Bimodules and Rota-Baxter Relations

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Abstract

In this paper we introduce and study Hom-type bimodules of some Hom-algebraic structures endowed with Rota-Baxter relations. We introduce bimodules over Homassociative Rota-Baxter algebras and give their various twisting and their connection with bimodules over Hom-preLie algebras. Then we introduce Rota-Baxter q-Hom-tridendriform algebras. Next we express axioms defining q-Hom-tridendriform algebras by mean of vector basis. Moreover we introduce bimodules over q-Hom-tridendriform algebras and give some examples, and prove that they are closed by twisting. Finally we give their connection with Hom-associative Rota-Baxter bimodules.

Keywords: Hom-preLie; Q-Hom-tridendriform and Hom-associative Rota-Baxter algebras; Rota-Baxter bimodules

AMS Subject Classification: 16W20, 17D25, 16T15

Introduction

Rota-Baxter operators have appeared in a wide range of areas in pure and applied mathematics. The paradigmatic example of a Rota-Baxter operator concerns the integration by parts formula. Rota-Baxter algebras find their application in probability and the study of fluctuation theory [1], combinatorics [2,3], quantum field theory [4], Lie algebra theory [5,6]. It is shown that the Rota-Baxter identity can be significant in the construction of Frobenius manifolds inherent to the integrable systems of hydrodynamic type [7]. They are closely related to dendriform algebra [8] which were introduced by Loday. Dendriform algebras have two binary operations, which dichotomize an associative multiplication. The motivation to introduce these algebraic structures comes from K-theory. Dendriform algebras are connected to several areas in mathematics and physics, including Hopf algebras, homotopy Gerstenhaber algebra, operads, homology, combinatorics, and quantum field theory, where they occur in the theory of renormalization of Connes and Kreimer. Rota-Baxter algebras are related to dendriform algebras via a pair of adjoint functors [9,10]. Roughly speaking, Rota-Baxter algebras are to dendriform algebras as associative algebras are to Lie algebras. q-tridendriform algebras are introduced [11] as an algebraic structure with three operations containing dendriform algebras and tridendriform algebras as particular cases. The author studied tridendriform algebra structures of the space of parking functions and and of the space of multipermutations. Tridendriform bialgebras was also studied by the author.

Hom-algebraic structures first arose in quasi-deformations of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures, in which the Jacobi condition is twisted. Other interesting Hom-type algebraic structures of many classical structures were studied as Hom-associative algebras, Hom-Lie admissible algebras and more general G-Hom-associative algebras [12], n-ary Hom-Nambu-Lie algebras [13], Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras [14], Hom-alternative algebras, Hom-Malcev algebras and Hom-Jordan algebras [15], L-modules, L-comodules and Hom-Lie quasi-bialgebras [16], Laplacian of Hom-Lie quasi-bialgebras [17].

Hom-algebraic structures were extended to the case of G-graded Lie algebras by study-ing Hom-Lie superalgebras, Hom-Lie admissible superalgebras [18], color Hom-Lie algebras [19], color Hom-Lie bialgebras and color Hom-Poisson bialgebras [20] and color Hom-Poisson algebras. Color Hom-Poisson algebras were introduced [21] as generalization of Hom-Poisson algebras [12]. The generalized left-Hom-symmetric algebras and generalized Hom-dendriform algebras as well as the corresponding modules are studied [22]. It is also proved that any generalized Hom-dialgebras give rise to generalized Hom-Leibniz-Poisson algebras and generalized Hom-Poisson dialgebras.

The aim of this paper is to introduce bimodules over some Hom-algebraic structures endowed with Rota-Baxter relations. The paper is organized as follows. In section one, we recall basic notions related to modules over Hom-associative and Hom-Lie algebras. Section two is devoted to Rota-Baxter bimodules over Hom-associative Rota-Baxter algebras and their connection with bimodules over Hom-preLie algebras. In section three we introduce q-Hom-tridendriform Rota-Baxter algebras. Most of the results (the various twisting) on q-Hom-tridendriform algebras are obviously analogs for Hom-tridendriform algebras and the proofs are identical nearly, so we omit them. We express axioms defining q-Hom-tridendriform algebras by mean of vector basis, which may be very useful in the classification setting. In section four introduce bimodules theory for q-Hom-tridendriform algebra and we prove the commutativity of the following diagram

\[
\text{HARBM} \rightarrow \text{HAM} \\
\downarrow \\
q \text{-HTDM} \leftarrow \text{HDM}
\]

where HARBM denote the category of Hom-associative Rota-Baxter bimodules, q-HTDM denote the category of q-Hom-tridendriform bimodules, HDM denote the category of Hom-dendriform bimodules and HAM denote the category of Hom-associative bimodules.

