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Research Article
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Analytical and Numerical Analysis of Functionally Graded Heat Conduction Based on Dirichlet Boundary Conditions

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Abstract

An analytical and numerical solution for the one dimensional of heat conduction in a slab exposed to different temperature at both ends is presented. The distribution of heat throughout the transient direction obeys to functionally graded (FG) temperature based on Dirichlet boundary conditions. The variation of functionally graded temperature can be described by any form of continuous function. In this case, where the external heat fluxes are not directly definite based on the Dirichlet or mixed boundary conditions, the fluxes that concluded over the slab faces are free to vary until the equilibrium condition is reached. By numerically solving the resulting heat-conduction equation, the distribution of temperature which vary with time through the slab is obtained. The obtained analytical results are presented graphically and the influence of the gradient variation of the temperature on shape formed with changed time is investigated.

Keywords: Functionally graded temperature; Heat conduction; Heat flux; Crank Nicolson

Introduction

Functionally graded materials (FGMs) are defined as the perfect materials in mechanical, thermal and corrosive resistant properties. Different from fiber-matrix laminated composites, FGMs do not have the problems of de-bonding resulting from large inter-laminar and thermal stresses. The impression of FGMs was firstly introduced by Japanese researchers in the mid-1980s, as ultra-high temperature resistant materials for various engineering fields for instance aircraft, space vehicles, and nuclear reactors. FGMs consist of two or more materials which microscopically inhomogeneous and spatial composite materials such as a pair of ceramic–metal. The mechanical properties of the material structure changes gradually throughout the thickness with varying continuously and smoothly from top to the bottom surface. Noda [1] showed many topics range from thermoelastic to thermoinelastic problems. He suggested that temperature dependent properties of the material should be taken into account in order to achieve more accurate analysis.

For a historical era, FGMs studying focused on the analyses of thermal stress in the ceramic coatings, static deformation, and forced vibration. Noda and Jin [2,3] presented a steady thermal stress for a crack elastic solid based on nonhomogeneous infinite, and concluded that effects of the thermal stress intensity for cracks in FGMs. Cho and Oden [4] used a Crank–Nicolson and Galerkin scheme to investigate a parametric study of thermal stress characteristics. Reddy [5] proposed a theoretical formulation and finite element models for the analysis of functionally graded plates (FGPs). Praveen and Reddy [6] studied responses of FGPs based on static and dynamic thermoelastic responses and concluded that, the differences of pure ceramic or metal plates depend on responses of FGPs. Yang and Shen [7] presented a free and forced vibration analyses of functionally graded plates subjected to impulsive lateral loads under thermal environments. Dirichlet boundary conditions (DBCs) generally hold two forms: homogeneous Dirichlet boundary conditions (HDBCs) and inhomogeneous Dirichlet boundary conditions (IDBCs). The former can be considered as a special case of the latter with zero imposed value. Zhang and Zhao [8] developed a weighted finite cell method (FCM) with high computing accuracy which extended to define boundary value function so that the inhomogeneous Dirichlet boundary conditions (IDBCs) are imposed exactly.

Theoretical Formulation

There are many models for expressing the variation of material properties in FGMs. The most commonly used of these models is the power law distribution. In this study, the new expression of heat transfer through in slab is assumed based on two parameters as:

$$\Phi(x)=\Phi_0\left[\frac{3}{2}(x+0.5)^2-\left(mx^2-0.5e^{kx}\right)\right] \tag{1}$$

Where, m and k are the parameters which are used to define the variation. ϕ_0 is a constant and related to the value of the variation function $\phi(\underline{x})$ at the left surface of the slab $(x=a)$ by:

$$\Phi_0=\varphi_0\frac{1}{0.375+0.5e^{ka}+a\left(1.5+(1.5-m)a\right)} \tag{2}$$

Here, $\varphi_0=\phi(x)$ is initial value of the variation at $x=a$ Eq. (1) is a nonlinear function and its variation (shape) is controlled by using two parameters. One may observe that, the adjustment of the parameters m and k is not easy for describing the desired variations. Figure 1 shows the variation of temperature of metal in the transverse direction of slab.

