \rangle \rangle = \langle \langle abc \rangle de \rangle - \langle c \langle bad \rangle e \rangle + \langle cd \langle abe \rangle \rangle$$

If its product satisfies  $\langle abc \rangle = -\langle cba \rangle$  and

$$\langle ab \langle cde \rangle \rangle = \langle \langle abc \rangle de \rangle + \langle c \langle bad \rangle e \rangle + \langle cd \langle abe \rangle \rangle$$

then this triple system is called an *anti-Jordan triple system*.

An  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system over  $\Phi$  is said to be *balanced* if there exists a bilinear form  $\langle \cdot, \cdot \rangle$  such that  $K(x, y) = \langle x|y \rangle \text{Id}$ , where  $\langle x|y \rangle \in \Phi^*$ .

**Remark 1.1.** For a balanced  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system, we have the following relation:

$$K(a, b) = L(b, a) - \varepsilon L(a, b), \quad \langle a|b \rangle = -\varepsilon \langle b|a \rangle = -\delta \langle b|a \rangle \\ \varepsilon \delta \langle acb \rangle - \langle cba \rangle = \varepsilon \delta \langle a|b \rangle c - \langle b|c \rangle a, \quad \varepsilon \delta = 1$$

For convenience (in the section 3 of this paper), the notation  $\langle x|y \rangle$  will be used by means of  $2 \langle x|y \rangle$ . That is, the notation  $\langle aab \rangle = \langle aba \rangle = \langle a|a \rangle b$  imply the balanced property.

For the  $\delta$ -Lie triple systems associated with  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple systems, we have the following.

**Proposition 1.1** ([7, 12]). *Let  $U(\varepsilon, \delta)$  be a  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system. If  $P$  is a linear transformation of  $U(\varepsilon, \delta)$  such that  $P \langle xyz \rangle = \langle PxPyPz \rangle$  and  $P^2 = -\varepsilon \delta \text{Id}$ , then  $(U(\varepsilon, \delta), [-, -, -])$  is a Lie triple system for the case of  $\delta = 1$  and an anti-Lie triple system for the case of  $\delta = -1$  with respect to the product*

$$[xyz] := \langle xPyz \rangle - \delta \langle yPxz \rangle + \delta \langle xPzy \rangle - \langle yPzx \rangle$$

**Corollary 1.1.** *Let  $U(\varepsilon, \delta)$  be a  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system. Then the vector space  $T(\varepsilon, \delta) := U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$  becomes a Lie triple system for the case of  $\delta = 1$  and an anti-Lie triple system for the case of  $\delta = -1$  with respect to the triple product defined by*

$$\left[ \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} L(a, d) - \delta L(c, b) & \delta K(a, c) \\ -\varepsilon K(b, d) & \varepsilon(L(d, a) - \delta L(b, c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

**Proposition 1.2.** *Let  $V$  be an anti-Jordan triple system (that is, it satisfies the condition (L1) with  $\varepsilon = 1$  and  $L(x,y)z = -L(z,y)x$ ). Then,  $V \oplus V$  becomes an anti-Lie triple system with respect to the product defined by*

$$\left[ \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} L(a,d) + L(c,b) & 0 \\ 0 & L(d,a) + L(b,c) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

From these results, it follows that the vector space

$$L(V) := \text{Inn Der } T \oplus T (= L(T, T) \oplus T)$$

where  $T$  is a  $\delta$  Lie triple system and  $\text{Inn Der } T : \{L(X, Y) | X, Y \in T\}_{\text{span}}$ , makes a Lie algebra ( $\delta = 1$ ) or Lie superalgebra ( $\delta = -1$ ) by

$$[D + X, D' + X'] = [D, D'] + L(X, X') + DX' - D'X$$

We denote by  $L(\varepsilon, \delta)$  the Lie algebras or Lie superalgebras obtained from these constructions associated with  $U(\varepsilon, \delta)$  and call these algebras a *standard embedding*.

A  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system  $U(\varepsilon, \delta)$  is said to be *unitary* if the linear span  $\mathbf{k}$  of the set  $\{K(a, b) | a, b \in U(\varepsilon, \delta)\}$  contains the identity endomorphism Id.

**Proposition 1.3** ([6, 7]). *For a unitary  $(\varepsilon, \delta)$ -Freudenthal-Kantor triple system  $U(\varepsilon, \delta)$  over  $\Phi$ , let  $T(\varepsilon, \delta)$  be the Lie or anti-Lie triple system and  $L(\varepsilon, \delta)$  be the standard embedding Lie algebra or superalgebra associated with  $U(\varepsilon, \delta)$ . The following are equivalent:*

- a)  $U(\varepsilon, \delta)$  is simple,
- b)  $T(\varepsilon, \delta)$  is simple,
- c)  $L(\varepsilon, \delta)$  is simple.