All vector spaces considered are supposed to be over fields of characteristics different from 2.
Preliminaries

In this section, we recall basic definitions of Rota-Baxter Hom-associative algebras and modules over Hom-Lie algebras.

Definition 1.1: [23] A Hom-associative algebra is a triple \((A, \cdot, \alpha)\) consisting of a linear space \(A\), a \(K\)-bilinear map \(\cdot: A \times A \rightarrow A\) and a linear space map \(\alpha: A \rightarrow A\) satisfying

\[
\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z) \quad \text{(Hom-associativity).}
\]

(1)

If in addition \(\alpha\) satisfies

\[
\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y) \quad \text{(multiplicativity),}
\]

(2)

then \((A, \cdot, \alpha)\) is said to be multiplicative.

When \(\alpha = \text{Id}_A\), \((A, \cdot, \text{Id}_A)\), simply denoted \((A, \cdot)\), is an associative algebra.

Example 1.1: Let \(\{e_1, e_2, e_3\}\) be a basis of a 3-dimensional vector space \(A\) over \(K\). The following multiplication \(\cdot\) and map on \(A\) define a Hom-associative algebra [24]:

\[
e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2 - e_1, \quad e_2 \cdot e_2 = e_2,
\]

(3)

\[
a(e_1) = a_1 e_2 + a_2 e_3, \quad a(e_2) = b_1 e_2 + b_2 e_3, \quad a(e_3) = e_1 e_2 + e_2 e_3.
\]

Where \(a_1, a_2, b_1, b_2, c_1, c_2\) are parameters in \(K\).

Definition 1.2: A Hom-Lie algebra is a triple \((L, [\cdot, \cdot], \beta)\) of a linear space \(L\), a \(K\)-bilinear map \([\cdot, \cdot]: L \times L \rightarrow L\) and a linear space map \(\beta: L \rightarrow L\) satisfying the identity [25]

\[
\beta(x \cdot y) = \beta(x) \cdot \beta(y) \quad \text{(Hom-Jacobi identity.)}
\]

(7)

In addition \(\beta\) satisfies [26]

\[
\beta(x \cdot y) = \beta(x) \cdot \beta(y) \quad \text{(Hom-associativity).}
\]

(8)

When \(\beta = \text{Id}_L\), \((L, [\cdot, \cdot], \text{Id}_L)\), simply denoted \((L, [\cdot, \cdot])\), is an associative algebra.

Definition 1.3: A Hom-Lie algebra is a triple \((L, [\cdot, \cdot], \alpha)\) consisting of a linear space \(L\), a \(K\)-bilinear map \([\cdot, \cdot]: L \times L \rightarrow L\) and a linear space map \(\alpha: A \rightarrow A\) satisfying the identity [26]

\[
[x, y] = -[y, x] \quad \text{(skew-symmetry)}
\]

(6)

\[
\alpha(x) + [\alpha(y), [x, y]] + [\alpha([x, y]), x] + [\alpha(x), [y, x]] = 0 \quad \text{(Hom-Jacobi identity.)}
\]

(7)

When \(\alpha = \text{Id}_L\), we obtain the definition of Lie algebras.

Lemma 1.1: Let \((A, \cdot, \alpha, R)\) be a multiplicative Hom-associative Rota-Baxter algebra of weight \(\lambda\) and \(\beta: A \rightarrow A\) be a morphism. Then \(A \beta = (A \beta, \beta^\alpha, \beta^R, R\alpha, R\beta, R\lambda)\) is also a multiplicative Hom-associative Rota-Baxter algebra of weight \(\lambda\) [25].

We have a similar conclusion for Hom-Lie algebras.

Now we define modules over Hom-associative and Hom-Lie algebras.

Definition 1.4: A Hom-module is a pair \((M, \beta)\) in which \(M\) is a \(K\)-vector space and \(\beta: M \rightarrow M\) is a linear map [27].

