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A Remark on the Hopf invariant for Spherical 4-braids

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Abstract

An approach by J.Wu describes homotopy groups $\pi_n(S^2)$ of the standard 2-sphere as isotopy classes of spherical n+1--strand Brunnian braids. The case n=3 is investigated for applications.

Introduction

An approach by Wu describes homotopy groups $\pi_n(S^2)$ of the standard 2-sphere as isotopy classes of spherical n+1-strand Brunnian braids, for more details, Theorem 1.2. This straightforward approach is not possible for n=3, i.e. for 4-strand braids the connection with $\pi_3(S^2)$ was unknown.

The homotopy group $\pi_3(S^2)$ in an infinite cyclic group, detected by the Hopf invariant

$$H:\pi_3(S^2)\mapsto \mathbb{Z}. \tag{1}$$

An element of $\pi_3(S^2)$ is represented by a mapping $h:S^3\mapsto S^2$, which is considered up to homotopy. The Hopf invariant $H(h)$ is well-defined as the integer linking number of two oriented curves $h^{-1}(a)$, $h^{-1}(b)$, where $a,b\in S^2$ be a pair of regular points of h. The Hopf invariant is very important for applications.

Proposition 7.1.1, sequence (17) gets an exact sequence, which algebraically describes the group $Brunn_4$ of 4-straight Brunnian braids [1]. The key point of our elementary geometrical construction is to construct an alternative epimorphism onto the group $\mathbb{Z}\times\mathbb{Z}$, see Definition. The kernel of this epimorphism is a well-defined subgroup $Brun_4 \subset \mathrm{Br}_4$ of Brunnian braids in a new sense (let us remark that $Brunn_4$ is not a subgroup of $Brun_4$. Define the Hopf invariant as a function of isotopy classes of spherical braids in $Brunn_4$. An idea of the construction was coming from Graham and Roman [2]. However, the results by Ellis and Mikhailov are not adopted for physical applications.

The Hamiltonian provides an elegant method for generating simple geometrical examples of complicated braids and links, as is presented in Mitchell A Berger [3].

The paper is motivated by the following problems:

- Derive applications of higher-order winding numbers to generate turbulent motions of vortices in two dimensions. For a special Hamiltonian motion of 3 vortices on the plane this is done in Mitchell A Berger [3]. (Problem 1).
- To unify the approach Ch.3 to $\pi^*(S^2)$ with the Wu's approach (Problem 2) [4].

Let us clarify Problem 2. Let F be the space of functions $f:\mathbb{R}^1\to\mathbb{R}^1$ with "right" boundary conditions at the infinity. The derivative of the order 1,2, and 3 of a function $f\in F$ can nowhere be vanished simultaneously. Define the mapping, $A:F\to\Omega(\mathbb{R}^3\setminus 0)$, by the formula $A(f)=\{x\mapsto(\frac{df(x)}{dx},\frac{d^2f(x)}{dx^2},\frac{d^3f(x)}{dx^3})\}$.

V.I.Arnold (1996) conjectured that the induced homomorphism A_n: $\pi_n(F)\to\pi_n(\Omega(S^2))\cong\pi_n+1\ (S^2)$ is an isomorphism for $n\geq 0$. This theorem was proved by V.A. Vassiliev in the special case n+2, and by Eliashberg and Mishachev in the general case.

The paper is organized as following. In Section 2 we recall required definitions concerning first-order stage of the construction and determine the linking numbers of spherical 4-component braids. In Section 3 the Hopf invariant for 4-component spherical braids is defined. This is a second-order particular defined invariant: to define this invariant we should assume that the all linking numbers (there are two) of components of a spherical braid are equal to zero. Results are formulated in Theorems 4, 6. The main result is the Corollary 8. In Section 4 we give proofs.

A possible application for turbulences (Problem 1)

Assume a motion of a large collection of n vortexes (or, particles) in a bounded domain U on the plane is investigated. The trajectories of vortexes (or, of particles) in the configuration space, i.e. in the Cartesian product, of the domain and the time, are represented by a braid F, components of the braid F correspond to vortexes in the collection. Assume that the windings numbers of components of the braid F are distributed as in the statement of Corollary. This means that the length of the segment (a,b), which is assumed sufficiently large, is bounded from below; the upper bound depends of the number n of vortexes in the collection. We may replace F by a colored braid, if b-a is sufficiently large, using the Arnol'd collection of the short paths, we have no loss of a generality.