For these standard embedding Lie algebras or superalgebras  $L(\varepsilon, \delta)$ , we have the following 5 grading subspaces:

$$L(\varepsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$$

where  $U(\varepsilon, \delta) = L_{-1}$ ,  $T(\varepsilon, \delta) = L_{-1} \oplus L_1$ ,  $\mathbf{k} = \{K(a, b)\}_{\text{span}} = L_{-2}$ .

This is one reason why we study about a characterization of triple systems, as properties of Lie superalgebras are closely related to those of the triple systems as well as Lie algebras.

## 2 Main results (proof of the Main Theorem)

From now, we will only consider a  $(-1, -1)$ -Freudenthal-Kantor triple system, unless otherwise specified, because the case of  $\delta = 1$  had considered in other papers ([3]–[7], [11]), that is, one deal with the relations

$$(abxyz) = ((abx)yz) - (x(bay)z) + (xy(abz)) \tag{2.1}$$

$$K(K(a, b)c, d) - L(d, c)K(a, b) - K(a, b)L(c, d) = 0 \tag{2.2}$$

where  $K(a, b)x = (axb) + (bxa)$  and  $L(a, b)x = (abx)$ .

**Remark 2.1.** We note that the relations (2.1) and (2.2) coincide with (1.1) and (1.2) with the case of  $\varepsilon = -1$ , and  $\delta = -1$ .

Let  $e$  be a tripotent, i.e  $(eee) = e$  and denote

$$L(x) = (eex), \quad R(x) = (xee), \quad Q(x) = (exe)$$

**Remark 2.2.** We note for the notation that

$$\begin{aligned} K(e, e)e &= (eee) + (eee) = 2e \\ K(x, e)e &= K(e, x)e = (xee) + (eex) \\ K(e, e)x &= 2Q(x) = 2(exe) \end{aligned}$$

From (2.1), we have

$$(xe(eee)) = ((xee)ee) - (e(exe)e) + (ee(xee))$$

and so

$$(xee) = ((xee)ee) - (e(exe)e) + (ee(xee))$$

Hence we get

$$R(x) = R^2(x) - Q^2(x) + LR(x) \tag{2.3}$$

On the other hand,

$$\begin{aligned} (ex(eee)) &= ((exe)ee) - (e(xee)e) + (ee(exe)) \\ Q(x) &= RQ(x) - QR(x) + LQ(x) \end{aligned} \tag{2.4}$$

Also,

$$(ee(xee)) = ((eex)ee) - (x(eee)e) + (xe(eee))$$

Therefore, we obtain

$$LR(x) = RL(x) \tag{2.5}$$

and moreover

$$\begin{aligned} (ee(exe)) &= ((eee)xe) - (e(eex)e) + (ex(eee)) \\ LQ(x) &= Q(x) - QL(x) + Q(x) \end{aligned}$$

hence,

$$LQ(x) = 2Q(x) - QL(x) \tag{2.6}$$

On the other hand, from (2.2) we have

$$\begin{aligned} K(K(x, e)e, e)e - L(e, e)K(x, e)e - K(x, e)L(e, e)e &= 0 \\ K(L(x) + R(x), e)e - L(L(x) + R(x))e - (L(x) + R(x))e &= 0 \end{aligned}$$

and so

$$L(L(x) + R(x)) + R(L(x) + R(x)) - L^2(x) - LR(x) - L(x) - R(x) = 0$$

Therefore we have

$$(R - \text{Id})(L(x) + R(x)) = 0 \tag{2.7}$$

Similarly, from (2.5) we have

$$(L + R)(R - \text{Id})(x) = 0 \tag{2.8}$$

**Lemma 2.1.** *There is no vector  $a \neq 0$  satisfying*

$$L(a) = -R(a) = -a \quad (2.9)$$

**Proof.** From (2.3), it follows that

$$R^2 - Q^2 + LR - R = 0$$

and if there is an element  $a$  satisfying (2.9), then we have

$$a - Q^2(a) - a - a = 0$$

i.e

$$Q^2(a) = -a \quad (2.10)$$

From (2.4), it follows that

$$\begin{aligned} Q(a) &= RQ(a) - Q(a) + LQ(a) \\ (R + L - 2\text{Id})Q(a) &= 0 \end{aligned} \quad (2.11)$$

From (2.6), it follows that

$$\begin{aligned} LQ(a) &= 2Q(a) + Q(a) \\ LQ(a) &= 3Q(a) \end{aligned} \quad (2.12)$$

