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An algebraic approach to the center problem for ODEs [1]

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Abstract

The classical Poincaré Center-Focus problem asks about the characterization of planar polynomial vector fields such that all their integral trajectories are closed curves whose interiors contain a fixed point, a center. This problem is reduced to a center problem for certain ODE . We present an algebraic approach to the center problem based on the study of the group of paths determined by the coefficients of the ODE.

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1 Introduction

We describe an algebraic approach to the center problem for the ordinary differential equation

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1}, \quad x \in I_T := [0, T] \tag{1.1}$$

with coefficients a_i from the Banach space $L^\infty(I_T)$ of bounded measurable complex-valued functions on I_T equipped with the supremum norm. Condition $\sup_{x\in I_T,\ i\in\mathbb{N}} \sqrt[i]{|a_i(x)|} < \infty$ guarantees that (1.1) has Lipschitz solutions on I_T for all sufficiently small initial values. By X we denote the complex Fréchet space of sequences $a = (a_1, a_2, \dots)$ satisfying this condition. We say that equation (1.1) determines a *center* if every solution v of (1.1) with a sufficiently small initial value satisfies $v(T) = v(0)$. By $\mathcal{C} \subset X$ we denote the set of centers of (1.1). The center problem is: *given $a \in X$ to determine whether $a \in \mathcal{C}$*. It arises naturally in the framework of the geometric theory of ordinary differential equations created by Poincaré. In particular, there is a relation between the center problem for (1.1) and the classical Poincaré Center-Focus problem for planar polynomial vector fields

$$\frac{dx}{dt} = -y + F(x, y), \quad \frac{dy}{dt} = x + G(x, y) \tag{1.2}$$

where F and G are polynomials of a given degree without constant and linear terms. This problem asks about conditions on F and G under which all trajectories of (1.2) situated in a small neighbourhood of $0 \in \mathbb{R}^2$ are closed. Passing to polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$ in (1.2) and expanding the right-hand side of the resulting equation as a series in r (for F, G with sufficiently small coefficients) we obtain an equation of the form (1.1) whose coefficients are trigonometric polynomials depending polynomially on the coefficients of (1.2). This reduces the Center-Focus Problem for (1.2) to the center problem for (1.1) with coefficients depending polynomially on a parameter.

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2 Group of paths

One of the main objects of our approach is a metrizable topological group $G(X)$ determined by the coefficients of equations (1.1) (the, so-called, *group of paths* in $\mathbb{C}^\infty$). It is defined as follows.

Let us consider X as a semigroup with the operations given for $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ by

$$a * b = (a_1 * b_1, a_2 * b_2, \dots) \in X \quad \text{and} \quad a^{-1} = (a_1^{-1}, a_2^{-1}, \dots) \in X$$

where for $i \in \mathbb{N}$

$$(a_i * b_i)(x) = \begin{cases} 2b_i(2x) & \text{if } \ 0 \le x \le T/2, \\ 2a_i(2x - T) & \text{if } \ T/2 < x \le T \end{cases} \qquad \text{and} \quad a_i^{-1}(x) = -a_i(T - x), \quad 0 \le x \le T$$

Let $\mathbb{C}^\infty$ be the vector space of sequences of complex numbers $(c_1, c_2, \dots)$ equipped with the product topology. For $a = (a_1, a_2, \dots) \in X$ by $\widetilde{a} = (\widetilde{a}_1, \widetilde{a}_2, \dots) : I_T \to \mathbb{C}^\infty$, $\widetilde{a}_k(x) := \int_0^x a_k(t)\, dt$ for all $k \in \mathbb{N}$, we denote a path in $\mathbb{C}^\infty$ starting at 0. The one-to-one map $a \mapsto \widetilde{a}$ sends the product $a * b$ to the product of paths $\widetilde{a} \circ \widetilde{b}$, that is, the path obtained by translating $\widetilde{a}$ so that its beginning meets the end of $\widetilde{b}$ and then forming the composite path. Similarly, $\widetilde{a^{-1}}$ is the path obtained by translating $\widetilde{a}$ so that its end meets 0 and then taking it with the opposite orientation.

