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Centralizers of Commuting Elements in Compact Lie Groups

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Abstract

The moduli space for a flat *G*-bundle over the two-torus is completely determined by its holonomy representation. When *G* is compact, connected, and simply connected, we show that the moduli space is homeomorphic to a product of two tori mod the action of the Weyl group, or equivalently to the conjugacy classes of commuting pairs of elements in *G*. Since the component group for a non-simply connected group is given by some finite dimensional subgroup in the centralizer of an n-tuple, we use diagram automorphisms of the extended Dynkin diagram to prove properties of centralizers of pairs of elements in *G*.

Keywords: Moduli space; Lie groups; Representation theory; Characteristic classes; Centralizers

Introduction

Classifying the moduli space of gauge equivalence classes of flat connections on a principal *G*-bundle over a compact Riemann surface Σ_g of genus g is of interest from various perspectives. For example, Atiyah-Bott [1] proved that this moduli space is equivalent to the finite dimensional representation space $\{\mathrm{Hom}(\pi_1(\Sigma_g),G)\}/G$ by constructing a symplectic structure on the moduli space by symplectic reduction from the infinite-dimensional sympletic manifold of all connections. If A is a flat connection on a K-bundle over T^3 then the holonomy of A is defined by the conjugacy classes of commuting triples in K. In topological field theory, vacua of Yang-Mills theory correspond to flat G bundles. Every nontrivial triple on T^3 equals an additional quantum vacuum state and determines a distinct component of the moduli space $\mathcal{M}_G$. If the triple is of rank zero (rigid), then it is unique up to G-equivalence and every element can be conjugated into the maximal torus T for G. If the triple is not of rank zero, then there is an entire family of triples with elements lying on some smaller torus inside the centralizer $Z(x, y, z)$. Kac-Smilga [2] proved that computing the number of quantum vacuum states over T^3 is equivalent to classifying commuting triples in a simple, compact, simply connected Lie group G. Witten [3] proved that the number of extra quantum vacuum states for a flat principal G-bundle over a spatial 3-torus T^3 is the topological invariant called the Witten Index which is equal to g, the dual Coxeter number of the Lie group G.

Our primary motivation comes from Borel-Friedman-Morgan [4]. Given G as a compact, connected, semisimple Lie group, they proved that principal G-bundles z with flat connections over a maximal two torus T^2 are classified up to restricted gauge equivalence by classifying commuting pairs of elements in the simply connected covering $\tilde{G}$ of G that commute up to the center. The first invariant is the nontrivial characteristic class $[w]\in H^2(T^2, \pi_1(G))=\pi_1(G)$. By identifying $\pi_1(G)$ with a subgroup of the center $\mathcal{C}G$ we fix a topological type of the bundle by $w(z)=c \in \mathcal{C}G$. Since the characteristic class is completely defined by the holonomy representation $\rho:\pi_1(T^2)\cong \mathbb{Z}\times\mathbb{Z} \to G$ where the images of ρ commute then for any lifts $\tilde{x},\tilde{y}\in\tilde{G}$, we have $[\tilde{x},\tilde{y}]=[w]=c$. Elements with this property are called c–pairs or "almost commuting".

Definition 1.1: A pair of elements $x, y\in G$ commutes if $[x, y]=1$. A c-pair in the simply connected covering $\tilde{G}$ of G is a pair of elements (x, y) where $x, y\in G$ such that $[x, y]=1$ and $[\tilde{x},\tilde{y}]=c\in\mathcal{C}\tilde{G}$.

To understand why a flat bundle is determined by its holonomy representation note the following. Let G be a compact, connected and not necessarily simply connected Lie group and $\pi:\tilde{G}\to G$ the universal covering map. Certainly the choice of a lift $\tilde{x}$ is unique up to an element in $\mathrm{Ker}(\pi)\cong\pi_1(G)$ which is identified as a subgroup of the center of the simply connected covering. Extending this for a c-pair: for $k\in\mathrm{Ker}(\pi)$, $[\tilde{x},\tilde{y}]=[k\tilde{x},k\tilde{y}]=c$ because $k\in\mathrm{Ker}(\pi)$ commutes with every element in $\tilde{G}$ and is also invariant under the choice of x, y. We may define conjugation by $\tilde{g}\in\tilde{G}$ to be $\tilde{g}[\tilde{x},\tilde{y}]\tilde{g}^{-1}=[\tilde{g}\tilde{x}\tilde{g}^{-1},\tilde{g}\tilde{y}\tilde{g}^{-1}]$ satisfying $\pi(\tilde{g}\tilde{x}\tilde{g}^{-1})=g\pi(\tilde{x})g^{-1}=gxg^{-1}$. This lift is independent of the choice of $c\in\mathcal{C}\tilde{G}$ and thus our c-pair is well-defined.