Otherwise, assume that the bound k of the distribution of full angles of windings numbers is much less then the number n of partials. Consider the normalized sum of squares of Hopf invariants

$$\Upsilon=\frac{N}{(b-a)l}\sum_{i=1}^{l}H^2(g_i),\quad N=C_n^4=\frac{n^4}{24}+O(n^3) \tag{2}$$

this sum is taken over all collection of admissible quadruples of components of F, the number l of admissible quadruples could be sufficiently large by Corollary 8. The following statements will be proved, or disproved, elsewhere:

- Υ is the universal constant of the motion, which depends no of the time scale and of the time interval [a, b] itself;
- The constant Υ is large (correspondingly, is small), if the motion of

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the system of vortexes (or, of partials) is turbulent (correspondingly, the system is closed to an integrable system);

- Assume the sum (2) is taken over all admissible quadruples, between which the distance is smaller then L. Then Υ (L) correlates with the spacial turbulent spectra of the motion up to the scale L.

Linking Numbers For Spherical Braids

By a spherical (ordered) n-braid we mean a collection of embeddings of the standard circles

$$f:\bigcup_{i=1}^{n} S_i^1 \subset S^2 \times S^1,$$

where the composition of this embedding with the standard projection $S^2 \times S^1 \to S^1$ on the second factor in the target space, restricted to an arbitrary component S_i^1, i=1,…,n is the identity mapping.

The set of all ordered spherical n-braids up to isotopy is denoted by Br_n. It is well-known that Br_n is a group.

For a fixed value $t \in S^1$, a braid f $\in Br_n$ intersects the level $S^2 \times t$ by an (ordered) collection of n points $\{Z_1(t),\dots Z_n(t)\}$. Let assume that n=4. Denote by

$$g = g(f): S_1^1 \cup S_2^1 \cup S_3^1 \subset S^2 \times S^1,$$

the 3-component braid, obtained from f by eliminating of the last component S_4^1.

Let us identify the sphere S^2 with the Riemann sphere, or with the complex projective line C. For a braid f let us consider the collection of Mobius transformations, which transforms the points z_1, z_2, z_3 into 0, 1, ∞ correspondingly:

$$F(z;t) = \frac{(z-z_1(t))(z_2(t)-z_3(t))}{(z-z_3(t))(z_2(t)-z_1(t))}.$$

The image $F(f)$ is a 4-strand braid with the constant components $\{z_1(t), z_2(t), z_3(t)\}=\{0, 1, \infty\}$. Denote this braid by

$$F(f) = f^{norm}. \tag{3}$$

The 3-strand braid g, constructed from f^{norm} is the constant in the points {0, 1, ∞}. The last component $f^{norm}(S^1)$ of $F(f)$ is represented by a closed path $z_4(t) \in \widehat{\mathbb{C}} \setminus \{0,1,\infty\}, t \in \mathbb{R}^1/2\pi$. Note that, generally speaking, braids f, f^{norm} are not isotopic. Moreover, if f is a Brunnian in the sense [2], f^{norm} is, generally speaking, not a Brunnian.

For a given (ordered) 4-component braid f let us define the linking number $Lk(f)$,

$$Lk: Br_4 \to \mathbb{Z}. \tag{4}$$

Consider the following 1-form

$$\omega_0 = \frac{1}{2\pi i}\frac{dz}{z}. \tag{5}$$

By definition we get

$$d\log(z) = \frac{1}{2\pi i}\frac{dz}{z},$$

where log (z) is given by the formula:

$$\log(z) = (2\pi i)^{-1}\int \frac{dz}{z},$$

assuming that log (1)=0, as a multivalued complex function.

Define $Lk(f)$ by the formula:

$$Lk(f) = \Re\int_0^{2\pi}\frac{dz_4(t)}{z_4(t)} = \int_{f_4^{norm}}\omega_0, \tag{6}$$

where $\Re$ is the real part of the integral. By construction, $Lk(f)$ is the winding number, i.e. the integer number of rotations of the path $z_4(t)$ of f^{norm} with respect to the origin and the infinity in C.