We set  $Q(a) = b$ , from (2.10) we have

$$Q^2(a) = Q(b) = -a \neq 0 \quad (2.13)$$

By (2.12),  $L(b) = 3b$  and by (2.11),

$$R(b) = -L(b) + 2b = -3b + 2b$$

Hence we have  $R(b) = -b$ . From (2.7),

$$(R - \text{Id})(L(b) + R(b)) = 0$$

and by  $L(b) = 3b$  and  $R(b) = -b$ , we get  $(R - \text{Id})(2b) = 0$ . Hence  $-2b - 2b = 0$  and so  $b = 0$ . Therefore we have

$$Q(b) = Q^2(a) = -a = 0 \quad (2.14)$$

This completes the proof. □

We use the following notations.

$$U_{R=-L} = \{x \in U \mid R(x) = -L(x)\}, \quad U_{R=\text{Id}} = \{x \in U \mid R(x) = x\}$$

**Lemma 2.2.** *The space  $U$  is a direct sum of subspaces  $U_{R=-L}$  and  $U_{R=\text{Id}}$ :*

$$U = U_{R=-L} \oplus U_{R=\text{Id}} \quad (2.15)$$

**Proof.** It follows from (2.7) and (2.8) that

$$U_{R=-L} \subset (R - \text{Id})U \quad (2.16)$$

$$U_{R=\text{Id}} \subset (R + L)U \quad (2.17)$$

Either of (2.16) or (2.17) implies

$$\dim U_{R=-L} \geq \dim U - \dim U_{R=\text{Id}} \quad (2.18)$$

We set  $P := U_{R=-L} \cap U_{R=\text{Id}}$ . To prove that  $P = 0$ , assume that  $P \neq 0$ . Then we get  $R = -L = \text{Id}$  on  $P$ . Hence we come to a contradiction with Lemma 2.1.  $\square$

**Remark 2.3.** We note that

$$U_{R=-L} = (R - \text{Id})U \quad (2.19)$$

$$U_{R=\text{Id}} = (R + L)U \quad (2.20)$$

**Lemma 2.3.** *The subspaces  $U_{R=-L}$  and  $U_{R=\text{Id}}$  are invariant with respect to operators  $L$  and  $R$ .*

**Proof.** Let us prove, for example,

$$RU_{R=\text{Id}} \subseteq U_{R=\text{Id}}$$

Using (2.5) and (2.20), we get

$$R(R + L)U = (R + L)RU \subseteq (R + L)U = U_{R=\text{Id}}$$

The other proofs are obtained in the same way.  $\square$

**Corollary 2.1.** *There is no vector  $a \neq 0$  such that*

$$L(a) = -a \quad (2.21)$$

**Proof.** If there is an element  $a \neq 0$  and let  $a = a_1 + a_2$ , where  $a_1 \in U_{R=-L}$  and  $a_2 \in U_{R=\text{Id}}$ . Then it follows from Lemma 2.3 that

$$L(a_1) = -a_1, \quad L(a_2) = -a_2$$

Considering the operator  $R$  on  $U_{R=-L}$  and  $U_{R=\text{Id}}$ , we also get

$$R(a_1) = a_1, \quad R(a_2) = a_2$$

Now, as one comes to a contradiction of Lemma 2.1, hence the Corollary holds.  $\square$

An *eigensubspace of the tripotent  $e$*  with respect to the eigenvalues  $\lambda, \mu$  is called a subspace  $U_{\lambda, \mu}$  of all vectors  $a$  satisfying

$$L(a) = \lambda a, \quad R(a) = \mu a \quad (2.22)$$

Now we prove that there are only two possibilities, i.e 1)  $\mu = -\lambda$ , 2)  $\mu = 1$ . To show this, we will denote by  $U_{\mu=-\lambda}$  the sum of all subspaces with the first possibility and by  $U_{\mu=1}$  with the second one.

**Lemma 2.4.** *Let  $a \in U_{\lambda\mu}$ . Then either of the following holds:  $a \in U_{\mu=-\lambda}$  or  $a \in U_{\mu=1}$ .*

**Proof.** Suppose  $a \notin U_{\mu=-\lambda}$  and  $a \notin U_{\mu=1}$ . By Lemma 2.2, we can see that

$$a = a_1 + a_2, \quad a_1 \in U_{R=-L}, a_2 \in U_{R=Id}$$

By the definition of  $U_{\lambda\mu}$ , we get

$$L(a_1 + a_2) = \lambda a_1 + \lambda a_2, \quad R(a_1 + a_2) = \mu a_1 + \mu a_2$$

This implies, according to Lemma 2.3,

$$L(a_1) = \lambda a_1, \quad L(a_2) = \lambda a_2, \quad R(a_1) = \mu a_1, \quad R(a_2) = \mu a_2$$

Therefore, both  $a_1$  and  $a_2$  are common eigenvectors of  $L$  and  $R$ . This implies the fact that  $\mu = -\lambda$  and  $\mu = 1$ , and hence we get  $\lambda = -1$ . This contradicts the Corollary of Lemma 2.1.  $\square$

**Lemma 2.5.** *Let  $a \in U_{\lambda=-\mu}$ . Then one of the following two possibilities occurs:*

- a)  $a \in U_{00}$  and  $Q(a) = 0$ ,
- b)  $a \in U_{1,-1}$ ,  $Q(a) \in U_{1,-1}$  and  $Q^2(a) = a$ .