Exact Solution of the Heat Equation

Consider a one-dimensional diffusion equation which is a partial differential equation for the temperature $T(x,t)$:

$$c(x)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial T}{\partial x}\right)+S(x,t) \tag{3}$$

Where, $c(x)$ is the specific heat of the material, κ is the constant of proportionality (thermal conductivity) of the material and $S(x,t)$ represents a given source of heat energy per unit volume. The equation simplifies when κ and c are independent of position:

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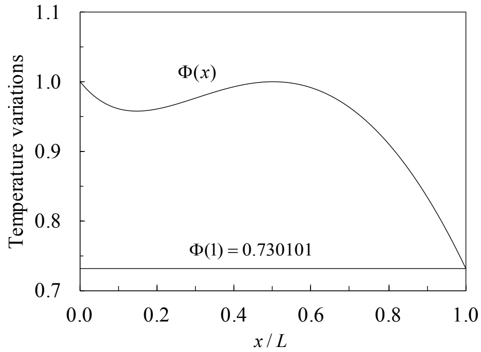

Figure 1: Temperature variation along the transverse position.

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial x^2} + \frac{S(x,t)}{c} \tag{4}$$

Where, χ is the material thermal diffusivity,

$$\chi = \frac{\kappa}{c} \tag{5}$$

The initial temperature in the slab heat equation is a first-order PDE in time:

$$T(x,0) = T_0(x) \tag{6}$$

The second order equation in space requires two boundary conditions to satisfy the solution. By putting the faces in a suitable thermal contact at a specified temperature $T_1(t)$ and $T_2(t)$ so the Dirichlet boundary conditions are achieved as:

$$\begin{aligned} T(0,t) &= T_1(t) \\ T(L,t) &= T_2(t) \end{aligned} \tag{7}$$

The equilibrium solution $\phi(x)$ that satisfies these boundary conditions, as well as the time-independent heat equation:

$$0 = \frac{\partial}{\partial x}\left(\kappa(x) \frac{\partial \Phi(x)}{\partial x} \right) + S(x,t) \tag{8}$$

Using the direct integration (analytically), equation goes to general solution:

$$\Phi(x) = C_1 + \int_0^x \left(\frac{C_2 - \int_0^{x'} S(x'')dx''}{k(x')} \right) dx' \tag{9}$$

The constants C_1 and C_2 are chosen to match the boundary conditions. For a temperature balance to exist, there can be no net heat energy infused into the section by the source: $\int_0^L S(x'')dx'' = 0$), else the temperature should increase or decrease as the overall energy content in the slab varies with time. The relation between the equilibrium temperature and the deviation from equilibrium $\Delta T(x,t)$ is:

$$T(x,t) = \Phi(x) + \Delta T(x,t) \tag{10}$$

By Substitution Eq. (10) into the heat equation Eq. (3) and application of Eq. (8) indicates that $\Delta T(x,t)$ satisfies the homogeneous heat equation:

$$c(x)\frac{\partial \Delta T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x) \frac{\partial \Delta T}{\partial x} \right) \tag{11}$$

With homogeneous boundary conditions, and initial condition:

$$\Delta T(x,0) = T_0(x) - \Phi(x) \tag{12}$$

We will only consider the case where κ and C are constants, so that Eq. (11) becomes the diffusion equation with no heat source,

$$\frac{\partial \Delta T}{\partial t} = \chi \frac{\partial^2 \Delta T}{\partial x^2} \tag{13}$$

We look for a solution of the form $\Delta T(x,t) = f(t)\psi(x)$. Substituting this expression into Eq. (13) and dividing by $f(t)\psi(x)$ yields,

$$\frac{1}{f(t)}\frac{\partial f}{\partial t} = \frac{\chi}{\psi(x)}\frac{\partial^2 \psi}{\partial x^2} \tag{14}$$