The permutation group $\Sigma(4)$ of the order 24 acts on the space of ordered spherical braids:

$$\Sigma(4) \times Br_4 \to Br_4. \tag{7}$$

The image of an ordered braid f by a transposition $\sigma:(1,2,3,4) \mapsto (\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ is well-defined by the corresponding re-ordering of components of f. Let us investigate the orbit of the linking numbers $Lk(f)$ with respect to (7). Simply say, we investigate how many independent linking numbers of components of braids are well-defined?

Let us consider the following exact sequences of groups:

$$0 \to A_4 \to \Sigma_4 \xrightarrow{sign} \mathbb{Z}/2 \to 0, \tag{8}$$

$$0 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to A_4 \to \mathbb{Z}/3 \to 0. \tag{9}$$

The subgroup $A_4 \subset \Sigma_4$ in the sequence (8) is represented by permutations, which preserve signs (equivalently, which is decomposed into an even number of elementary transpositions). The subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset A_4$ in the sequence (9) is generated by the permutations{(1,2)(3,4);(1,3)(2,4);(1,4)(2,3)}.

Let us consider 2-primary subgroup $K \subset \Sigma_4$ (the dihedral group of the order 8), which is defined as the extension of the subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$ from the sequence (9), which is included in the sequence (8). An epimorphism

$$\theta = (\theta_1, \theta_2): K \to \mathbb{Z}/2 \times \mathbb{Z}/2, \tag{10}$$

is defined as follows: $\theta_1(\sigma)$=1 (the group $Z/2$ is in the multiplicative form), if σ preserves a (non-ordered) partition (1,3)(2,4), and $\theta_1(\sigma)$, and $\theta_1(\sigma)$=-1, otherwise. Therefore θ_1 is an epimorphism with the kernel $\mathbb{Z}/2 \times \mathbb{Z}/2$ from the left subgroup of the sequence (9). The epimorpism $\theta_2(\sigma)$ is determined by the sign of a permutation σ, this is the restriction of the right epimorphism in the sequence (8) to the subgroup $K \subset \Sigma_4$. The kernel $Ker(\theta) \cong \mathbb{Z}/2$ is the center of the dihedral group K.

Lemma 1

1. The function (4) is invariant with respect to the action (7) by an arbitrary permutation, which in the kernel of θ in (10).

2. The function (4) is skew-invariant with respect to the action by a permutation, which is in the kernel of θ_1 (the composition of θ with the projection on the first factor, but not in the kernel of θ_2 (the composition of θ with the projection on the second factor).

3. Denote by $\tilde{f} \in Br_4$ the ordered braid, which is obtained from $f \in Br_4$ by the action (7) by the element (1,2) is the product of the generators of the factors). There exists an ordered braids $f \in Br_4$, for which the linking numbers $Lk(f)$, $Lk(\tilde{f})$ are arbitrary integers.

From Lemma one may deduce the following corollary.

Corollary 2

1. For an arbitrary braid $f \subset Br_4$ the linking number $Lk(f)$, is well-defined as the differences of the winding number of the component 2 between the components 1 and 3 with the winding number of the component 4 between the components 1 and 3.

2. For a braid , where $f \in Br_4$ is an arbitrary, $\tilde{f}$ is defined in Lemma

1, the linking number $Lk(\tilde{f})$ is well-defined as the difference of the winding number of the component 2 between the components 1 and 3 with the winding number of the component 4 between the components 2 and 3.

Corollary (2) motivates the following definition.

Definition 3

Let $f \in Br_4$ *be a (ordered) spherical braid. Define the total linking number* $LK(f) \in \mathbb{Z} \oplus \mathbb{Z}$ by the following formula:

The total linking number is a well-defined homomorphism

$LK : Br_4 \mapsto \mathbb{Z} \oplus \mathbb{Z}.$

Hopf Invariant of Braids

Let $f \in Br_4$ be a (ordered) spherical braid with the trivial total linking number: $Lk(f)=0$. Such braids generate the subgroup in the group Br_4, denote this subgroup by $Brunn_4 \subset Br_4$. Let us remark that this subgroup does not coincide with the subgroup of Brunnian braids $Brun_4$, defined in Berrick et al. [1], Theorem 1.2.