For completeness, we recall some definitions found in [5] on Dynkin diagrams and root/coroot systems that we will use throughout the paper. Let Φ be a reduced irreducible root system for a compact connected Lie group G, and let $\Delta=\{a_1,\ldots,a_n\}$ be a choice of simple roots for G. Let d be the highest root of Φ with respect to Δ. Set $\tilde{a}=-d$ and let $\tilde{\Delta}=\Delta\cup\{\tilde{a}\}$ be the extended set of simple roots. Then $\Delta^{\vee}$ is the set of coroots $a^{\vee}$ inverse to each root $a\in\Delta$. If we define A to be the unique alcove containing the origin in the positive Weyl chamber associated to Δ then there is a bijection between the walls of A and $\tilde{\Delta}$. Therefore $\tilde{\Delta}$ is the set of nodes for the extended Dynkin diagram $\widetilde{D(G)}$. For each element $c\in\mathcal{C}G$ the differential $w_c\in\mathcal{W}$ of the action of the center on the alcove is a linear map normalizing $\tilde{\Delta}\subset\mathfrak{h}$ and the action of w_c on the nodes of $\widetilde{D(G)}$ is a diagram automorphism. Given a maximal torus $T\subset G$, denote Lie $(T)=\mathfrak{h}$ and the exponential map identifies T with $\mathfrak{h}/Q^{\vee}$ where $Q^{\vee}=\sum\mathbb{Z}a_i^{\vee}$ is the lattice associated to the coroots dual to a choice of simple roots $a_i\in\Delta$ for G. Denote the affine Weyl group by W_{aff}. The alcove is defined over the maximal torus $T\subseteq G$ as $A=\mathfrak{h}/W_{aff}(\Phi)\subseteq$ h where $W_{aff}(\Phi)$ acts simply transitively on the set of alcoves in the vector space V; thus there is an induced action of the center $\mathcal{C}G$ on the alcove A.

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Component Group of the Centralizer of Commuting Pairs

The work in [4] gave an explicit characterization of the moduli space of c-pairs in terms of the extended coroot diagram of a simply connected group G and the action of the Weyl group on that diagram. This beneficial relationship between the root/coroot system and holonomy plays an essential role. Let G be a compact connected Lie group. When G is disconnected, there is the appearance of a c-"1-chain" coming from a component group which may be a finite group of a certain order. A group G is reductive if any representation is a direct sum of irreducible representations.s Notice that when a group G is compact, it is equivalent to being reductive. We will use the following theorem by Borel.

Theorem 2.1 (Theorem 5): Let G be a compact, connected and simply connected Lie group. Then $Z(x)$ is connected.

The following demonstrates the relationship between the conjugacy classes for commuting pairs (x, y) and flat G-bundles over T^2:

Proposition 2.2: Assume that G is a compact, connected and simply connected Lie group. For any maximal torus S, we have that $\{\mathrm{Hom}(\pi_1(T^2, x), G)/G\}\to(S\times S)/\mathcal{W}$ is a homeomorphism.

Proof. Fix generators (γ_x, γ_y) for $\pi_1(T^2, x)$. Notice that we have a representation $\rho:\pi_1(T^2)\to G$ where $\rho(\gamma_x)=x, \rho(\gamma_y)=y$ and that these images define the commutator in G. In fact, the representation determines the commutator in the following sense. Let T be the maximal torus in G. Then for some $g\in G$, $gxg^{-1}\in T$ and $gyg^{-1}\in T$ since every element in G can be conjugated into the maximal torus. We want to show that both $x, y \in T$. To do this, define conjugation by $g\in G$ for the pair (x, y) by $g(x, y)g^{-1}=(gxg^{-1}, gyg^{-1})=(x', y')$ where $(x, y)\in T\times T$ and $(x', y')\in T\times T$. The fact that G is simply connected implies that $Z_G(x)$ is connected (2.1). Thus we may restrict to the connected component of the identity $Z^0(x)$. Since $x\in Z_G(x)$ we must show that $T\subseteq Z_G(x)$ because this would imply that both $x, y\in T$. By definition of the representation, the image $[x, y]=1$ so that $y\in T$ is conjugate to x which implies we may project y to an element $z_y\in\mathcal{W}$. If we conjugate the pair (x, y) by $(\zeta_y^{-1}g)\in G$, then $(\zeta_y^{-1}g)(x,y)(\zeta_y^{-1}g)^{-1}=\zeta_y^{-1}(x',y')\zeta\in T'\times T'$. Thus we have shown that $(\zeta_y^{-1}g)$ conjugates elements from $T\times T$ to $T'\times T'$ and $\zeta_y^{-1}g\in\mathcal{W}$.