Definition 1.5: Let \((A, \cdot, \alpha)\) be a Hom-associative algebra and \((M, \beta)\) be a Hom-module. A \(\alpha\)-left \(A\)-module structure on \(M\) consists of a \(K\)-bilinear map \(\alpha \otimes A \rightarrow M\) such that for any \(m \in M, x, y \in L\) [27]

\[
\beta(x \cdot m) = \alpha(x) \cdot \beta(m).
\]

(10)

\[
[x, y] \cdot m = \alpha(x) \cdot (y \cdot m) - \alpha(y) \cdot (x \cdot m).
\]

(11)

When \(\beta = \text{Id}_M\) and \(\alpha = \text{Id}_A\), we recover the definition of Lie modules [19].

The following result shows that \(A\)-modules extend to \(L(A)\)-modules for the same module structure map.

Lemma 1.2: Let \((A, \cdot, \alpha)\) be a Hom-associative algebra and \((M, \cdot, \beta)\) be an \(A\)-module. Then \(M\) is an \(L(A)\)-module for the structure map \(\cdot\) [16].

Bimodules Over Hom-associative Rota-Baxter and Hom-pre Lie algebras

In this section we introduce bimodules over Hom-associative Rota-Baxter algebras. We give their various twisting and their connection with bimodules over Hom-preLie algebras.

Definition 2.1: Let \((A, \cdot, \alpha, R)\) be a Hom-associative Rota-Baxter algebra of weight and let \((M, \cdot, \beta)\) be a Hom-module. A Hom-associative Rota-Baxter bimodule structure on \(M\) consists of:

A left \(A\)-action \([A \otimes M \rightarrow M : (x \otimes m) \mapsto x \cdot m\),

A right \(A\)-action \([M \otimes A \rightarrow M : (m \otimes x) \mapsto m \cdot x\),

A Rota-Baxter operator \(R_\beta: M \otimes M \rightarrow M\) of weight \(\lambda\) such that the following conditions hold for \(x, y \in A\) and \(m \in M\)

\[
\beta(x \cdot m) = \alpha(x) \cdot \beta(m),
\]

(12)

\[
\beta(m \cdot x) = \beta(m) \cdot \alpha(x),
\]

(13)

\[
\alpha(x) \cdot (y \cdot m) = (x \cdot y) \cdot \beta(m),
\]

(14)

\[
(m \cdot x) \cdot (y \cdot m) = (m \cdot (x \cdot y)) \cdot \beta(m),
\]

(15)

\[
\alpha(x) \cdot (m \cdot y) = (x \cdot m) \cdot \alpha(y),
\]

(16)

\[
R_\beta(m \cdot x) = \beta^\alpha R_\beta(m),
\]

(17)

\[
R_\beta(x \cdot m) = R_\beta(R_\beta(R_\beta(m) \cdot x + x \cdot m - R_\beta(m) \cdot x + \lambda x \cdot m)),
\]

(18)

\[
R_\beta(m \cdot x) < R(x) = R_\beta(R_\beta(m) \cdot x + x \cdot m - R_\beta(m) \cdot x + \lambda m \cdot x).
\]

(19)

The quintuple \((M, \cdot, \alpha, R, R_\beta)\) is then called the Hom-associative Rota-Baxter bimodule over \(M\), Hom-associative Rota-Baxter algebra.

Remark 2.1: We have a similar definition for modules over Hom-Lie Rota-Baxter algebras.

The following result is an extension of Lemma 2.5 [29]. It asserts that bimodules over Hom-associative Rota-Baxter algebras are stable under twisting.

Theorem 2.1: Let \((M, \cdot, \alpha, R, R_\beta)\) be a Hom-associative Rota-Baxter bimodule over the Hom-associative Rota-Baxter algebra \((A, \cdot, \alpha, R, R_\beta)\) and be an endomorphism of \(A\). Let, \([A : A \otimes M \rightarrow M, \cdot] ; [A : A \otimes M \rightarrow M, \otimes] ; [A : A \otimes M \rightarrow M, \alpha] ; [A : A \otimes M \rightarrow M, \beta] ; [A : A \otimes M \rightarrow M, \lambda] \] be bilinear maps defined by

\[
g = \beta^\alpha, \quad \gamma = \alpha^\alpha, \quad \alpha = \beta^\beta (Id_A \otimes M)\] and \(\alpha = \beta^\beta (Id_M \otimes \alpha^\alpha)\).

