An algebraic approach to the center problem for ODEs

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Abstract

The classical Poincaré Center-Focus problem asks about the characterization of planar polynomial vector fields such that all their integral trajectories are closed curves whose interiors contain a fixed point, a center. This problem is reduced to a center problem for certain ODE. We present an algebraic approach to the center problem based on the study of the group of paths determined by the coefficients of the ODE.

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1 Introduction

We describe an algebraic approach to the center problem for the ordinary differential equation

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)v^{i+1}, \quad x \in I_T := [0, T]$$

(1.1)

with coefficients $a_i$ from the Banach space $L^\infty(I_T)$ of bounded measurable complex-valued functions on $I_T$ equipped with the supremum norm. Condition $\sup_{x \in I_T, i \in \mathbb{N}} \sqrt{|a_i(x)|} < \infty$ guarantees that (1.1) has Lipschitz solutions on $I_T$ for all sufficiently small initial values. By $X$ we denote the complex Fréchet space of sequences $a = (a_1, a_2, \ldots)$ satisfying this condition. We say that equation (1.1) determines a center if every solution $v$ of (1.1) with a sufficiently small initial value satisfies $v(T) = v(0)$. By $C \subset X$ we denote the set of centers of (1.1). The center problem is: given $a \in X$ to determine whether $a \in C$. It arises naturally in the framework of the geometric theory of ordinary differential equations created by Poincaré. In particular, there is a relation between the center problem for (1.1) and the classical Poincaré Center-Focus problem for planar polynomial vector fields

$$\frac{dx}{dt} = -y + F(x, y), \quad \frac{dy}{dt} = x + G(x, y)$$

(1.2)

where $F$ and $G$ are polynomials of a given degree without constant and linear terms. This problem asks about conditions on $F$ and $G$ under which all trajectories of (1.2) situated in a small neighbourhood of $0 \in \mathbb{R}^2$ are closed. Passing to polar coordinates $(x, y) = (r \cos \phi, r \sin \phi)$ in (1.2) and expanding the right-hand side of the resulting equation as a series in $r$ (for $F, G$ with sufficiently small coefficients) we obtain an equation of the form (1.1) whose coefficients are trigonometric polynomials depending polynomially on the coefficients of (1.2). This reduces the Center-Focus Problem for (1.2) to the center problem for (1.1) with coefficients depending polynomially on a parameter.

2 Group of paths

One of the main objects of our approach is a metrizable topological group $G(X)$ determined by the coefficients of equations (1.1) (the, so-called, group of paths in $C^\infty$). It is defined as follows.

Let us consider $X$ as a semigroup with the operations given for $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ by

$$a \ast b = (a_1 \ast b_1, a_2 \ast b_2, \ldots) \in X \quad \text{and} \quad a^{-1} = (a_1^{-1}, a_2^{-1}, \ldots) \in X$$

where for $i \in \mathbb{N}$

$$(a_i \ast b_i)(x) = \begin{cases} 2b_i(2x) & \text{if } 0 \leq x \leq T/2, \\ 2a_i(2x - T) & \text{if } T/2 < x \leq T \end{cases} \quad \text{and} \quad a_i^{-1}(x) = -a_i(T - x), \quad 0 \leq x \leq T$$

Let $C^\infty$ be the vector space of sequences of complex numbers $(c_1, c_2, \ldots)$ equipped with the product topology. For $a = (a_1, a_2, \ldots) \in X$ by $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots) : I_I \rightarrow C^\infty$, $\tilde{a}_k(x) := \int_0^x a_k(t) \, dt$ for all $k \in \mathbb{N}$, we denote a path in $C^\infty$ starting at 0. The one-to-one map $a \mapsto \tilde{a}$ sends the product $a \ast b$ to the product of paths $\tilde{a} \circ \tilde{b}$, that is, the path obtained by translating $\tilde{a}$ so that its beginning meets the end of $\tilde{b}$ and then forming the composite path. Similarly, $a^{-1}$ is the path obtained by translating $\tilde{a}$ so that its end meets 0 and then taking it with the opposite orientation.