Theorem 4

There exists a well-defined homomorphism

$$H : Brunn_4 \to Z, \tag{11}$$

called the Hopf invariant. The homomorphism (11) is invariant with respect to the action (7) by an arbitrary permutation, which in the kernel of θ_2 in (10) (this homomorphism is defined as the sign of a permutation of straights), and is skew-invariant with respect to the action by a permutation, which is not in the kernel of θ_2.

Definition of the hopf invariant

In this section we present the construction, which is closed to Theorem 3 of Mitchell A Berger [3], using differential topology instead of homology algebra. Let $f \in Brunn_4$ be an arbitrary. Consider the braid f^{norm}, given by Mitchell A Berger [3]. Recall, the braid $g \in Br_3$, which consists of the straits (1-3) of f^{norm}, is the constant braid at the points 0, 1, ∞ in correspondingly. Consider the strait (4) of the braid f^{norm}. This strait is represented by an oriented closed path $z_4 : S^1 \to \hat{C} \setminus \{0 \cup 1 \cup \infty\}$. This path determines a cycle, which is an oriented boundary, because of the condition $LK(f^{norm})=0$.

Let us prove that $Lk(f)=0$. Denote the group of Mobius transformations by M. The standard inclusion $SO(3) \subset M$ is well-defined. This inclusion is a homotopy equivalence, therefore we get $\pi_1(M) = \pi_1(SO(3) = \mathbb{Z}/2$. This proves that $LK(2f^{norm})=LK(2f)$. Because $LK(2f^{norm})=2LK(2f^{norm})$, $LK(2f)=2Lk(f)$, we get $LK(f^{norm})=Lk(f)$. The equality $Lk(f)=0$ is proved.

Consider the inclusions

$I_0 : C \setminus \{0 \cup 1 \cup \infty\} \subset C \setminus \{1 \cup \infty\},$

$I_\infty : C \setminus \{0 \cup 1 \cup \infty\} \subset C \setminus \{0 \cup 1\},$

$I_1 : C \setminus \{0 \cup 1 \cup \infty\} \subset C \setminus \{0 \cup \infty\}.$

Because $H_1(\hat{\mathbb{C}} \setminus \{1 \cup \infty\}; \mathbb{Z}) = \pi_1(\hat{\mathbb{C}} \setminus \{1 \cup \infty\}) = \mathbb{Z}$, the condition $LK(f^{norm})=0$ implies $I_0,\#([i])=0$, for the homomorphism

$I_{0,\#} : \pi_1(C \setminus \{0 \cup 1 \cup \infty\}) \to \pi_1(C \setminus \{0 \cup \infty\}).$

Analogously $I_{\infty,\#}([i])=0$, $I_1,\#([i])=0$.

There exist the following 3 maps of the standard 2-disk

$$e_0 : D_0^2 \to \hat{C} \setminus \{1 \cup \infty\}, \quad e_0|_{\partial D^2} = z_4,$$

$$e_\infty : D_\infty^2 \to \hat{C} \setminus \{0 \cup 1\}, \quad e_\infty|_{\partial D^2} = z_4,$$

$$e_1 : D_1^2 \to \hat{C} \setminus \{0 \cup \infty\}, \quad e_1|_{\partial D^2} = z_4.$$

Consider a 2-sphere, which is represented by a gluing $D_0^2 \cup_\partial D_\infty^2$ of the disks $D_0^2 \cup D_\infty^2$ along the common boundary, which is identified with the circle S_4^1. Denote this sphere by S_1^2. Analogously define spheres $S_0^2 = D_\infty^2 \cup_\partial D_1^2$, $S_\infty^2 = D_1^2 \cup_\partial D_0^2$. Because the target spaces of the mappings e_0, e_∞, e_1 are aspherical, the corresponding mapping is well-defined up to homotopy.