Then \((M, \cdot, \alpha, \cdot, R, R_\beta)\) is a Hom-associative Rota-Baxter
bimodule of weight λ over \((A, \beta, \beta' \alpha, R)\).

**Proof:** We prove the compatibility condition (16). For any \(x,y \in A, m \in M\), one has
\[
\alpha(x) \gamma_m (m \ll y) = \alpha((x \gamma_m) y) = \alpha(x) \gamma_m (m \ll y) = \alpha^2(x) \gamma_m (m \ll y) = \alpha^2(x) \gamma_m (x \ll m) = \alpha(x).
\]
And
\[
R(x) \gamma_m R_m(m) = R((x \gamma_m) m) = R(x \gamma_m) m = R(x) \gamma_m (m \ll y) = R(x) \gamma_m (m \ll y).
\]

The rest of this section is dedicated to bimodules over Hom-preLie algebras from Hom-associative Rota-Baxter bimodules.

**Definition 2.2:** A non-associative algebra \(S\) with the linear map \(\alpha: S \rightarrow S\) and the multiplicative \((x, y, z) \mapsto x \cdot y \cdot z\) is called Hom-preLie algebra if the following Hom-preLie identity is satisfied
\[
(x \cdot y) 
\]
\[
(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot (y \cdot z) = 0,
\]
for all \(x, y, z \in S\).

**Theorem 2.3:** Let \((A, M, R)\) be a Hom-preLie-Rota-Baxter algebra, \(A**\) be a Hom-preLie-Rota-Baxter algebra, \(A**\) be an endomorphism of \(A\). Then \((A**, M, R)\) is a Hom-preLie-Rota-Baxter algebra over \(A\).

**Example 2.1:** Any multiplicative Hom-associative Rota-Baxter algebra is a Hom-preLie-Rota-Baxter algebra over itself. The above theorem shows that Hom-Lie Rota-Baxter modules over Hom-Lie Rota-Baxter algebra are closed under twisting.

**Theorem 2.4:** Let \((A, \beta, R_M)\) be a multiplicative Hom-associative Rota-Baxter algebra and \((M, \beta, R_M)\) be a Hom-associative Rota-Baxter (left) module over \(A\). Then \((M, \beta, R_M)\) is a left module over the Hom-Lie Rota-Baxter algebra associated to \(A\) as in Lemma 2.4.

**Corollary 2.5:** Let \((M, \beta, R_M)\) be a Hom-Lie (left) module of weight \(\lambda\) over the Hom-Lie Rota-Baxter algebra \(L_{\lambda} = L_{\lambda}[-\lambda, -\lambda]\). Then \((M, \beta, R_M)\) is another Hom-Lie Rota-Baxter (left) module over \(L_{\lambda}\).

The rest of this section is dedicated to bimodules over Hom-preLie algebras from Hom-associative Rota-Baxter bimodules.

**Definition 2.3:** Let \((A, \beta, R_M)\) be a Hom-preLie-Rota-Baxter algebra. An S-bimodule \(M\) is a Hom-preLie-Rota-Baxter module of weight \(\lambda\) over the Hom-preLie algebra \(A\) if the following conditions are satisfied
\[
(x \cdot y) 
\]
\[
(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot (y \cdot z) = 0,
\]
for all \(x, y, z \in S\).

**Theorem 2.6:** Let \((A, \beta, R_M)\) be a Hom-preLie-Rota-Baxter algebra. An S-bimodule \(M\) is a Hom-preLie-Rota-Baxter module of weight \(\lambda\) over the Hom-preLie algebra \(A\) if the following conditions are satisfied
\[
(x \cdot y) 
\]
\[
(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot (y \cdot z) = 0,
\]
for all \(x, y, z \in S\).

**Definition 2.4:** Let \((A, \beta, R_M)\) be a multiplicative Hom-associative Rota-Baxter algebra and \((M, \beta, R_M)\) be a Hom-associative Rota-Baxter (left) module over \(A\). Then \((M, \beta, R_M)\) is a left module over the Hom-Lie Rota-Baxter algebra associated to \(A\) as in Lemma 2.4.