For $a \in X$ consider the basic iterated integrals

$$I_{i_1, \ldots, i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \ldots \leq s_k \leq T} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1 \quad (2.1)$$

By the Ree shuffle formula the linear space generated by all such functions on $X$ is an algebra. For $a, b \in X$ we write $a \sim b$ if all basic iterated integrals vanish at $a \ast b^{-1}$. Then $a \sim b$ if and only if $I_{i_1, \ldots, i_k}(a) = I_{i_1, \ldots, i_k}(b)$ for all basic iterated integrals, see [1]. In particular, $\sim$ is an equivalence relation on $X$. By $G(X)$ we denote the set of equivalence classes. Then $G(X)$ is a group with the product induced by the product $\ast$ on $X$. By $\pi : X \rightarrow G(X)$ we denote the map determined by the equivalence relation. By the definition each iterated integral $I_i$ is constant on fibres of $\pi$ and therefore it determines a function $\hat{I}_i$ on $G(X)$ such that $I_i = \hat{I}_i \circ \pi$. The functions $\hat{I}_i$ are referred to as iterated integrals on $G(X)$. These functions separate the points on $G(X)$.

Next, we equip $G(X)$ with the weakest topology $\tau$ in which all basic iterated integrals $\hat{I}_{i_1,\ldots,i_k}$ are continuous. Then $(G(X), \tau)$ is a topological group. Moreover, $G(X)$ is metrizable, contractible, residually torsion free nilpotent (i.e., finite dimensional unipotent representations of $G(X)$ separate the points on $G(X)$) and is the union of an increasing sequence of compact subsets, see [2].

By $G_f(X)$ we denote the completion of $G(X)$ with respect to the metric $d$. Then $G_f(X)$ is a topological group which is called the group of formal paths in $C^\infty$.

3 Representation of paths by noncommutative power series

Let $C \langle X_1, X_2, \ldots \rangle$ be the associative algebra with unit $I$ of complex noncommutative polynomials in $I$ and free noncommutative variables $X_1, X_2, \ldots$ (i.e., there are no nontrivial relations between these variables). By $C \langle X_1, X_2, \ldots \rangle[[t]]$ we denote the associative algebra of formal power series in $t$ with coefficients from $C \langle X_1, X_2, \ldots \rangle$. Let $S \subset C \langle X_1, X_2, \ldots \rangle$ be the multiplicative semigroup generated by $I, X_1, X_2, \ldots$. Consider a grading function $w : S \rightarrow \mathbb{Z}_+$ determined by the conditions

$$w(I) = 0, \quad w(X_i) = i \quad (i \in \mathbb{N}) \quad \text{and} \quad w(x \cdot y) := w(x) + w(y), \quad \forall x, y \in S$$
This splits $S$ in a disjoint union $S = \bigsqcup_{n=0}^{\infty} S_n$, where $S_n = \{ s \in S : w(s) = n \}$. By $\mathcal{A} \subset \mathbb{C} \langle X_1, X_2, \ldots \rangle[[t]]$ we denote the subalgebra of series $f$ of the form

$$f = \sum_{n=0}^{\infty} f_n t^n \quad \text{where} \quad f_n \in V_n : = \text{span}_\mathbb{C}(S_n), \quad n \in \mathbb{Z}_+$$

(3.1)

We equip $\mathcal{A}$ with the weakest topology in which all coefficients in (3.1) considered as functions in $f \in \mathcal{A}$ are continuous. Since the set of these functions is countable, $\mathcal{A}$ is metrizable. Moreover, if $d$ is a metric on $\mathcal{A}$ compatible with the topology, then $(\mathcal{A}, d)$ is a complete metric space. Also, by the definition the multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous in this topology.

By $G \subset \mathcal{A}$ we denote the closed subset of elements $f$ of form (3.1) with $f_0 = I$. Then $(G, \cdot)$ is a topological group. Its Lie algebra $L_G \subset \mathcal{A}$ consists of elements $f$ of form (3.1) with $f_0 = 0$.

(For $f, g \in L_G$ their product is defined by the formula $[f, g] := f \cdot g - g \cdot f$.) Also, the map $\exp : L_G \to G$, $\exp(f) := e^f = \sum_{n=0}^{\infty} f^n/n!$, is a homeomorphism.

Further, for an element $a = (a_1, a_2, \ldots) \in X$ consider the equation

$$F'(x) = \left( \sum_{i=1}^{\infty} a_i(x) t^i X_i \right) F(x), \quad x \in I_T$$

(3.2)

This can be solved by Picard iteration to obtain a solution $F_n : I_T \to G$, $F_n(0) = I$, whose coefficients in expansion in $X_1, X_2, \ldots$ and $t$ are Lipschitz functions on $I_T$. We set

$$E(a) := F_n(T), \quad a \in X$$

(3.3)

By the definition we have

$$E(a \ast b) = E(a) \cdot E(b), \quad a, b \in X$$

(3.4)