Consider the following commutative diagram of inclusions:

$$\begin{array}{ccc} C \setminus \{0 \cup \infty \cup 1\} & \subset & C \setminus \{0 \cup \infty\} \\ \cap & & \cap \\ C \setminus \{\infty \cup 1\} & \subset & C \setminus \{\infty\} \end{array} \tag{12}$$

Consider the mappings $e_0 : D_0^2 \to C \setminus \{1 \cup \infty\}$, $e_1 : D_1^2 \to C \setminus \{0 \cup \infty\}$ to the left bottom and to the right upper spaces of the diagram (12) correspondingly. The mapping $e_0 \cup_\partial e_1 : S_\infty^2 \to C \setminus \{\infty\}$ is well defined by gluing of the two mappings e_0, e_1 along the common mapping i of the boundaries. Consider the standard 3-ball D_∞^3 (with corners along the curve S_4^1) with the boundary $\partial D_\infty^3 = S_\infty^2$. The mapping $e_0 \cup \partial\, e_1$ can be extended to the mapping

$$d_\infty : D_\infty^3 \to \hat{C} \setminus \{\infty\}. \tag{13}$$

The target space of this mapping is the right bottom space of the diagram (12). Because the target space of the mapping d_∞ is contractible, the mapping d_∞ is well-defined up to homotopy. By the analogous constructions the following mappings

$$d_1 : D_1^3 \to \hat{\mathbb{C}} \setminus \{1\}, \tag{14}$$

$$d_0 : D_0^3 \to \hat{\mathbb{C}} \setminus \{0\} \tag{15}$$

are well-defined.

The mappings (13), (14), (15) determine the mapping

$$h = h(f) : S^3 \to S^2 \tag{16}$$

as follows. Take a 3-sphere S^3, which is catted into 3 balls D_∞^3, D_1^3, D_0^3 along the common circle $S_4^1 \subset S^3$. The sphere S^3 is represented as the join $S_4^1 * S_a^1$ of the two standard circle. On the circle S_a^1 take 3 points $x_0, x_1, x_\infty \in S_a^1$. The subsets $S_4^1 * [x_0, x_1] \subset S^3, S_4^1 * [x_1, x_\infty] \subset S^3, S_4^1 * [x_\infty, x_0] \subset S^3$ are 3 copies of 3D disks, which are glued along corresponding subdomains in its boundaries.

Let us identify $D_\infty^3 \cong S_4^1 * [x_0, x_1], D_0^3 \cong S_4^1 * [x_1, x_\infty], D_1^3 \cong S_4^1 * [x_\infty, x_0]$. The boundary ∂D_∞^3 is identified with the balls $S_4^1 * \{0\} \cong D_0^2$, $S_4^1 * \{1\} \cong D_1^2$, which are glued along the common boundary S_4^1. The boundary ∂D_0^3 is identified with the balls $S_4^1 * \{1\} \cong D_1^2$, $S_4^1 * \{\infty\} \cong D_\infty^2$, which are identify along the common boundary S_4^1. The boundary ∂D_1^3 is identified with the balls $S_4^1 * \{\infty\} \cong D_\infty^2$, $S_4^1 * \{0\} \cong D_0^2$, which are identified along the same boundary S_4^1. The mappings d_0, d_1, d_∞ on the corresponded balls are well-defined by the formulas (13-15) correspondingly. This mappings define the mapping (16) on the 3-sphere.

Definition 5: The Hopf invarian $H(f)$ for a braid $f \in Brunn_4$ in the formula (11) is defined as the Hopf invariant of the mapping h by the formula (1). The mapping $h=h(f)$ is explicitly defined from the braid f by the formula (16).

A formula to calculate the Hopf invariant

Let us introduces an explicit formula to calculate the Hopf invariant for a braid f $\in$ $Brunn_4$. Consider the complex plane C. The 4-th strain of the braid f^{norm} determines a curve on the plane without two points {0.1}, which is denoted by

$$\gamma : S^1 \to \mathbb{C} \setminus \{0 \cup 1\}. \tag{16}$$

Let us consider the closed 1-form (4). Define a complex 1-form

$$\omega_1 = \frac{1}{2\pi i}\frac{dz}{z-1}. \tag{17}$$

Define a real (multivalued) function λ_0 by integration along the path γ (*t*), $t \in [0,t] \subset S^1$ of the real part of the form (18) as following:

$$\lambda_0(t) = \Re\int_0^t \omega_0. \tag{18}$$

Define a real (multivalued) function λ_1 by integration along the path of the real part of the form (17) as following:

$$\lambda_1(t) = \Re\int_0^t \omega_1. \tag{19}$$

To take the multivalued functions (18), (19) well-defined, assume that the path γ starts at the point $2 \in \mathbb{C} : \lambda_0(0)=2, \lambda_1(0)=2$.