Which can be separated into two ODEs with respect to independent x and t. By putting of each separated equations of Eq. (14) are equal to a constant λ,

$$\frac{1}{f(t)}\frac{\partial f}{\partial t} = \lambda \tag{15}$$

$$\frac{\chi}{\psi(x)}\frac{\partial^2 \psi}{\partial x^2} = \lambda \tag{16}$$

After applying Dirichlet boundary conditions, solution of Eq. (16) for the eigenmodes is,

$$\psi(x) = D \sin \frac{n\pi x}{L} \tag{17}$$

And;

$$\lambda = \lambda_n = -\chi(n\pi/L)^2, \quad n = 1,2,3,\cdots \tag{18}$$

The solution of Eq. (15) provides the time-dependent amplitude for each eigenmode as,

$$f(t) = A e^{\lambda t} \tag{19}$$

Therefore, the general solution of the heat equation away from equilibrium is,

$$\Delta T(x,t) = \sum_{n=1}^{\infty} A_n e^{\lambda_n t} \sin \frac{n\pi x}{L} \tag{20}$$

The constant D is absorbed into the Fourier coefficient A_n which can be found by matching the initial condition, Eq. (12),

$$\Delta T(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = T_0(x) - T_{eq}(x) \tag{21}$$

Equation (21) is a Fourier sine series and the coefficient A_n is determined as,

$$A_n = \frac{2}{L}\int_0^L \left[T_0(x) - \psi(x) \right] \sin \frac{n\pi x}{L} dx \tag{22}$$

Dirichlet boundary conditions in a uniform slab are applied to get equations (20) and (22) which is the solution for the deviation from equilibrium. In addition, Eq. (22) is solved numerically by using trapezoidal method which can be seen in the numerical example.

The Crank-Nicolson Method

An implicit scheme of Crank-Nicolson which is based on the central approximation of Eq. (13) at the point $\left(x_i, t_j + \frac{1}{2}\Delta t \right)$ as shown in Figure 2,

$$\frac{T_i^{j+1} - T_i^j}{2\frac{\Delta t}{2}} = \chi \frac{(T_{i+1}^{j+\frac{1}{2}} - 2T_i^{j+\frac{1}{2}} + T_{i-1}^{j+\frac{1}{2}})}{\Delta x^2} \tag{23}$$

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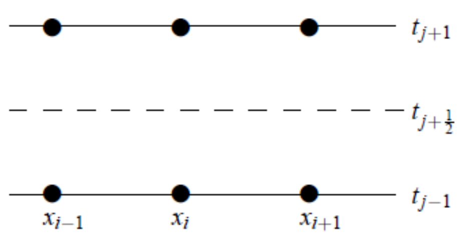

Figure 2: Crank-Nicolson method scheme.

Space derivative approximation which is used for is just an average of approximations in points (x_i, t_j) and (x_i, t_{j+1}),

$$\frac{T_i^{n+1}-T_i^n}{\Delta t} = \chi\frac{(T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1})+(T_{i+1}^n-2T_i^n+T_{i-1}^n)}{2\Delta x^2} \tag{24}$$

Introducing $\alpha = \chi\Delta t/\Delta x^2$ one can rewrite Eq. (24) as,

$$-\alpha T_{i+1}^{j+1} + 2(1+\alpha)T_i^{j+1} - \alpha T_{i-1}^{j+1} = \alpha T_{i+1}^j + 2(1-\alpha)T_i^j + \alpha T_{i-1}^j \tag{25}$$

The terms which appear in the right-hand side of Eq. (25) are known. Hence, Eq. (25) form a tridiagonal linear system *(AT=b)* which can be solve simultaneously to find the temperature at every node at any point in time.

The complex stability analysis will be procedure in some simple steps. The equation (24) can be written as,

$$\frac{T_i^{n+1}-T_i^n}{\Delta t} = \frac{\chi}{2\Delta x^2}\left(T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1}+T_{i+1}^n-2T_i^n+T_{i-1}^n\right) \tag{26}$$

The equation (26) has a consistency,

$$\frac{\chi}{2}\left(T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1}+T_{i+1}^n-2T_i^n+T_{i-1}^n\right) = \frac{\chi}{2}\left(\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+1} + O(\Delta x^2) + \frac{\partial^2 T}{\partial x^2}\bigg|_i^{n} + O(\Delta x^2)\right) = \chi\frac{\partial^2 T}{\partial x^2}\bigg|_i^{n+\frac{1}{2}} + O(\Delta t^2+\Delta x^2) \tag{27}$$

$$\frac{T_i^{n+1}-T_i^n}{\Delta t} = \frac{\partial T}{\partial t}\bigg|_i^{n+\frac{1}{2}} + O(\Delta t^2) \tag{28}$$