**Proof.** First, suppose that  $Q(a) = 0$ . Then (2.3) is equivalent to

$$R^2(x) + LR(x) - R(x) = 0$$

that is

$$L^2(x) - L^2(x) - R(x) = 0 \tag{2.23}$$

Thus  $\lambda = 0$  and  $\mu = 0$ , and one come to the case (a).

Let  $Q(a) \neq 0$ . From (2.6) it follows that

$$LQ(a) = (2 - \lambda)Q(a). \tag{2.24}$$

From (2.4) it follows that

$$Q(a) = RQ(a) + \lambda Q(a) + LQ(a)$$

i.e

$$(R + L)Q(a) - (\lambda + 1)Q(a) = 0 \tag{2.25}$$

Subtracting (2.25) from (2.24), we get

$$RQ(a) = -Q(a) \tag{2.26}$$

Thus this implies  $Q(a) \in U_{2-\lambda,-1}$  and from Lemma 2.4 follows  $Q(a) \in U_{\mu=-\lambda}$  or  $Q(a) \in U_{\mu=1}$ .

In the first case, we have  $2 - \lambda = 1$ , i.e  $\lambda = 1$ , which implies that both  $a \in U_{1,-1}$  and  $Q(a) \in U_{1,-1}$ . Hence,  $R(a) = -a$ ,  $L(a) = a$ , so  $Q^2(a) = a$ , by (2.3). Thus in this case, we come to the case b).

In the second case,  $Q(a) \in U_{\mu=1}$ . Then it follows from (2.26) that  $RQ(a) = -Q(a)$ . This case does not appear. This completes the proof.  $\square$

**Lemma 2.6.** *Let  $a \in U_{\mu=1}$ . Then one of the following two possibilities occurs:*

- a')  $a \in U_{01}$  and  $Q(a) = 0$ ,
- b')  $a \in U_{11}$ ,  $Q(a) \neq 0 \in U_{11}$  and  $Q^2(a) = a$ .

**Proof.** First suppose that  $Q(a) = 0$ . Then (2.3) is equivalent to  $L(a) = 0$ . This means that  $\lambda = 0$ . Thus one come to the case  $a'$ .

Suppose that  $Q(a) \neq 0$ . Then by (2.6),

$$LQ(a) = (2 - \lambda)Q(a) \quad (2.27)$$

and by (2.4)

$$RQ(a) + LQ(a) = 2Q(a) \quad (2.28)$$

Subtracting (2.28) from (2.27), we get

$$RQ(a) = \lambda Q(a) \quad (2.29)$$

Thus  $Q(a) \in U_{2-\lambda, \lambda}$  and from Lemma 2.4 follows  $Q(a) \in U_{\mu=-\lambda}$  or  $Q(a) \in U_{\mu=1}$ .

In the first case, we have

$$RQ(a) = -\lambda Q(a) \quad (2.30)$$

Thus from (2.29) and (2.30) we get  $\lambda = 0$  which implies  $Q(a) \in U_{00}$  and  $Q(a) \in U_{2,0}$ , because  $Q(a) \in U_{\mu=-\lambda}$  and  $Q(a) \in U_{2-\lambda, \lambda}$ . Thus this case does not appear.

In the second case, we have

$$RQ(a) = Q(a) \quad (2.31)$$

From (2.29) and (2.31) we have  $\lambda = 1$ . The case of  $\lambda = 1$  means that both  $a \in U_{11}$  and  $Q(a) \in U_{11}$ ,  $Q^2(a) = a$ , by (2.3). This comes to the case  $b'$ . This completes the proof.  $\square$

Combining Lemma 2.5 and Lemma 2.6, we have the following.

**Theorem 2.1.** *Let the three linear operators  $L, R, Q$  be defined on a  $(-1, -1)$ -Freudenthal-Kantor triple system. Then the space  $U$  is a direct sum of four subspaces;*

$$U = U_{00} \oplus U_{1,-1} \oplus U_{01} \oplus U_{11} \quad (2.32)$$

where

- 1)  $\forall a \in U_{\lambda\mu}$  we denote  $L(a) = \lambda a$ ,  $R(a) = \mu a$ ,
- 2)  $a \in U_{00}$  or  $a \in U_{01}$  if  $Q(a) = 0$ ,
- 3)  $a \in U_{1,-1}$  and  $Q^2(a) = a$  if  $Q(a) \in U_{1,-1}$ ,
- 4)  $a \in U_{11}$  and  $Q^2(a) = a$  if  $Q(a) \in U_{1,1}$ .