Define a closed 1-form Ψ (*t*) along a curve $\gamma(t) \in \mathbb{C} \setminus \{0 \cup 1\}$ by the following formula:

$$\psi(t) = \lambda_0(t)\omega_1 + \lambda_1(t)\omega_0. \tag{20}$$

Let us consider a function, which is well-defined as the real part of the integral

$$\Psi(T) = \Re\int_0^T \psi(t)d\gamma, \quad t \in [0,T] \subset S^1, \Psi(0) = 0. \tag{21}$$

Theorem 6

The Hopf invariant of a braid f $\in$ $Brunn_4$ in the formula (11), which is defined by Definition 5, is calculated by the formula:

$$-H(f) = \Psi(2\pi) = \frac{1}{2}\Re\int_0^{2\pi}\psi(t)d\gamma, \tag{22}$$

where γ is the closed path, determined by the 4-th straight of the braid f^{norm} by the formula (17).

From Theorem 6 we get a corollary.

Corollary 7

1. The Hopf invariant (11) is an epimorphism.

2. Assume there is a braid $f \in Brunn_4$ for which the braid f^{norm} is represented by a commutator of the straight (4) with straights (1) and (2) (such a braid is called the Borromean rings). Then $H(f)=\pm 1$, where the sign in the formula depends on the sign of the commutator.

Proof of corollary

It is sufficient to prove --2. The right-hand side of the formula (21) coincides with the formula (28) [1], which is simplified for the considered example. The Berger's formula is applied for the 3-uple configuration space, this gives the opposite sign for the last term in the formula (21) with respect to the origin formula. For the Borromean ring the formula (22) is non-trivial. The right side of the formula gives $H(f)=1$ for the right Borromean rings. Corollary is proved.

The following Corollary is the main result of the paper. The author hope that this result is the initial step toward the solution of the first problem, mentioned in Introduction.

Corollary 8

Assume we have a classical -conponent colored non-ordered braid F, n>>4, for which all pairwise winding (integer) numbers of components are distributed to the segment: $\{-2k\pi, \dots -2\pi, 0, 2\pi, \dots 2k\pi\}$, *0<k<<n. Let G is the spherical braid, which is defined as the image of F by the stereography projection* $\mathbb{R}^2 \times I \to S^2 \times I$. *Then there exist at least* $K = \dfrac{n^4}{24(2k+1)^2} + O(n^3) \quad (K(n) \to +\infty \text{ when } n \to +\infty)$ *4-component subbraids fi ⊂ F, for which LK(g_i)=0, $g_i \subset$ G,. In particular, the squares* $H^2(g_i) \in \mathbb{N}$ *are well-defined.*

Proof of corollary

Proof is evident: the number *K* of subbraids $g_i \subset F$ with trivial total linking number $LK(g_i)=0$ is explicitly estimated from below using integers *k*, *n*.

Proofs

Proof of lemma

Proof of Statement 1. Take an oriented 3--manifold M^3. Take two disjoin oriented cycles $C_I \subset M^3$, $C_{II} \subset M_3$, which represent the trivial homology class

$$0 = [C_I] = [C_{II}] \in H_1(M^3; Z). \tag{23}$$

The linking number *link* (L_I, L_{II}) is a well-defined integer the algebraic intersection coefficient of the boundary $(\Gamma_I, \partial\Gamma_I) \subset M^3$, $\partial\Gamma_I = C_I$. The linking number *link* (C_I, C_{II}) is well defined, because of the condition.

Obviously, *link* (C_I, C_{II})=*link* (C_{II}, C_I), because the collections of signed points $A_I = \Gamma_I \cap C_{II}$ and $A_{II} = C_I \cap \Gamma_{II}$ represent the same cycle $[A_I]=[A_{II}] \in H_0(M^3; Z)$. The boundary of $-[A_I] \cup [A_{II}]$ is given by the oriented curve $\Gamma_I \cap \Gamma_{II}$.