Conversely, $\mathcal{W}$ acts by simultaneous conjugation on $S\times S$ so that if $g\in\mathcal{W}$ is a reflection, then $gS\in N_G(S)/S$. Thus $\mathcal{W}\times S\to S$ by $(gS, t)\mapsto gtg^{-1}$ and thus we have its action on the pair $\mathcal{W}\times(S\times S)\to S\times S$ by $(gS,t,h)\mapsto(gtg^{-1}, ghg^{-1})$. Define the commutator by $[t, h]=[gtg^{-1}, ghg^{-1}]$. Since elements in S commute, if $\langle\gamma_x\gamma_y\rangle$ generates $\pi_1(T^2)$ and $\rho(\gamma_x\gamma_y)=[gxg^{-1}, gyg^{-1}]=1$ then the holonomy determines the commutator and vice versa. Thus we have defined conjugacy of pairs by sending a pair homeomorphically to $S\times S$ because the representation modulo conjugation by G yields a commutator.

Given a commuting n-tuple $\bar{x}=(x_1,\ldots,x_n)$, the next corollary follows immediately because the fundamental group of the centralizer $Z(x_1)$ is trivial, and for in a simply connected group, the component group $\pi_0 Z(x_1)$ is contained in the fundamental group of the semisimple subgroup $\pi_1 DZ(x_1)$.

Corollary 2.3: When G is a group of type A_n, C_n every commuting n-tuple can be conjugated into the maximal torus T in G so that the moduli space has the form $\mathfrak{M}_G=T^n/\mathcal{W}$.

The corollary can also be seen directly as follows. If $\bar{x}=(x_1,\ldots,x_n)$ is a commuting n-tuple such that $[\tilde{x}_1,\tilde{x}_2]=c\in\mathcal{C}G$ and $[\tilde{x}_1,\tilde{x}_i]=1, i=2,3,\ldots,n,$ choosing $\tilde{x}_1$ in the alcove over the torus implies that $\tilde{x}_2$ projects to a Weyl element and therefore conjugates back into the maximal torus; every other element has trivial commutator and thus can be conjugated to the maximal torus. This also works when (x_i, x_j) for $1\le i\le j\le n$ is an arbitrary n-tuple because the lifts $[\tilde{x}_i,\tilde{x}_j]=c_{ij}\in\mathcal{C}G$ and for the cases of type A_n, C_n the center is generated by one cyclic element. Hence only one pair in the n-tuple determines what happens to the other elements in the n-tuple.

Corollary 2.4: Let $\bar{x}=(x_1,\ldots,x_n)$ be a commuting n-tuple and S a maximal torus in G. If G is simply connected then the component group $\pi_0 Z(S)$ is a subgroup of $\mathbb{Z}/n_i\mathbb{Z}$, where $n_i\le 6$ and corresponds to the coroot integer for $x\in G$ which is associated to the node in the extended Dynkin diagram $\widetilde{D(G)}$.

By [4], Lemma 3.1.5: for $x_n\in Z(x_1,\ldots,x_{n-1})$, $Stab_{W(\Phi(x_1,\ldots,x_{n-1}))}(x_n)\cong Stab_{\pi_1 DZ(x_1,\ldots,x_{n-1})}(\tilde{x}_n)$. Thus the component group $\pi_0 Z(x_1,\ldots, x_{n-1})$ is a subgroup of the fundamental group $\pi_1 Z(x_1,\ldots, x_{n-1})$, which in turn is a finite subgroup of the center $\mathcal{C}\tilde{Z}(x_1,\ldots,x_{n-1})$. If the fundamental group of the centralizer $Z(x_1)$ is trivial, then $Z(x_1, x_2)$ is a torus T^2 and hence $\pi_0 Z(x_1, x_2)$ is trivial. So suppose that $\pi_1 DZ(x_1)\neq\{1\}$. If G is simply connected, then $Z(x_1)$ is connected and thus $\pi_0 Z(x_1, x_2)\subseteq\pi_1 DZ(x_1)$. Even if G is not simply connected but still connected, choosing x_2 to lie in the connected component $Z^0(x_1)$ of the centralizer of $Z(x_1)$ will yield the same result.