**Proof.** To prove (2.32), it is enough to prove

$$U_{R=-L} = U_{\lambda=-\mu} \quad \text{and} \quad U_{R=\text{Id}} = U_{\mu=1} \quad (2.33)$$

That is, the direct sum

$$U = U_{00} \oplus U_{01} \oplus U_{11} \oplus U_{1,-1}$$

follows from Lemma 2.5 and Lemma 2.6. The equalities (2.33) mean that the operator  $L$  has no Jordan blocks of second degree, i.e there are no vectors  $a_1, a_2 \neq 0$  such that

$$L(a_1) = \lambda a_1, \quad L(a_2) = \lambda a_2 + a_1 \quad (2.34)$$



Indeed, according to Lemma 2.2 and Lemma 2.3, we can consider the two cases:  $a_1, a_2 \in U_{R=-L}$  and  $a_1, a_2 \in U_{R=\text{Id}}$ , separately.

In the case of  $a_1, a_2 \in U_{R=-L}$ , we have

$$R(a_1) = -\lambda a_1, \quad R(a_2) = -\lambda a_2 - a_1 \quad (2.35)$$

It follows from (2.6) that

$$LQ(a_1) = 2Q(a_1) - QL(a_1) = (2 - \lambda)Q(a_1) \quad (2.36)$$

$$LQ(a_2) = (2 - \lambda)Q(a_2) - Q(a_1). \quad (2.37)$$

Using (2.36) and (2.37), we obtain from (2.4)

$$RQ(a_1) = Q(a_1) + QR(a_1) - LQ(a_1), \quad RQ(a_2) = Q(a_2) + QR(a_2) - LQ(a_2)$$

and so

$$RQ(a_1) = Q(a_1) - \lambda Q(a_1) - (2 - \lambda)Q(a_1) = -Q(a_1)$$

Furthermore, using (2.35),

$$\begin{aligned} [RQ(a_2)Q(a_2) + Q(-\lambda a_2 - a_1) - LQ(a_2)] &= Q(a_2) - \lambda Q(a_2) - Q(a_1) - (2 - \lambda)Q(a_2) + Q(a_1) \\ &= -Q(a_2) \end{aligned}$$

That is, we have

$$RQ(a_1) - Q(a_1) \quad \text{and} \quad RQ(a_2) - Q(a_2) \quad (2.38)$$

Equations (2.36), (2.37) and (2.38) imply that

$$\begin{aligned} (L + R)Q(a_1) &= (1 - \lambda)Q(a_1), \quad (L + R)Q(a_2) = (1 - \lambda)Q(a_2) - Q(a_1) \\ RQ(a_1) &= -Q(a_1), \quad RQ(a_2) = -Q(a_2) \end{aligned}$$

Now, applying (2.7) to  $Q(a_2)$  and using the above relations, we obtain

$$\begin{aligned} (R - \text{Id})(L + R)Q(a_2) &= R((1 - \lambda)Q(a_2)) - R(Q(a_1)) - (1 - \lambda)Q(a_2) + Q(a_1) \\ &= -(1 - \lambda)Q(a_2) + Q(a_1) + (\lambda - 1)Q(a_2) + Q(a_1) \\ &= (2\lambda - 2)Q(a_2) + 2Q(a_1) = 0 \end{aligned}$$

From Lemma 2.5 and Lemma 2.6, the cases of  $\lambda = 1$  and  $Q(a_1) = 0$  are impossible. Hence one comes to  $Q(a_1) = 0$  and  $Q(a_2) = 0$ . Therefore, by Lemma 2.5, we have  $a_1 \in U_{00}$  and  $\lambda = 0$ , and by (2.35),  $R(a_2) = -a_1$ . We come to a contradiction with (2.34).

In the second case,  $a_1, a_2 \in U_{R=\text{Id}}$ , and we have

$$R(a_1) = a_1, \quad R(a_2) = a_2. \quad (2.39)$$

Thus it follows that

$$LQ(a_1) = 2Q(a_1) - \lambda Q(a_1) \quad (2.40)$$

$$LQ(a_2) = 2Q(a_2) - Q(\lambda a_2 + a_1) = (2 - \lambda)Q(a_2) - Q(a_1) \quad (2.41)$$

From (2.4), it follows that

$$Q(x) = RQ(x) - QR(x) + LQ(x)$$

and so

$$RQ(a_1) = Q(a_1) + QR(a_1) - LQ(a_1)$$

Then using (2.39), (2.40) and (2.41) we get

$$\begin{aligned} RQ(a_1) &= Q(a_1) + Q(a_1) - (2Q(a_1) - \lambda Q(a_1)) = \lambda Q(a_1) \\ RQ(a_2) &= Q(a_2) + QR(a_2) - LQ(a_2) = Q(a_2) + Q(a_2) - (2 - \lambda)Q(a_2) + Q(a_1) \\ &= \lambda Q(a_2) + Q(a_1) \end{aligned}$$