Also, an explicit calculation leads to the formula

$$E(a) = I + \sum_{n=1}^{\infty} \sum_{i_1 + \cdots + i_k = n} I_{i_1, \ldots, i_k} (a) X_{i_1} \cdots X_{i_k} t^n$$

(3.5)

From the last formula one obtains that there is a homomorphism $\widehat{E} : G(X) \to G$ such that $E = \widehat{E} \circ \pi$, that is,

$$\widehat{E}(g) = I + \sum_{n=1}^{\infty} \sum_{i_1 + \cdots + i_k = n} \hat{I}_{i_1, \ldots, i_k} (g) X_{i_1} \cdots X_{i_k} t^n, \quad g \in G(X)$$

(3.6)

Formula (3.6) shows that $\widehat{E} : G(X) \to G$ is a continuous embedding. Moreover, one can determine a metric $d_1$ on $\mathcal{A}$ compatible with topology such that $\widehat{E} : (G(X), d) \to (G, d_1)$ is an isometric embedding. Therefore $\widehat{E}$ is naturally extended to a continuous embedding $G_f(X) \to G$ (denoted also by $\widehat{E}$). By definition, $\widehat{E} : G_f(X) \to G$ is an injective homomorphism of topological groups and $\widehat{E}(G_f(X))$ is the closure of $\widehat{E}(G(X))$ in the topology of $G$.

In what follows we identify $G(X)$ and $G_f(X)$ with their images under $\widehat{E}$.

4 Lie algebra of the group of formal paths

Recall that each element $g \in L_G$ can be written as $g = \sum_{n=1}^{\infty} g_n t^n$, $g_n \in V_n$, $n \in \mathbb{N}$. We say that $g$ is a Lie element if each $g_n$ belongs to the free Lie algebra generated by $X_1, \ldots, X_n$. In this case each $g_n$ has the form

$$g_n = \sum_{i_1 + \cdots + i_k = n} c_{i_1, \ldots, i_k} [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \cdots]]$$

(4.1)
with all \(c_{i_1,\ldots,i_k} \in \mathbb{C}\). (Here the term with \(i_k = n\) is \(c_n x_n\).)

Let \(L_n \subset V_n\) be the subspace of elements \(g_n\) of form (4.1). Then

\[
\dim_{\mathbb{C}} L_n = \frac{1}{n} \sum_{d|n} (2^{n/d} - 1) \cdot \mu(d)
\]

(4.2)

where the sum is taken over all numbers \(d \in \mathbb{N}\) that divide \(n\), and \(\mu : \mathbb{N} \to \{-1, 0, 1\}\) is the Möbius function.

By \(L_{\text{Lie}}\) we denote the subset of Lie elements of \(L_G\). Then \(L_{\text{Lie}}\) is a closed (in the topology of \(A\)) Lie subalgebra of \(L_G\). The following result was proved in [3].

**Theorem 4.1.** The exponential map \(\exp : L_G \to G\) maps \(L_{\text{Lie}}\) homeomorphically onto \(G f(X)\).

Thus \(L_{\text{Lie}}\) can be regarded as the Lie algebra of \(G f(X)\).

### 5 Center Problem for ODEs

Let \(\mathbb{C}[[z]]\) be the algebra of formal complex power series in \(z\). By \(D, L : \mathbb{C}[[z]] \to \mathbb{C}[[z]]\) we denote the differentiation and the left translation operators defined on \(f(z) = \sum_{k=0}^{\infty} c_k z^k\) by

\[
(Df)(z) := \sum_{k=0}^{\infty} (k+1)c_{k+1} z^k, \quad (Lf)(z) := \sum_{k=0}^{\infty} c_{k+1} z^k
\]

(5.1)

Let \(\mathcal{A}(D, L)\) be the associative algebra with unit \(I\) of complex polynomials in \(I, D\) and \(L\). By \(\mathcal{A}(D, L)[[t]]\) we denote the associative algebra of formal power series in \(t\) with coefficients from \(\mathcal{A}(D, L)\). Also, by \(G_0(D, L)[[t]]\) we denote the group of invertible elements of \(\mathcal{A}(D, L)[[t]]\) consisting of elements whose expansions in \(t\) begin with \(I\).