For $a \in X$ consider the *basic iterated integrals*

$$I_{i_1,\dots,i_k}(a) := \int \cdots \int_{0 \le s_1 \le \cdots \le s_k \le T} a_{i_k}(s_k) \cdots a_{i_1}(s_1)\, ds_k \cdots ds_1 \tag{2.1}$$

By the Ree shuffle formula the linear space generated by all such functions on X is an algebra. For $a, b \in X$ we write $a \sim b$ if all basic iterated integrals vanish at $a * b^{-1}$. Then $a \sim b$ if and only if $I_{i_1,\dots,i_k}(a) = I_{i_1\dots,i_k}(b)$ for all basic iterated integrals, see [1]. In particular, $\sim$ is an equivalence relation on X. By $G(X)$ we denote the set of equivalence classes. Then $G(X)$ is a group with the product induced by the product $*$ on X. By $\pi : X \to G(X)$ we denote the map determined by the equivalence relation. By the definition each iterated integral $I.$ is constant on fibres of π and therefore it determines a function $\widehat{I}.$ on $G(X)$ such that $I. = \widehat{I}. \circ \pi$. The functions $\widehat{I}.$ are referred to as iterated integrals on $G(X)$. These functions separate the points on $G(X)$.

Next, we equip $G(X)$ with the weakest topology τ in which all basic iterated integrals $\widehat{I}_{i_1,\dots,i_k}$ are continuous. Then $(G(X), \tau)$ is a topological group. Moreover, $G(X)$ is metrizable, contractible, residually torsion free nilpotent (i.e., finite dimensional unipotent representations of $G(X)$ separate the points on $G(X)$) and is the union of an increasing sequence of compact subsets, see [2].

By $G_f(X)$ we denote the completion of $G(X)$ with respect to the metric d. Then $G_f(X)$ is a topological group which is called the *group of formal paths* in $\mathbb{C}^\infty$.

3 Representation of paths by noncommutative power series

Let $\mathbb{C}\langle X_1, X_2, \dots\rangle$ be the associative algebra with unit I of complex noncommutative polynomials in I and free noncommutative variables $X_1, X_2, \dots$ (i.e., there are no nontrivial relations between these variables). By $\mathbb{C}\langle X_1, X_2, \dots\rangle[[t]]$ we denote the associative algebra of formal power series in t with coefficients from $\mathbb{C}\langle X_1, X_2, \dots\rangle$. Let $S \subset \mathbb{C}\langle X_1, X_2, \dots\rangle$ be the multiplicative semigroup generated by $I, X_1, X_2, \dots$. Consider a grading function $w : S \to \mathbb{Z}_+$ determined by the conditions

$$w(I) = 0, \quad w(X_i) = i \quad (i \in \mathbb{N}) \quad \text{and} \quad w(x \cdot y) := w(x) + w(y), \quad \forall x, y \in S$$

This splits S in a disjoint union $S = \sqcup_{n=0}^{\infty} S_n$, where $S_n = \{s \in S : w(s) = n\}$. By $\mathcal{A} \subset \mathbb{C}\langle X_1, X_2, \dots \rangle[[t]]$ we denote the subalgebra of series f of the form

$$f = \sum_{n=0}^{\infty} f_n t^n \quad \text{where} \quad f_n \in V_n := \mathrm{span}_{\mathbb{C}}(S_n), \quad n \in \mathbb{Z}_+ \tag{3.1}$$

We equip $\mathcal{A}$ with the weakest topology in which all coefficients in (3.1) considered as functions in $f \in \mathcal{A}$ are continuous. Since the set of these functions is countable, $\mathcal{A}$ is metrizable. Moreover, if d is a metric on $\mathcal{A}$ compatible with the topology, then $(\mathcal{A}, d)$ is a complete metric space. Also, by the definition the multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous in this topology.

By $G \subset \mathcal{A}$ we denote the closed subset of elements f of form (3.1) with $f_0 = I$. Then $(G, \cdot)$ is a topological group. Its Lie algebra $\mathcal{L}_G \subset \mathcal{A}$ consists of elements f of form (3.1) with $f_0 = 0$. (For $f, g \in \mathcal{L}_G$ their product is defined by the formula $[f, g] := f \cdot g - g \cdot f$.) Also, the map $\exp : \mathcal{L}_G \to G$, $\exp(f) := e^f = \sum_{n=0}^{\infty} f^n/n!$, is a homeomorphism.