Similarly to the first case, by applying (2.7) to  $Q(a_2)$  and using the above relations, we obtain

$$\begin{aligned} (R - \text{Id})(L + R)Q(a_2) &= (R - \text{Id})((2 - \lambda)Q(a_2) - Q(a_1) + \lambda Q(a_2) + Q(a_1)) \\ &= 2(R - \text{Id})Q(a_2) \\ &= 2(\lambda Q(a_2) + Q(a_1)) - 2Q(a_2) \end{aligned}$$

Thus we get

$$2(\lambda - 1)Q(a_2) + Q(a_1) = 0 \tag{2.42}$$

Hence, from the same method as for the first case, we have  $Q(a_1) = 0$  and  $Q(a_2) = 0$ . Therefore by Lemma 2.6, it follows that  $a_1 \in U_{01}$  and  $a_2 \in U_{01}$ ,  $\lambda = 0$ . This implies a contradiction of the assumption on (2.34). This completes the proof.  $\square$

**Remark 2.4.** We note that the case of more than one tripotent elements will be discussed elsewhere. In particular, if there exists a tripotent element  $e$  such that  $K(e, e)x = \langle e|e \rangle x$  and  $\langle e|e \rangle = 1$  then we have the balanced case  $U = U_{11} \oplus U_{1,-1}$ .

### 3 Examples of (-1,-1)-Freudenthal-Kantor triple systems

In this section, we will consider the standard embedding Lie superalgebras of the  $B(m, n)$  and  $D(m, n)$  types associated with the anti-Lie triple system and the (-1,-1)-Freudenthal-Kantor triple system. Furthermore, we will study a Peirce decomposition of such types.

**Theorem 3.1** ([10]). *Let  $U$  be a vector space of  $\text{Mat}(k, n; \Phi)$ . Then the space  $U$  is a unitary (-1,-1)-Freudenthal-Kantor triple system with respect to the triple product*

$$\langle xyz \rangle = z {}^t yx + y {}^t xz - x {}^t yz$$

where  ${}^t x$  denotes the transpose matrix of  $x$ .

**Remark 3.1.** For this triple system, by straightforward calculations, using the results in Sec. 1, we have the following:

- 1)  $k = 2m$  ( $m \geq 2$ ):  $L(U) \cong D(m, n)$  type's Lie superalgebra and

$$\dim L(U) = 2(n + m)^2 - m + n$$

- 2)  $k = 2m + 14$  ( $m \geq 0$ ):  $L(U) \cong B(m, n)$  type's Lie superalgebra and

$$\dim L(U) = 2(n + m)^2 + 3n + m$$

That is, summarizing these, we have the following.

**Theorem 3.2.** *Let  $U$  be the same triple system as that described in Theorem 3.1 and  $L(U)$  be the standard embedding Lie superalgebras associated with  $U = \text{Mat}(k, n; \Phi)$ . Then  $L(U)$  are Lie superalgebras of type  $D(m, n)$  or  $B(m, n)$  if  $k=2m$  or  $k= 2m+1$ , respectively.*

For a bilinear trace form of a (-1,-1)-Freudenthal-Kantor triple system, we have the formula as follows [7]:

$$\gamma(x, y) := \frac{1}{2} \text{Trace}\{2(R(x, y) + R(y, x)) + L(x, y) + L(y, x)\}$$

where  $R(x, y)z = \langle zxy \rangle$  and  $L(x, y)z = \langle xyz \rangle$ .

Thus for  $U = \text{Mat}(k, n; \Phi)$  of the above Theorem 3.1, by straightforward calculations, we obtain the identity

$$\gamma(x, y) = c_{x,y}(2n + 2 - k)$$

where  $c_{x,y}$  is a constant element in  $\Phi$  with dependent  $x, y \in U$ .

This implies that the trace form  $\gamma(x, y)$  is degenerate if  $m = n + 1$  (the case of  $k = 2m$ ). Thus this fact is related to the degenerate property of the Killing form of the Lie superalgebra  $D(n + 1, n)$ . For the correspondence between the trace form of the anti-Lie triple system  $U \oplus U$  and the trace form (Killing form) of the standard embedding Lie superalgebra  $L(U)$  associated with (-1,-1)-FKTS  $U$  we refer to the author’s previous paper [7] and do not go into detail here.

**Remark 3.2.** For the construction of balanced types of Lie algebras and superalgebras, that is, in the case of  $\dim \mathbf{k} = \dim L_{-2} = \dim L_2 = 1$ , we refer to [1, 4, 13].