Take $M^3=S^2 \times S^1$. Take an arbitrary braid f^{norm}. The cycle C_I is represented by the images of the following two closed paths $[z_1(t)]=0 \times -S^1$, $[z_3(t)]=\infty \times S^1$, $t \in [0,2\pi]$, where the path $z_1(t)$ is taken with the opposite orientation along S^1. The cycle C_{II} is represented by the two closed paths $[z_2(t)]=0 \times -S^1$, [z4 (t)] $\subset S^2 \times S^1$, where the path $z_2(t)$ is taken with the opposite orientation along S^1.

Take $\sigma_\alpha=(1,2)(3,4)$, σ_α is the generator of Ker(θ). It is easy to see that $Lk(f^{norm})=link(C_I, C_{II})$, $Lk(\sigma_\alpha \times f^{norm})=link(C_{II}, C_I)$. Statement 1 is proved.

Proof of Statement 2. Assume $\sigma_b=(1,3)$, the case $\sigma_b=(2,4)$ is analogous. Then $Lk(f^{norm})$=link(C_I, C_{II}), Lk(σb × f^{norm})=*link*($-C_I$, C_{II}). Therefore we get $Lk(f^{norm})=-Lk$(σb × f^{norm}). Statement 2 is proved.

Lemma 1 is proved.

Proof of Corollary 2

Statements 1,2 are obvious.

Proof of Statement 3. The straights $\{z_1(t), z_2(t), z_3(t), z_4(t)\}$ determines 6 pairs of -cycles in $S^2 \times S^1$. A function of winding numbers of component is given by a linear combinations of linking numbers between the corresponding pairs of cycles. To prove that such a function is well-defined, we have to assume that the each cycle is a boundary. Denote the cycle, generated by the pair of paths $-z_i$ (*t*), z_j (*t*) by $C_{i,j}$. We have the following identity: link ($C_{1,2}$, $C_{3,4}$)+*link* ($C_{2,3}$, $C_{1,4}$)+link ($C_{3,1}$, $C_{2,4}$)=0, and the analogous 3 identities, which are obtained by the permutation of the indexes. Therefore we get a collection of 2 independent well-defined linking numbers. Statement is proved. Corollary 2 is proved.

Proof of theorem

Let us prove that the homomorphism (11) is skew-invariant with respect to the action (7) by an odd permutation. Assume that the permutation σ is given by an elementary transposition of straights with number (1-3) say by the transposition σ=(1,2). Then by the formula (16), the mappings $h(f)$ is related with the mapping $h(\sigma\times f)$ by the composition with the reflection $S^3 \to S^3$, which translates the curve $S^1_4 \subset S^3$ to itself, and permutes the points x_0, x_1 on the circle S^1_a. The reflection changes the homotopy class of h to the opposite. This proves Theorem in this case.

Assume that σ=(1, 4) (the cases σ=(2, 4), or (3,4) are analogous). Then we may calculate the Hopf invariants of the mappings $h(f)$ and $h(\sigma\times f)$, using the formula (22) (Theorem 6 is proved below). The Mobius group is locally contractible. Therefore, the ordered braid $(\sigma\times f)$ is isotope to the braid f in which the components (1,4) are re-numbered. The restriction of the considered isotopy on the common straight at $\infty \in C$ is the identity.

By Statement 2 of Corollary 7, the Hopf invariant $h(\sigma\times f^{norm})$ is defined as the length of commutators of the straight (1) with straights (4) and (2).

The Hopf invariant for $h(f)$ coincides with the commutator of the straight (4) with the straights (1,4). Therefore the Hopf invariant for $h(\sigma\times f)$ is opposite to the Hopf invariant for $h(f)$, because the sign of the commutator is changed by a permutation of components.

Theorem 4 is proved.

Proof of Theorem 6

Consider the mapping $h: S^3 \to S^2 = \hat{C}$, which is defined by the formula (16). Take two normalized volume forms $\Omega_0, \Omega_1 \in \Lambda^2(S^2)$:

$$\iint_{S^2}\Omega_0 = \iint_{S^2}\Omega_1 = 1.$$

The forms Ω_0, Ω_1 are defined as the standard ill-supported forms at the points 0, 1, correspondingly. The Hopf invariant (11) is calculated by the formula:

$$H(f) = \frac{1}{2}\iiint_{S^3} h^*(\Omega_0)\wedge\beta_1 + h^*(\Omega_1)\wedge\beta_0, \qquad (24)$$

where $x \in S^3$, $h^*(\Omega_0) \in \Lambda^2(S^3)$ is the pull-back of $\Omega_0 \in \Lambda^2(S^2)$ by h: $S^3 \to S^2$, $\beta_0 \in \Lambda^1(S^3)$ is an arbitrary 1-form, such that $d(\beta_0)=h^*(\Omega_0)$, the 1-form $\beta_1 \in \Lambda^1(S^3)$ is defined analogously to β_0.