Further, for an element $a = (a_1, a_2, \dots) \in X$ consider the equation

$$F'(x) = \left(\sum_{i=1}^{\infty} a_i(x)\, t^i X_i \right) F(x), \quad x \in I_T \tag{3.2}$$

This can be solved by Picard iteration to obtain a solution $F_a : I_T \to G$, $F_a(0) = I$, whose coefficients in expansion in $X_1, X_2, \dots$ and t are Lipschitz functions on I_T. We set

$$E(a) := F_a(T), \quad a \in X \tag{3.3}$$

By the definition we have

$$E(a * b) = E(a) \cdot E(b), \quad a, b \in X \tag{3.4}$$

Also, an explicit calculation leads to the formula

$$E(a) = I + \sum_{n=1}^{\infty} \left(\sum_{i_1 + \cdots + i_k = n} I_{i_1,\dots,i_k}(a) X_{i_1} \cdots X_{i_k} \right) t^n \tag{3.5}$$

From the last formula one obtains that there is a homomorphism $\widehat{E} : G(X) \to G$ such that $E = \widehat{E} \circ \pi$, that is,

$$\widehat{E}(g) = I + \sum_{n=1}^{\infty} \left(\sum_{i_1 + \cdots + i_k = n} \widehat{I}_{i_1,\dots,i_k}(g) X_{i_1} \cdots X_{i_k} \right) t^n, \quad g \in G(X) \tag{3.6}$$

Formula (3.6) shows that $\widehat{E} : G(X) \to G$ is a continuous embedding. Moreover, one can determine a metric d_1 on $\mathcal{A}$ compatible with topology such that $\widehat{E} : (G(X), d) \to (G, d_1)$ is an isometric embedding. Therefore $\widehat{E}$ is naturally extended to a continuous embedding $G_f(X) \to G$ (denoted also by $\widehat{E}$). By definition, $\widehat{E} : G_f(X) \to G$ is an injective homomorphism of topological groups and $\widehat{E}(G_f(X))$ is the closure of $\widehat{E}(G(X))$ in the topology of G.

In what follows we identify $G(X)$ and $G_f(X)$ with their images under $\widehat{E}$.

4 Lie algebra of the group of formal paths

Recall that each element $g \in \mathcal{L}_G$ can be written as $g = \sum_{n=1}^{\infty} g_n t^n$, $g_n \in V_n$, $n \in \mathbb{N}$. We say that g is a *Lie element* if each g_n belongs to the free Lie algebra generated by $X_1, \dots, X_n$. In this case each g_n has the form

$$g_n = \sum_{i_1 + \cdots + i_k = n} c_{i_1,\dots,i_k} [X_{i_1}, [X_{i_2}, [\cdots, [X_{i_{k-1}}, X_{i_k}] \cdots]]] \tag{4.1}$$

with all $c_{i_1,\dots,i_k} \in \mathbb{C}$. (Here the term with $i_k = n$ is $c_n X_n$.)

Let $L_n \subset V_n$ be the subspace of elements g_n of form (4.1). Then

$$dim_{\mathbb{C}} L_n = \frac{1}{n} \sum_{d|n} (2^{\,n/d} - 1) \cdot \mu(d) \tag{4.2}$$

where the sum is taken over all numbers $d \in \mathbb{N}$ that divide n, and $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function.

By $\mathcal{L}_{Lie}$ we denote the subset of Lie elements of $\mathcal{L}_G$. Then $\mathcal{L}_{Lie}$ is a closed (in the topology of $\mathcal{A}$) Lie subalgebra of $\mathcal{L}_G$. The following result was proved in [3].

Theorem 4.1. *The exponential map* $\exp : \mathcal{L}_G \to G$ *maps* $\mathcal{L}_{Lie}$ *homeomorphically onto* $G_f(X)$.

Thus $\mathcal{L}_{Lie}$ can be regarded as the Lie algebra of $G_f(X)$.

5 Center Problem for ODEs

Let $\mathbb{C}[[z]]$ be the algebra of formal complex power series in z. By $D,\ L : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ we denote the differentiation and the left translation operators defined on $f(z) = \sum_{k=0}^{\infty} c_k z^k$ by

$$(Df)(z) := \sum_{k=0}^{\infty} (k+1) c_{k+1} z^k, \quad (Lf)(z) := \sum_{k=0}^{\infty} c_{k+1} z^k \tag{5.1}$$

Let $\mathcal{A}(D, L)$ be the associative algebra with unit I of complex polynomials in I, D and L. By $\mathcal{A}(D, L)[[t]]$ we denote the associative algebra of formal power series in t with coefficients from $\mathcal{A}(D, L)$. Also, by $G_0(D, L)[[t]]$ we denote the group of invertible elements of $\mathcal{A}(D, L)[[t]]$ consisting of elements whose expansions in t begin with I.