Proposition 2.5: Let G be simply connected and let $\Delta=\{a_1,\ldots, a_n\}$ be a choice of simple roots. Let $\Delta_x=\{\tilde{a},a_1,\ldots,a_k\}, k\le n,$ be a choice of simple roots for $Z(x)$ and let $\mathrm{h}(x)\subseteq\mathrm{h}$ be the real linear span of the coroots dual to the roots in Δ_x. Then there is an exact sequence $1\to Q^\vee(x)\to Q^\vee\cap\mathrm{h}(x)\to\mathbb{Z}/n_i\mathbb{Z}\to 1$.

Proof. By definition of the fundamental group, $n_i=gcd\,(g_{k+1},\ldots, g_n)$ knowing that all the coroot integers for both the classical and exceptional groups are less than or equal to six, $n_i\le 6$. Dividing each of the coroot integers in any group G by n_i we may define a new integer $g_{r'}=g_r/n_i$ for $r>k$. By definition, this element will have order n_i in the central subgroup $Q^\vee\cap\mathfrak{h}/Q^\vee(x)$ and thus is a generator for the cokernel. Hence $Q^\vee\cap\mathfrak{h}/Q^\vee(x)\cong\mathbb{Z}/n_i\mathbb{Z}$.

Proposition 2.6: For an arbitrary compact, connected simple group G and for a commuting n-tuple $\bar{x}=(x_1,\ldots,x_n)$, the component group of the centralizer of the n-tuple can be defined in terms of the roots as:

$$\pi_0 Z(\bar{x})=\frac{Stab_{L/Q^\vee}(\bar{x})}{\mathcal{W}(\Phi(\bar{x}))}.$$

Proof. If G is not simply connected, then under complexification $T_C=\mathfrak{h}/L$, where $Q^\vee\subseteq L\subseteq P^\vee$. If $\Phi(x)$ is a subset of roots which annihilate x we must determine how $Stab_{\mathcal{W}}(x)$ is defined with respect to this smaller subset of roots. If x corresponds to some node a_i in the extended Dynkin diagram such that $g_{a_i}\neq 1$ then $\Phi(x)=\{a_k\in\tilde{\Delta}\mid k\neq 1\}$. Since $Q\subseteq L\subseteq P^\vee$ we have the nesting of tori $\mathrm{h}/P^\vee\subseteq\mathfrak{h}/L\subseteq\mathfrak{h}/Q^\vee$. Thus for the lift $\mathfrak{h}/\mathrm{L}\to\mathfrak{h}/Q^\vee$ sending $x\mapsto\tilde{x}$ its kernel consists of all the roots in L not in $Q^\vee$ i.e. $L/Q^\vee$. Therefore the roots which annihilate x are the same as those annihilating $\tilde{x}$.