To make a stability analysis, the scheme will be written into two form stages:

$$\frac{T_i^{n+\frac{1}{2}}-T_i^n}{\frac{\Delta t}{2}} = \chi\frac{T_{j+1}^n-2T_j^n+T_{j-1}^n}{\Delta x^2} \rightarrow Z_1(\theta) = 1-4\chi\frac{\Gamma}{2}\sin^2\frac{\theta}{2}$$

$$\frac{T_i^{n+1}-T_i^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \chi\frac{T_{j+1}^{n+1}-2T_j^{n+1}+T_{j-1}^{n+1}}{\Delta x^2} \rightarrow Z_2(\theta) = \frac{1}{1+4\chi\frac{\Gamma}{2}\sin^2\frac{\theta}{2}} \tag{29}$$

Where, $\Gamma = \dfrac{\Delta t}{\Delta x^2}$ and is called of grid ratio.

Therefore;

$$w^{n+\frac{1}{2}} = Z_1(\theta)w^n, \qquad w^{n+1} = Z_2(\theta)w^{n+\frac{1}{2}} = \underbrace{Z_1 Z_2}_{Z(\theta)} w^n \tag{30}$$

Eq. (30) leads to the consequence that Z(θ) ≤ 1, so this scheme is an unconditionally stable and $\|\varepsilon\| = O(\Delta x^2 + \Delta t^2)$ then this system has a second order of convergence. The consistency error is obtained by substituting the exact solution T in the discrete system. This means the numeric solution $T_{\Delta x}$ converges to exact solution in a given norm if $\|\varepsilon_{\Delta x}\| = \|T - T_{\Delta x}\|$ satisfies $\lim_{\Delta x\to 0}\|\varepsilon_{\Delta x}\| = 0$.

The consistency between the solutions of the continuous and discrete problems does not guarantee also have a tendency to zero. On the other hand, the difference between the differential and discrete operators on a smooth sufficient function tends to zero.

Numerical Example

In this section, the distribution of temperature throughout the transverse direction of slab is tested based on assumed temperature gradient. The properties of slab are selected as the parameters in Eq (1): $k = -4.210933$, $m = 2.743578$ and $\Phi_0 = 1.142857$ to ensure the start point at the left side equal 1.0 and the end point goes to 0.730101. Other properties, L=1.0 m, n=40, χ=1.0 m^2/sec, Δt=0.0001 sec and Δx=L/n. The difference between an analytical and trapezoidal method used for integral part in Eq. (22) and the values applied to find the temperature in Eq. (10) based on time step t=0.02sec as shown in Table 1.

Figures 3-6 represent the shape of temperature distribution which taking with increasing time (t=0.02, 0.1, 0.2 and 0.4). By fixing the temperature variation based Crank-Nicolson (CN) method, the comparison with the exact is done with showing two-time step t=0.3 sec and t=0.4 sec as in Figures 7 and 8. For example, the time elapsed to get (CN) temperature variation for 0.5, 0.55, 0.575 and 0.6 are respectively: 0.535268, 0.519372, 0.566319 and 0.505062.

Conclusions

In this paper, simple method to solve the transient response problem of a functionally graded temperature variation in the 1-D

x/L	Analytical	Trapezoidal
0.0	1.000000	1.000000
0.1	0.588196	0.588655
0.2	0.294891	0.295522
0.3	0.129512	0.130024
0.4	0.0610078	0.061314
0.5	0.0489602	0.049157
0.6	0.0727702	0.073010
0.7	0.139666	0.140043
0.8	0.272146	0.272607
0.9	0.479501	0.479836
1.0	0.730101	0.730101

Table 1: Comparison between analytical and trapezoidal method for heat equation based on time step *t*=0.02 sec that integral appears in Eq.22.