Evidently, the 1-forms β_0 in the integral (24) is represented in its cohomology class by a cocycle, which satisfies the condition $h^*(\Omega_0)=d\beta_0=0$ inside the ball D^3_0. This follows from the fact that the curve $h^{-1}(0)$ is outside the ball D^3_0. In the formula (24) the first term is well-defined up to gauge transformation $\beta_0 \mapsto \beta_0 + grad\varphi_0$. We may put β_0=0 in D^3_0, and keep β_0 on $D^2_1 = D^3_0 \cap D^3_\infty$.

Analogously, $d\beta_1$=0 in the ball D^3_1. In the second term in the integral (24), using $\beta_1 \mapsto \beta_1 + grad\varphi_1$, we get β_1=0 in D^3_1, and keep β_1 on D^2_0. Then we get the following simplification of (24):

$$H(f) = \frac{1}{2}\iiint_{D^3_\infty} h^*(\Omega_0)\wedge\beta_1 + h^*(\Omega_1)\wedge\beta_0.$$

In the ball D^3_∞ the 3-form $h^*(\Omega_0)\wedge\beta_1$ is exact, we get $\alpha_1 \in \Lambda^2(D^3_\infty)$, $d\alpha_1 = h^*(\Omega_0)\wedge\beta_1$. Moreover, we may put $\alpha_1 = \beta_0\wedge\beta_1$ over $D^2_1 = D^3_\infty \cap D^3_0$, $\alpha_0 = 0$ over $D^2_0 = D^3_\infty \cap D^3_1$.

In the ball D^3_∞ the 3-form $h^*(\Omega_1)\wedge\beta_0$ is exact, we get $\alpha_0 \in \Lambda^2(D^3_\infty)$, $d\alpha_0 = h^*(\Omega_1)\wedge\beta_0$. We may put $\alpha_0 = \beta_1\wedge\beta_0 = -\beta_0\wedge\beta_1$ over D^2_0, and α_0=0 over D^2_1.

Apply the 3D Gauss-Ostrogradsky formula, we get

$$\iiint_{D^3_\infty} h^*(\Omega_0)\wedge\beta_1 = \iint_{D^2_1}\beta_0\wedge\beta_1,$$

$$\iiint_{D^3_\infty} h^*\Omega_1\wedge\beta_0 = -\iint_{D^2_0}\beta_0\wedge\beta_1.$$

The 2-form $\beta_0\wedge\beta_1 \in \Lambda^2(D^2_1)$ is exact. Because in the disk D^2_1 the 0-form λ_1 is well defined, and $d\lambda_1=\beta_1$, we get: $d(\lambda_1\beta_0) = -\beta_0\wedge\beta_1$.

Analogously, the 2-form $\beta_0\wedge\beta_1 \in \Lambda^2(D^2_0)$ is exact. Because in the disk D^2_0 the 0-form λ_0 is well defined, and $d\lambda_0=\beta_0$, we get: $d(\lambda_0\beta_1) = \beta_0\wedge\beta_1$.

Apply the 2D Green formula (singular points of β_0, β_1 give no contribution to the integral over the boundary) we get:

$$\iint_{D^2_1}\beta_0\wedge\beta_1 = -\int_\gamma \lambda_1\beta_0,$$

$$\iint_{D^2_0}\beta_0\wedge\beta_1 = \int_\gamma \lambda_0\beta_1,$$

where $\gamma = D^2_1 \cap D^2_0$.

The integral (24) is simplified as

$$H(f) = -\frac{1}{2}\int_\gamma \lambda_0\beta_1 + \lambda_1\beta_0.$$

This formula coincides with the formula (22). Theorem 6 is proved.

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