Let $\mathcal{W}(\Phi(x))$ be a subgroup in $\mathcal{W}$ defined by a subroot system when viewed as characters which annihilate x. The faithful action of $\mathcal{W}_{aff}$ on $\mathfrak{h}/Q^\vee$ yields a split exact sequence $1\to P^\vee/Q^\vee\to\mathcal{W}_{aff}\to\mathcal{W}\to 1$ and since the kernel is central, $\mathcal{W}_{aff}=P^\vee/Q^\vee\times\mathcal{W}$ is a direct product because the action of the Weyl group is trivial on the center. Restrict the Weyl group to $L:\mathcal{W}_{aff}^L=L/Q^\vee\times\mathcal{W}$. The torus action of $L/Q^\vee$ on $\mathfrak{h}/Q^\vee$ provides the quotient $(\mathrm{h}/Q^\vee)/(L/Q^\vee)=\mathfrak{h}/L$. This implies that $Stab\mathcal{W}(x)\subseteq LQ^\vee\times\mathcal{W}$ and the projection $\pi:(\mathrm{h}/Q)\to(L/Q)$ satisfies $\pi^{-1}(\tilde{x})=x$ as the unique lift to the alcove. Therefore we may define $Stab_{L/Q^\vee}(x)=Stab_{\mathcal{W}_{aff}}(\tilde{x})$ in the sense that the roots which annihilate $\tilde{x}$ can be used to define a subset $S\subseteq L/Q^\vee$, where $S=\pi_0Z(x_1,\ldots,x_n)$. This allows for a component group larger than the fundamental group and therefore it is not necessarily cyclic. Since $S\subseteq\mathcal{C}G$ it induces a well-defined cyclic permutation on the vertices in the alcove and its fixed space $\mathfrak{h}^S$ may be something other than the barycenter. L is defined as follows. For $Z(x_1)$ the vector space is $\mathfrak{h}$ and its coroot lattice is the entire $Q^\vee$. Because we are considering commuting elements, we choose $x_2\in Z(x_1)$ to lie in $\mathfrak{h}(x_1)/L_1$ where $\mathfrak{h}(x_1)$ is the vector space associated to $DZ(x_1)$. Because $Z(x_1)$ is not necessarily connected, the associated lattice is $Q^\vee\subseteq L_1\subseteq P^\vee$. By induction, the element $x_n\in\mathfrak{h}(x_1,\ldots,x_{n-1})/L_{n-1}$ and $Q^\vee\subseteq L_1\subseteq\cdots\subseteq L_{n-1}\subseteq P^\vee$ so that $L=L_{n-1}$ as the associated lattice to the centralizer of the prior $n-1$ elements. From the definition of these lattices, when they are quotiented out by the coroot lattice, they will either be a cyclic subgroup of the center whose order divides the order of the center or will be the entire center. Thus we have the above conclusion since $S\subseteq L/Q^\vee\subseteq P^\vee/Q^\vee$.

Properties of Centralizers

In general, the component group of the centralizer of an ordered n-tuple is some subquotient of the Weyl group and lies in the connected component $Z^0(x_1,\ldots,x_n)$. Let $\tilde{\Delta}=\{\tilde{a},a_1,\ldots,a_n\}$ be the set of extended simple roots for a Lie group G. Any closed subset of the extended simple roots for G gives a subdiagram of the extended diagram. We are interested in the subset of roots Δ_x that annihilate the n-tuple. In particular, the simple root system for the centralizer $Z(x)$ for any $x\in G$ is defined as by $\Delta_x=\{a\in\Delta|\alpha(x)\in\mathbb{Z}\}$ which has an associated Weyl group $\mathcal{W}(\Phi(x))$.Let $\overline{x}=(x_1,\ldots,x_n)$ be a commuting n-tuple. Any element $x\in\pi_0Z_G(\overline{x})$ can be represented by $g\in Z(\overline{x})$ since g normalizes $Z(\overline{x})$ and therefore via conjugation defines a map $\{x\}\times I\to G$ defining the path components of G. Let S_{Δ_x} be the maximal torus in the centralizer $Z(x_1,\ldots,x_n)$ generated by the roots in Δ_x given by its Lie algebra $\mathfrak{s}=\bigcap_{a\in\Delta_x}\mathrm{K}er(a)$. Since G is reductive, there is a standard decomposition given by $G=(\mathcal{C}G)^0\times_F DG$ where $(\mathcal{C}G)^0$ is a central torus in the semisimple subgroup DG of G. and $F=(\mathcal{C}G)^0\cap DG$ is a finite subgroup of the center of DG. Thus we get a decomposition of the the centralizer into $Z(S_{\Delta_x})=S_{\Delta_x}\times_{F_{\Delta_x}}DZ(S_{\Delta_x})$. The obstruction to a lift will lie in the center of $Z(x_1)$ which means that given the equivalence classes above, the obstruction will lie in F. The action of F must be nontrivial in order to get the semidirect product by F.

Example 3.1: Let $G=SU(2)$. There is a nontrivial central action $c\in\mathbb{Z}_2$ on the alcove over A_1 given by switching the two vertices, leaving the barycenter as the only fixed point. As described above, we may consider $SU(2)\times_{\mathbb{Z}_2}SU(2)$ where the nontrivial central element acts "diagonally" on each $SU(2)$ component, switching the vertices of the alcove over each copy of A_1. We denote by $A_1\times A_1$ the join of the alcoves over each A_1. In this case, the join of the two 1-simplices is a 2-simplex given as a square with the barycenter $b=\{b_1, b_2\}$ as the only fixed point under the central action. The join can be thought of as the Minkowski sum of two simplices: $S_1+S_2=\{s_1+s_2: s_1\in S_1, s_2\in S_2\}$.