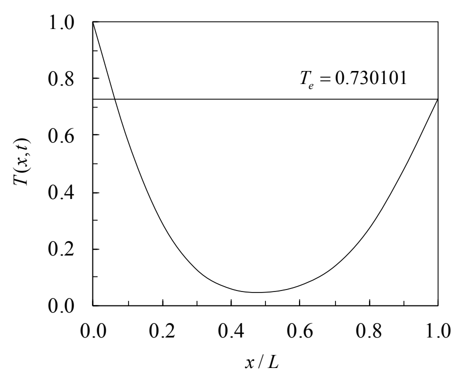

Figure 3: Temperature distribution for *t*=0.02 sec.

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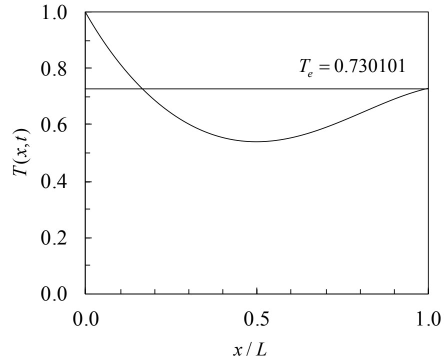

Figure 4: Temperature distribution for *t*=0.1 sec.

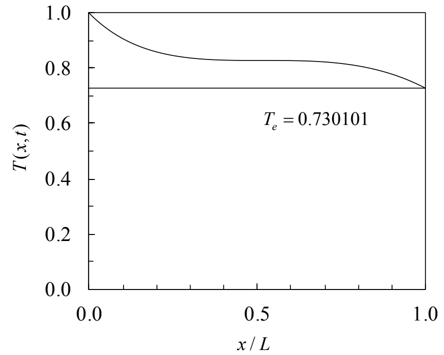

Figure 5: Temperature distribution for *t*=0.2 sec.

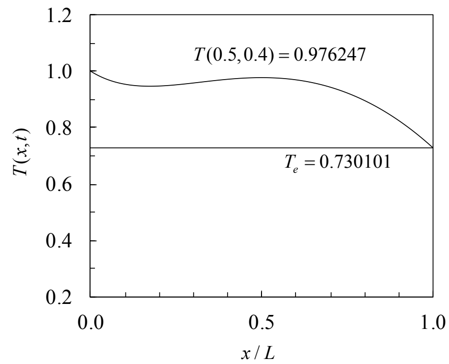

Figure 6: Temperature distribution for *t*=0.4 sec.

slab has been proposed. The assumed temperature is embedded into the diffusion equation with no source of heat then exact solution is obtained. In addition, the integral part of A_n coefficient in Eq. (22) is solved numerically using trapezoidal method and the difference is exhibited in Table 1. In this table, the differences of temperature between the two methods are too small. Crank-Nicolson (CN) method is applied to solve the diffusion equation based on the given variation $\phi(x)$. The assumption of variation is selected for two variables k and m. The temperature at any point goes to take the assumed variation with time. By fixing the numerical method at the shape of variation function, the times that the exact solution is needed to get the numerical solution are computed. Using these methods, the time required to reach

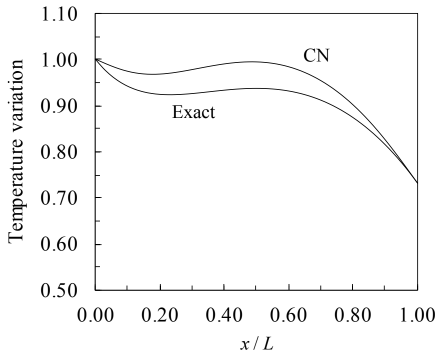

Figure 7: Temperature variation for *t*=0.3 sec based on exact and Crank Nicolson method.

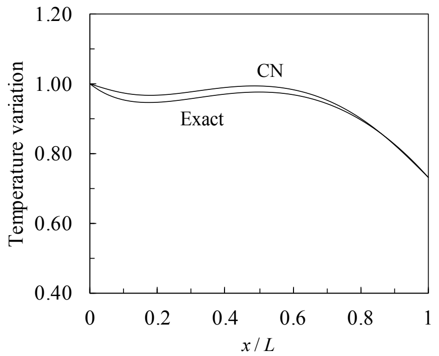

Figure 8: Temperature variation for *t*=0.4 sec based on exact and Crank Nicolson method.

the assumed model can be calculated at any position with the given Dirichlet boundary condition.

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