**Example 3.1.** For the triple system given in Theorem 3.1, we have the following decomposition. Let  $U = \text{Mat}(k, n; \Phi)$  be the triple system defined by

$$\langle xyz \rangle = z {}^t yx + y {}^t xz - x {}^t yz$$

Here, let  $k \geq n > l$  and we set

$$e := \begin{pmatrix} E_l & 0 \\ 0 & 0 \end{pmatrix} \quad \text{is } (k, n) \text{ matrix, } \quad E_l \text{ is } (l, l) \text{ identity matrix}$$

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A : (l, l), \quad B : (l, n - l), \quad C : (k - l, l), \quad D : (k - l, n - l) \text{ matrix}$$

Then we have

$$\begin{aligned} x &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{A- {}^t A}{2} & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} \frac{A+ {}^t A}{2} & 0 \\ 0 & 0 \end{pmatrix} \\ &\in U_{00} \oplus U_{01} \oplus U_{1,-1} \oplus U_{11} \end{aligned}$$

In particular, for this triple system, let  $k \geq n = l$ , and we set

$$e := \begin{pmatrix} E_l \\ 0 \end{pmatrix}, \quad (k, l) \text{ matrix}$$

Then we have  $eex = x$ , for all  $x \in U$  and  $xee = e {}^t ex + e {}^t xe - x$ . Thus by straightforward calculations, we can obtain the decomposition

$$x = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{A+ {}^t A}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A- {}^t A}{2} \\ B \end{pmatrix} \in U_{11} \oplus U_{1,-1}$$

As the final topic in this section, we shall give several simple examples of Peirce decompositions of triple systems.

In a generalized Jordan triple system equipped with  $xyx = yxy = \langle x|x \rangle y$  and  $\langle x|y \rangle = \langle y|x \rangle$ , that is, this means the balanced property defined in section one, by straightforward calculations, we have the following.

**Proposition 3.1.** *Let  $U$  be a balanced generalized Jordan triple system. Then the decomposition is given by  $U = U_{11} \oplus U_{1,-1}$ , where  $U_{11} = \{x \in U | R(x) = x\}$  and  $U_{1,-1} = \{x \in U | R(x) = -x\}$ .*

**Proof.** Indeed, from  $R^2(x) = x$ , we have

$$x = \frac{x + R(x)}{2} + \frac{x - R(x)}{2}$$

and the proof is verified. □

**Remark 3.3** ([13]). For the standard embedding Lie superalgebra  $L(U)$  associated with a simple balanced  $(-1,-1)$ -FKTS  $U$ , we have the following decomposition. For convenience, we set

$$U_{11} := U^+ = \{x|x = \Phi e\}, \quad U_{1,-1} := U^- = \{x| \langle x|e \rangle = 0\}$$

where  $eee = e, \langle e|e \rangle = 1$ . In this Lie superalgebra,  $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ , and we have

$$\begin{aligned} L_{-1} &= U = U^+ \oplus U^- \\ L_{-2} &= \{K(x, y) | x, y \in U\}_{span} \\ L_0 &= L(U, U) = \{L(x, y) | x, y \in U\}_{span} \\ &= L(U^+, U^+) \oplus L(U^+, U^-) \oplus L(U^-, U^+) \oplus L(U^-, U^-) \end{aligned}$$

and

$$\dim L(U) = 2 + 2 \dim U + \dim L(U, U)$$

For the examples of balanced  $(-1,-1)$ -Freudenthal-Kantor triple systems  $U$  and the standard embedding Lie superalgebras  $L(U)$ , we refer to [1] and [13].

On the other hand, for a quadratic triple system [14], that is, in a triple system equipped with  $xyx = yxx = \langle x|x \rangle y$  and  $\langle x|y \rangle = \langle y|x \rangle$ , we have the following.

**Proposition 3.2.** *Let  $U$  be a quadratic triple system. Then the decomposition is given by  $U = U_{11}^+ \oplus U_{11}^-$ , where  $U_{11}^+ = \{x \in U | exe = x\}$  and  $U_{11}^- = \{x \in U | exe = -x\}$ .*

**Example 3.2.** Let  $U$  be a triple system satisfying  $(xyz) = \langle y|z \rangle x$  and  $\langle x|y \rangle = \langle y|x \rangle$ . Then this triple system  $U$  is a  $(-1, -1)$ -Freudenthal-Kantor triple system, but it is not balanced. Furthermore, we have  $U = U_{11} \oplus U_{01}$ , i.e

$$x = \frac{x + Q_e(x)}{2} + \frac{x - Q_e(x)}{2}$$

where  $Q_e(x) = exe = \langle x|e \rangle e$ , if  $\langle e|e \rangle = 1$ , then  $xee = x \forall x \in U$ .