Further, consider equation (1.1) corresponding to an $a = (a_1, a_2, \dots) \in X$:

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1}, \quad x \in I_T \tag{5.2}$$

Using a linearization of (5.2) we associate to this equation the following system of ODEs:

$$H'(x) = \left(\sum_{i=1}^{\infty} a_i(x) D L^{i-1} t^i \right) H(x), \quad x \in I_T \tag{5.3}$$

Solving (5.3) by Picard iteration we obtain a solution $H_a : I_T \to G_0(D, L)[[t]]$, $H_a(0) = I$, whose coefficients in the series expansion in D, L and t are Lipschitz functions on I_T. It was established in [1] that (5.2) determines a center (i.e., $a \in \mathcal{C}$) if and only if $H_a(T) = I$. This implies the following result, see [3].

Theorem 5.1. *We have*

$$a \in \mathcal{C} \iff \sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k} I_{i_1,\dots,i_k}(a) \equiv 0, \quad \forall i \in \mathbb{N} \tag{5.4}$$

where $p_{i_1,\dots,i_k}$ *is a complex polynomial of degree* k *defined by the formula*

$$p_{i_1,\dots,i_k}(t) = (t - i_1 + 1)(t - i_1 - i_2 + 1)(t - i_1 - i_2 - i_3 + 1) \cdots (t - i + 1)$$

Let $G[[r]]$ be the set of formal complex power series $f(r) = r + \sum_{i=1}^{\infty} d_i r^{i+1}$. Let $d_i : G[[r]] \to \mathbb{C}$ be such that $d_i(f)$ is the $(i+1)$st coefficient in the series expansion of f. We equip $G[[r]]$ with the weakest topology in which all d_i are continuous functions and consider the multiplication $\circ$ on $G[[r]]$ defined by the composition of series. Then $G[[r]]$ is a separable topological group. Moreover, it is contractible and residually torsion free nilpotent. By $G_c[[r]] \subset G[[r]]$ we denote the subgroup of power series locally convergent near 0 equipped with the induced topology. Next, we define the map $P : X \to G[[r]]$ by the formula

$$P(a) := r + \sum_{i=1}^{\infty} \left(\sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k}(i) \cdot I_{i_1,\dots,i_k}(a) \right) r^{i+1} \tag{5.5}$$

Then $P(a * b) = P(a) \circ P(b)$ and $P(X) = G_c[[r]]$. Moreover, let $v(x;r;a)$, $x \in I_T$, be the Lipschitz solution of equation (5.2) with initial value $v(0;r;a) = r$. Clearly for every $x \in I_T$ we have $v(x;r;a) \in G_c[[r]]$. It is proved in [1] that $P(a) = v(T;\cdot;a)$ (i.e., $P(a)$ is the *first return map* of (5.2)). In particular, we have

$$a \in \mathcal{C} \iff \sum_{i_1+\cdots+i_k=i} p_{i_1,\dots,i_k}(i) \cdot I_{i_1,\dots,i_k}(a) \equiv 0, \quad \forall i \in \mathbb{N} \tag{5.6}$$

Also, (5.5) implies that there is a continuous homomorphism $\widehat{P} : G(X) \to G[[r]]$ such that $P = \widehat{P} \circ \pi$ (where $\pi : X \to G(X)$ is the quotient map). We extend it by continuity to $G_f(X)$ retaining the same symbol for the extension. Then $\widehat{\mathcal{C}} := \pi(\mathcal{C}) = Ker\widehat{P}$ is a normal subgroup of $G_f(X)$. By $\widehat{\mathcal{C}}_f$ we denote its closure in $G_f(X)$. This group is called the *group of formal centers* of equation (1.1).

Theorem 5.2 ([3])**.** *The Lie algebra $\mathcal{L}_{\widehat{\mathcal{C}}_f} \subset \mathcal{L}_{Lie}$ of $\widehat{\mathcal{C}}_f$ consists of elements*

$$\sum_{n=1}^{\infty} \left(\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k} [X_{i_1}, [X_{i_2}, [\cdots, [X_{i_{k-1}}, X_{i_k}] \cdots]]] \right) t^n$$

such that

$$\sum_{i_1+\cdots+i_k=n} c_{i_1,\dots,i_k} \cdot \gamma_{i_1,\dots,i_k} = 0, \quad \forall n \in \mathbb{N}, \quad \textit{where} \quad \gamma_n = 1 \quad \text{and}$$
$$\gamma_{i_1,\dots,i_k} = (-1)^{k-1}(i_k - i_{k-1})(i_{k-1} + i_k - i_{k-2}) \cdots (i_2 + \cdots + i_k - i_1) \quad \text{for} \quad k \geq 2$$

In particular, the map $\exp : \mathcal{L}_{\widehat{\mathcal{C}}_f} \to \widehat{\mathcal{C}}_f$ *is a homeomorphism.*

For further results and open problems we refer to papers [1]–[3] and references therein.

References

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