Further, consider equation (1.1) corresponding to an \(a = (a_1, a_2, \ldots) \in X:\)

\[
\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)v^{i+1}, \quad x \in I_T
\]

(5.2)

Using a linearization of (5.2) we associate to this equation the following system of ODEs:

\[
H'(x) = \left( \sum_{i=1}^{\infty} a_i(x)DL^{i-1}t \right) H(x), \quad x \in I_T
\]

(5.3)

Solving (5.3) by Picard iteration we obtain a solution \(H_a : I_T \to G_0(D, L)[[t]]\), \(H_a(0) = I\), whose coefficients in the series expansion in \(D, L\) and \(t\) are Lipschitz functions on \(I_T\). It was established in [1] that (5.2) determines a center (i.e., \(a \in C\)) if and only if \(H_a(T) = I\). This implies the following result, see [3].

**Theorem 5.1.** We have

\[
a \in C \iff \sum_{i_1+\cdots+i_k = i} \sum_{k=1}^{\infty} p_{i_1,\ldots,i_k}(a) \equiv 0, \quad \forall i \in \mathbb{N}
\]

(5.4)

where \(p_{i_1,\ldots,i_k}\) is a complex polynomial of degree \(k\) defined by the formula

\[
p_{i_1,\ldots,i_k}(t) = (t-i_1+1)(t-i_2+1)(t-i_1-i_2-i_3+1) \cdots (t-i+1)
\]
Let $G[[r]]$ be the set of formal complex power series $f(r) = r + \sum_{i=1}^{\infty} d_i r^{i+1}$. Let $d_i : G[[r]] \to \mathbb{C}$ be such that $d_i(f)$ is the $(i+1)$st coefficient in the series expansion of $f$. We equip $G[[r]]$ with the weakest topology in which all $d_i$ are continuous functions and consider the multiplication $\circ$ on $G[[r]]$ defined by the composition of series. Then $G[[r]]$ is a separable topological group. Moreover, it is contractible and residually torsion free nilpotent. By $G_c[[r]] \subset G[[r]]$ we denote the subgroup of power series locally convergent near 0 equipped with the induced topology. Next, we define the map $P : X \to G[[r]]$ by the formula

$$P(a) := r + \sum_{i=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = i} p_{i_1,\ldots,i_k}(i) \cdot I_{i_1,\ldots,i_k}(a) \right) r^{i+1}$$

(5.5)

Then $P(a \ast b) = P(a) \circ P(b)$ and $P(X) = G_c[[r]]$. Moreover, let $v(x; r; a)$, $x \in I_T$, be the Lipschitz solution of equation (5.2) with initial value $v(0; r; a) = r$. Clearly for every $x \in I_T$ we have $v(x; r; a) \in G_c[[r]]$. It is proved in [1] that $P(a) = v(T; \cdot; a)$ (i.e., $P(a)$ is the first return map of (5.2)). In particular, we have

$$a \in \mathcal{C} \iff \sum_{i_1 + \cdots + i_k = i} p_{i_1,\ldots,i_k}(i) \cdot I_{i_1,\ldots,i_k}(a) \equiv 0, \ \forall i \in \mathbb{N}$$

(5.6)

Also, (5.5) implies that there is a continuous homomorphism $\hat{P} : G(X) \to G[[r]]$ such that $P = \hat{P} \circ \pi$ (where $\pi : X \to G(X)$ is the quotient map). We extend it by continuity to $G_f(X)$ retaining the same symbol for the extension. Then $\hat{C} := \pi(C) = \text{Ker} \hat{P}$ is a normal subgroup of $G_f(X)$. By $\hat{C}_f$ we denote its closure in $G_f(X)$. This group is called the group of formal centers of equation (1.1).

**Theorem 5.2 ([3]).** The Lie algebra $\mathcal{L}_{\hat{C}_f} \subset \mathcal{L}_{\text{Lie}}$ of $\hat{C}_f$ consists of elements

$$\sum_{n=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = n} c_{i_1,\ldots,i_k} [X_{i_1}, [X_{i_2}, \cdots, [X_{i_{k-1}}, X_{i_k}] \cdots ]] \right) t^n$$

such that

$$\sum_{i_1 + \cdots + i_k = n} c_{i_1,\ldots,i_k} \cdot \gamma_{i_1,\ldots,i_k} = 0, \ \forall n \in \mathbb{N}, \ \text{where} \ \gamma_n = 1 \ \text{and}$$

$$\gamma_{i_1,\ldots,i_k} = (-1)^{k-1} (i_k - i_{k-1})(i_{k-1} + i_k - i_{k-2}) \cdots (i_2 + \cdots + i_k - i_1) \ \text{for} \ k \geq 2$$

In particular, the map $\exp : \mathcal{L}_{\hat{C}_f} \to \hat{C}_f$ is a homeomorphism.

For further results and open problems we refer to papers [1]–[3] and references therein.

**References**


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