**Example 3.3** ([8], anti-Jordan triple system). Let  $U$  be a vector space with an anti-symmetric bilinear form such that  $\langle x|y \rangle = - \langle y|x \rangle$ . Then  $U$  is a  $(1,-1)$  Jordan triple system with respect to the triple product

$$xyz = \langle x|y \rangle z + \langle y|z \rangle x - \langle z|x \rangle y$$

This product means an anti-J.T.S.

Furthermore, by means of an element  $e$  such that  $eee = 0$ , we have the decomposition  $U = U_0 \oplus U_1$ , where  $U_0 = \{x | \langle e|x \rangle = 0\}$  and  $U_1 = \{x | x = \Phi e\}$ .

This Lie superalgebra decomposition is  $L(U) = L_{-1} \oplus L_0 \oplus L_1$ , where  $L_{-1} = U$  and  $L_1$  is the copy of  $U$ ,  $L_0 = L(U_i, U_i)_{span}$ .

Also, it holds that  $L(x_i, x_j)x_k \in U_{i+j+k}$ , where  $(i + j + k)$  is mod 2, and  $U_0U_0U_0 = 0$ .

## Acknowledgment

The author would like to thank Dr D. Mondoc for several suggestions in preparation of this paper.

## References

- [1] A. Elduque, N. Kamiya, and S. Okubo. Simple (-1,-1)-balanced Freudenthal-Kantor triple systems. *Glasgow Math. J.* **45** (2003), 353–372.
- [2] A. Elduque, N. Kamiya, and S. Okubo. (-1,-1)-balanced Freudenthal-Kantor triple systems and non commutative Jordan algebras. *J. Algebra*, **294** (2005), 19–40.
- [3] N. Kamiya. A structure theory of Freudenthal-Kantor triple systems. *J. Algebra*, **110**(1987), 108–123.
- [4] N. Kamiya. On examples of Peirce decompositions of eneralize Jordan triple systems of 2nd order – balanced cases. In "Noncommutative Geometry and Representation Theory in Mathematical Physics. Proc. Satellite Con. to 4th ECM, July 5–10, 2004, Karlstad, Sweden", J Fuchs, J. Mickelson, G. Rosenblioum, A. Stolin and A. Westerberg, Eds. *Contemp. Math.*, **391** (2005), 157–166.
- [5] N. Kamiya. A construction of all simple Lie algebras over  $\mathbb{C}$  from balanced Freudenthal-Kantor triple systems. *Contribute to General Algebras*, **7** (1991), 205–213. Holder-Pichler-Tempsky, Vienna.
- [6] N. Kamiya. On Freudenthal-Kantor triple systems and generalized structurable algebras. In "Proc. Nonassociative algebras and its applications", S. González, Ed. *Kluwer Acad. Publ. Math. and its Appl.* **303** (1994), 198–203.
- [7] N. Kamiya. A structure theory of Freudenthal-Kantor triple systems II. *Comment. Math. Univ. St. Paul.*, **38** (1989), 41–60.
- [8] N. Kamiya, On  $(\varepsilon, \delta)$  Freudenthal-Kantor triple systems. In "International symposium on nonassociative algebras and related topics", World Scientific Publ., N.J., 1991, 65–75.
- [9] N. Kamiya. Examples of Peirce decomposition of generalized Jordan triple systems of second order. Balanced classical cases. *RIMS, Kyoto Univ.*, **1503** (2006), 1–9.
- [10] N. Kamiya. A construction of Lie superalgebras  $B(m, n)$  and  $D(m, n)$  from triple systems. *Bulg. J. Phys.*, **33** (2006), 128–139.
- [11] I. Kantor and N. Kamiya, A Peirce decomposition for generalized Jordan triple systems of second order. *Comm. Algebra*, **31** (2003), 5875–5913.
- [12] N. Kamiya and S. Okubo. On  $\delta$  Lie supertriple systems associated with  $(\varepsilon, \delta)$  Freudenthal-Kantor supertriple systems. *Proc. Edinburgh Math. Soc.*, **43** (2000), 243–260.
- [13] N. Kamiya and S. Okubo, Construction of Lie superalgebras  $D(2, 1 : \alpha)$ ,  $G(3)$ , and  $F(4)$  from some triple systems. *Proc. Edinburgh Math. Soc.*, **46** (2003), 87–98.
- [14] N. Kamiya and S.Okubo. On composition, quadratic and some triple systems. *Lect. Notes Pure Appl. Math.*, **246** (2006), 205–231, Chapman & Hall/CRC.
- [15] S. Okubo and N. Kamiya, Quasi-classical Lie superalgebras and Lie supertriple systems. *Comm. Algebra*, **30** (2002), 3825–3850.

*Received October 17, 2007*

*Revised March 02, 2008*