**Corollary 2.5:** Let \((M, \beta, R_M)\) be a Hom-Lie (left) module of weight \(\lambda\) over the Hom-Lie Rota-Baxter algebra \(L_{\lambda} = L_{\lambda}[-\lambda, -\lambda]\). Then \((M, \beta, R_M)\) is another Hom-Lie Rota-Baxter (left) module over \(L_{\lambda}\).
Let $(M,\triangleright,\triangleright\triangleright)$ be a Hom-Lie Rota-Baxter algebra of weight 0 and $(M,\triangleright,\triangleright)$ be the sub-adjacent Hom-Lie module. Let $\alpha$ be a twisting map and $R$ be a Hom-associative Rota-Baxter algebra of weight 1. We define the operation on $A$ by
\[
\alpha(x)\triangleright\triangleright y = R(x)\triangleright y + 2x\cdot y,
\]
for $x,y,z \in D$. It is clear that for any $q \neq 0$, $(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ turn to Hom-tridendriform algebra when ever $(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ is a q-Hom-tridendriform algebra.

**Example 3.1:** i) Any q-tridendriform algebra is a q-Hom-tridendriform algebra with $q = 1$.

two binary operations) which is a Hom-Poisson Rota-Baxter algebra (with two binary operations) which is a Hom-Poisson Rota-Baxter algebra. A morphism of q-Hom-tridendriform algebra is a linear map $\beta: D \rightarrow D$ satisfying
\[
R^\alpha x \triangleright y = R(x \triangleright R(y) + R(x) \triangleright y + lx \cdot y),
\]
for any $x,y,z \in D$. A q-Hom-tridendriform algebra is a q-Hom-Poisson Rota-Baxter algebra of weight $\lambda\in \mathbb{K}$.

**Remark 3.1:** We have a similar definition for Hom-Poisson Rota-Baxter algebra (with two binary operations) which is a Hom-Poisson algebra endowed with a linear operator that commutes with the twisting map and satisfies the Rota-Baxter relations for both associative and Hom-Lie product. By a direct computation we can show that every Hom-associative Rota-Baxter algebra has a non-commutative Hom-Poisson Rota-Baxter algebra structure in which the Hom-Poisson bracket is the commutator bracket. It is also easy to prove that Rota-Baxter Poisson algebras turn Hom-Poisson Rota-Baxter algebras by twisting the Poisson Rota-Baxter algebra structure. By a straightforward calculation, we may prove that any Hom-Lie Rota-Baxter algebra is a Hom-Poisson Rota-Baxter algebra of the same twisting map and Rota-Baxter operator.

**Definition 3.2:** A q-Hom-tridendriform algebra $(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ is also a q-Hom-tridendriform algebra.

**Theorem 3.2:** Let $(M,\triangleright,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ be a Hom-associative Rota-Baxter bimodule of weight 1 over the Hom-associative Rota-Baxter algebra $(A,\triangleright,\triangleright)$ of weight 1. Let us define bilinear maps $\alpha,\beta,\beta_1,\beta_2,\beta_3$ in which is a q-Hom-tridendriform algebra.

**Definition 3.3:** Let $\lambda,\theta \in \mathbb{K}$.

\[
\beta_\lambda x \triangleright y = \beta(x)\triangleright y + \lambda x \cdot y,
\]
for any $x,y,z \in D$. It is clear that for any $q \neq 0$, $(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ turn to Hom-tridendriform algebra when ever $(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ is a q-Hom-tridendriform algebra.

**Example 3.1:** i) Any q-tridendriform algebra is a q-Hom-tridendriform algebra with $q = 1$.

**Theorem 3.4:** Let $(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright)$ be a q-Hom-tridendriform algebra. Then, for any parameters $\lambda,\theta \in \mathbb{K}$,

\[
(D,\triangleright\triangleright,\triangleright,\triangleright\triangleright) = \lambda \triangleright\triangleright_{\lambda,\theta} + \lambda \triangleright\triangleright_{\lambda,\theta} + \lambda \triangleright\triangleright_{\lambda,\theta} := \lambda \alpha q := \theta \alpha q
\]

is also a q-Hom-tridendriform algebra.
Lemma 3.1: The axioms in Definition 3.1 are respectively equivalent to

\[
\begin{align*}
\alpha_{ijk}a_{lm}^n &= \alpha_{i,jk}a_{lm}^n + \alpha_{ij,k}a_{lm}^n + q \alpha_{ijk}a_{lm}^n, \\
\beta_{ijk}a_{lm}^n &= \beta_{i,jk}a_{lm}^n + \beta_{ij,k}a_{lm}^n, \\
\gamma_{ijk}a_{lm}^n &= \gamma_{i,jk}a_{lm}^n + \gamma_{ij,k}a_{lm}^n + q \gamma_{ijk}a_{lm}^n, \\
\delta_{ijk}a_{lm}^n &= \delta_{i,jk}a_{lm}^n + \delta_{ij,k}a_{lm}^n, \\
\epsilon_{ijk}a_{lm}^n &= \epsilon_{i,jk}a_{lm}^n + \epsilon_{ij,k}a_{lm}^n + q \epsilon_{ijk}a_{lm}^n,
\end{align*}
\]

for any \(i, j, k, l, m, n \in \mathbb{N}\) with \(i, j, k, l, m, n \leq N\).

Lemma 3.2: Let \(R\) be a Rota-Baxter operator of weight \(\lambda\) on a q-Hom-tridendriform algebra of dimension \(N\). Then in terms of basis elements, relations (31)-(34) are equivalent to the following:

\[
\begin{align*}
\alpha(x) &= \beta(m) = (x \cdot y + x \cdot y + q \cdot x \cdot y) \circ \beta(m), \\
(\beta \circ x) &= \beta(m) = (x \cdot y \circ x + q \cdot x \cdot y) \circ \beta(m), \\
(\beta \circ y \circ y) &= \beta(m) = (x \cdot y \circ y + q \cdot x \cdot y) \circ \beta(m), \\
(\beta \circ x \cdot y) &= \beta(m) = (x \cdot y \circ x \cdot y + q \cdot x \cdot y) \circ \beta(m),
\end{align*}
\]

for any \(i, j, k, l, m, n \in \mathbb{N}\) with \(i, j, k, l, m, n \leq N\).

Theorem 3.2: The linear map \(R\) is a Rota-Baxter operator of weight \(\lambda\) on an N-dimensional q-Hom-tridendriform algebra if and only if, relations (35)-(45) hold.

\[
\begin{align*}
(\beta \circ x) &= \beta(m) = (x \cdot y \circ x + q \cdot x \cdot y) \circ \beta(m), \\
(\beta \circ y \circ y) &= \beta(m) = (x \cdot y \circ y + q \cdot x \cdot y) \circ \beta(m), \\
(\beta \circ x \cdot y) &= \beta(m) = (x \cdot y \circ x \cdot y + q \cdot x \cdot y) \circ \beta(m), \\
(\beta \circ y \circ y) &= \beta(m) = (x \cdot y \circ y + q \cdot x \cdot y) \circ \beta(m),
\end{align*}
\]

for any \(i, j, k, l, m, n \in \mathbb{N}\) with \(i, j, k, l, m, n \leq N\).

Remark 4.1: 1. When \(\alpha = \text{Id}_D\), we get the definition of bimodule over q-tridendriform algebra.

2. When \(\alpha = \beta = 0\), we recover bimodule over Hom-dendriform algebra [10].

3. When \(\alpha = \beta = 0\) and \(\alpha = \text{Id}_D\), we recover bimodule over dendriform algebra [2].

Example 4.1: a) Any q-Hom-tridendriform algebra is a bimodule over itself.

b) Let \((M_1, \cdot_1, \circ_1, \alpha_1)\) and \((M_2, \cdot_2, \circ_2, \alpha_2)\) be two bimodules over a q-Hom-tridendriform algebra \((D, \cdot_1, \circ_1, \alpha_1)\). Then the direct product \(M = M_1 \times M_2\) is a bimodule over the q-Hom-tridendriform algebra \(D\) with structure maps \(
\begin{align*}
\alpha(x, (m_1, m_2)) &= (\alpha_1(x, m_1), \alpha_2(x, m_2)), \\
\beta(x, (m_1, m_2)) &= (\beta_1(x, m_1), \beta_2(x, m_2)),
\end{align*}
\)

and \(\beta(m_1, m_2) = (\beta_1(m_1), \beta_2(m_2))\).

Theorem 4.1: Let \((M, \cdot, \circ, \alpha)\) be a bimodule over q-tridendriform algebra \((D, \cdot, \circ, \alpha)\). Let \(\alpha : D \to D\) be a q-tridendriform algebra endomorphism and \(\beta : M \to M\) be a linear map such that...
\[ f(x < m) = a(x) < b(m), \quad f(m < x) = b(m) < a(x), \quad f(b(m) < a(x)) = f(x < m) = a(x) < b(m), \quad f(m < x) = b(m) < a(x). \]

Define
\[ g : D \otimes M \to M \quad \text{and} \quad g : D \otimes M \to M \quad \text{and} \quad g : D \otimes M \to M \]
\[ x \otimes m \mapsto a(x) \otimes b(m), \quad x \otimes m \mapsto a(x) \otimes b(m), \quad x \otimes m \mapsto a(x) \otimes b(m). \]

Then, \((M, <, \beta, \lambda, \alpha, \beta, \beta)\) is a bimodule over the multiplicative q-Hom-tridendriform algebra \( D = (D, <, \beta, \lambda, \alpha, \beta) \), where \(<, \alpha, \beta, \lambda, \alpha, \beta, \beta\) are the terms occurring in Definition 4.1 may be written
\[ (x \alpha) \otimes (y \beta) = \beta^2(x \alpha \beta), \quad (x \alpha) \otimes (y \beta) = \beta^2(x \alpha \beta), \quad (x \alpha) \otimes (y \beta) = \beta^2(x \alpha \beta). \]

Thus, the proof finishes by using identities (46)-(66) whenever \( \alpha = Id_D \) and \( \beta = Id_M \).

Now we have the following series of lemmas.

It is known [25] that if \((A, \star, \alpha, R)\) is a Hom-associative Rota-Baxter algebra. Then, \((A, \star, \alpha, R)\) is a Hom-associative Rota-Baxter algebra, where \( x \star y = R(x) \cdot y + x \cdot R(y) + \lambda \cdot x \cdot y \). Then we have the following proposition.

**Lemma 4.1:** Let \((M, <, \beta, R)\) be a Hom-associative Rota-Baxter bimodule of weight \( \lambda \) over the Hom-associative Rota-Baxter algebra \((A, \star, \alpha, R)\). Define bilinear maps \( \langle M \otimes A \to M \) and \( \cdot : A \otimes M \to M \) by
\[ x \otimes m \mapsto R(x) \cdot m \quad \text{and} \quad m \otimes x \mapsto R(M) \cdot m \quad \text{and} \quad R(x) \cdot m = R(M) \cdot m \quad \text{and} \quad R(x) \cdot m = R(M) \cdot m. \]

Then \((M, <, \beta, R, M)\) is a Hom-associative Rota-Baxter bimodule of weight \( \lambda \) over the Hom-associative algebra \((A, \star, \alpha)\).

**Proof:** First, for any \( x \in A, m \in M \),
\[ R(x) \otimes R(M) = R^2(x) \cdot R(m) \quad \text{and} \quad R(x) \otimes R(M) = R^2(x) \cdot R(m) \quad \text{and} \quad R(x) \otimes R(M) = R^2(x) \cdot R(m). \]

Next, for any \( x, y \in A, m \in M \), one has
\[ (x \star y) \otimes R(M) = R^2(x) \cdot R(m) \quad \text{and} \quad R(x) \cdot R(M) = R^2(x) \cdot R(M). \]
\( x \triangleright m = x \triangleright m + q_m \triangleright x \), \( x \triangleright m = x \triangleright m \), \( x \triangleright m = x \triangleright m \), \( m \triangleright x = m \triangleright x \).

Then \((M, \langle \cdot, \cdot \rangle, \triangleright)\) is a Hom-dendriform bimodule over the Hom-dendriform algebra \((A, \triangleright, q, \alpha, \triangleright)\).

**Lemma 4.4:** [10] Let \((M, \langle \cdot, \cdot \rangle, \triangleright)\) be a bimodule over the Hom-dendriform \((D, \triangleright, \alpha, \triangleright)\). Define bilinear maps \( \langle M \otimes A \rightarrow M \) and \( \triangleright : A \otimes M \rightarrow M \) by

\[
x \langle m = x \triangleright m + x \triangleright m, \quad m \triangleright x = m \triangleright x + m \triangleright x
\]

Then \((M, \langle \cdot, \cdot \rangle, \triangleright)\) is a bimodule over the Hom-associative algebra \((A, \triangleright, +, q, \alpha)\).

Let us Denote by HARBM: the category of Hom-associative Rota-Baxter bimodules, q-HTDM : the category of Hom-tridendriform bimodules and HA : the category of Hom-associative bimodules. Then, the above discussion may be summarized in the following theorem.

**Theorem 4.2:** The following diagram is commutative

\[
\begin{array}{c}
\text{HARBM} \rightarrow \text{HAM} \\
\downarrow \downarrow \\
\text{q-HTDM} \leftrightarrow \text{HDM}
\end{array}
\]

**Proof:** The top horizontal arrow and the bottom horizontal arrow follow from Lemma 4.1 and Lemma 4.4 respectively. The left vertical arrow is established in Lemma 4.2 and the right vertical arrow is the functor constructed in Lemma 4.3.

**Corollary 4.3:** Let \((M, \langle \cdot, \cdot \rangle, \triangleright, R_M)\) be a Rota-Baxter bimodule of weight over the Hom-associative Rota-Baxter algebra \((A, \triangleright, a, R)\). Let us define bilinear maps

\[
x \langle m = x \triangleright R_M(m) + \lambda x \rangle m \quad \text{and} \quad x \triangleright m = R(x) \triangleright m \\
m \triangleright x = m \triangleright x + \lambda m \triangleright x \quad \text{and} \quad m \triangleright x = R_M(m) \langle x \rangle.
\]

Then, \((M, \langle \cdot, \cdot \rangle, \triangleright)\) is a bimodule over the Hom-dendriform algebra.

**Corollary 4.4:** Let \((M, \langle \cdot, \cdot \rangle, \triangleright, \alpha, \triangleright)\) be a bimodule over the q-Tridendriform algebra \((D, \langle \cdot, \cdot \rangle, \alpha, \triangleright)\). Let us define

\[
x \triangleright x = m \triangleright m + m \triangleright m + q_m \triangleright m, \\
m \triangleright x = x \triangleright m + m \triangleright m + q_m \triangleright m
\]

Then, \((M, \langle \cdot, \cdot \rangle, \triangleright, \alpha, \triangleright)\) is a bimodule over the Hom-associative algebra \((D, \triangleright, \alpha, \triangleright)\).

**Remark 4.2:** Whenever the Rota-Baxter relations are verified, in Lemma 4.3 (resp. Lemma 4.4) for \( \langle q, \triangleright \rangle \) and (resp. \( \langle \cdot, \cdot \rangle, \triangleright \)), then so is for \( \langle \cdot, \cdot \rangle \) and \( \triangleright \) (resp. \( \langle \cdot, \cdot \rangle, \triangleright \)).

**Further Discussion and Conclusions**

In this paper we give structure theorems of Rota-Baxter algebras over q-Tridendriform algebras and the corresponding bimodules, their twisting, and their connection with Hom-associative Rota-Baxter algebras and Hom-preLie algebras.

1. A similar analysis may be made for bimodules over flexible, admissible and alternative q-Tridendriform Rota-Baxter algebras. Also, the procedure to twist classical algebraic structures to obtain the Hom-version may be applied to Rota-Baxter opera-tor on Hom-Leibniz algebras and L-Hom-tridendriform algebra.

2. Thanks to Theorem 3.2 one may think of the classification of q-Tridendriform Rota-Baxter algebras of weight 0 and 1 (and the corresponding modules). But due to the size and the non-linearity of system (35)-(45), we hope to return to these questions elsewhere.

3. One of the issue is to establish graded case (and the corresponding modules) of the following commutative diagram

\[
\begin{array}{c}
q - \text{HTDA} \rightarrow \text{HA} \\
\downarrow \downarrow \\
\text{HDA} \rightarrow \text{HPLA}
\end{array}
\]


**References**

7. Szablowski BM, Classical r-matrix like approach to Frobenius manifolds WDVV equations and flat metrics. to appear in J Phys A.
20. Bakayoko I, Oh SQ. Color Hom-Lie bialgebras and color Hom-Poisson bialgebras.