In order to determine the component group of the centralizer of an n-tuple in a non-simply connected group we note that the finite diagonal subgroup contained in the center of each centralizer $Z(x_1,\ldots x_k)$, for some k, at some point becomes the component group and therefore defines the singularities in the moduli space. For the classical groups, $Z(x_1)$ will be a product of type A_n, B_n or D_n and for the exceptional groups, $Z(x_1, x_2)$ will be of type A_n, D_n. Therefore, it sufficies to consider the diagonal group action of the fundamental group $\pi_1DZ(x_1)$ on groups of these types. Since the fundamental group is a subgroup of the center of the simply connected covering, for type B_n we only consider the $\mathbb{Z}/2\mathbb{Z}$ action on the alcove given by flipping two vertices; the action of any higher order central cyclic group is trivial.

Example 3.2: Consider a group of type B_3. Select $\tilde{x}$ to correspond to the vertex in the 3-simplex associated to the trivalent node $\alpha_2^\vee$. The centralizer $Z(x)$ is the set of those elements which annihhilate x. In the Lie algebra, these elements are precisely the generators for the maximal torus $Lie(S)=\mathfrak{h}$ in the Lie algebra $LDZ(x)$. When we remove the node $\alpha_2^\vee$ from the diagram, we are left withn 3 nodes, each orthogonal to each other. Thus we get $LDZ(x)=SU(2)\times SU(2)\times SU(2)$. By [6], if a root system has a node x with torsion prime p then there exists a diagonal element $c\in\mathcal{C}Z(x)$ of order p. For B_3, $p=2$ and hence the finite diagonal subgroup is $\Delta=(-1,-1,-1)$ of order 2 in the center $\mathcal{C}\widetilde{DZ}(x)=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Δ acts on $LDZ(x)$ by flipping the two vertices corresponding to each copy of $SU(2)$. We write $DZ(x)=LDZ(x)/\Delta=\times_3 SU(2)/\mathbb{Z}_2$.

In order to obtain a commuting triple, the choice for y must come out of $DZ(x)$ since $y\in Z(x)$ normalizes $Z(x)$. If $\tilde{y}$ is chosen so that it does not lie in the fixed point space under the diagonal action, the component group $\pi_0Z(x, y)$ is trivial. However, if $\tilde{y}$ lies somewhere in the fixed space under the action of the center, there will be a nontrivial component group $\pi_0Z(x,y)$. The alcove for $SU(2)=T/\mathcal{W}=T^1/\mathbb{Z}_2$ is a 1-simplex. Thus the 3-simplex determined by the join of three 1-simplices is a cube and is $DZ(x)$. If $\tilde{y}$ is in the interior of the cube, then since $\mathbb{Z}_2$ stabilizes y, and we have $Z(x, y)=T^3\times\mathbb{Z}_2$. The maximal torus in $Z(x, y)$ is $(\mathbb{Z}_2)^3$. Select $z\in\mathbb{Z}_2$ because we require $[y, z]=1$ but $[\tilde{y},\tilde{z}]=c\in\mathcal{C}G$. Therefore, $Z(x, y, z)=(\mathbb{Z}_2)^4$.

Notice that the centralizer of a commuting triple will depend on the choices for each of the elements x, y, and z. If instead, we select $\tilde{x}$ to correspond to the node $\alpha_3^\vee$, then the Lie algebra of the semisimple part of the centralizer is $LDZ(x)=SU(4)$. The real dimension of the maximal torus is zero thus $Z(x)=DZ(x)$. The diagonal element must be in the center $\mathcal{C}\,SU(4)=\{\omega I:\omega^4=1\}$. In this case, Δ is simply the trivial action (multiplication by +1).

Definition 3.3: Define the rank $rk(x_1,\ldots,x_n)$ of an n-tuple to be the rank of $Z(x_1,\ldots,x_n)$. An n-tuple has rank zero if and only if $Z(x_1,\ldots,x_n)$ is a finite group. A c-pair (x, y) is in normal form with respect to the maximal torus T in the alcove A if $x\in T$ is the image under the exponential map of $\tilde{x}\in\mathfrak{h}^c$ and $y\in N_G(T)$ projects to $w_c\in\mathcal{W}$. Note $w_c\in W$ is the differential action of $c\in\mathcal{C}G$ that, as a group of affine isometries of the Lie algebra t of the maximal torus T normalizes the alcove A.

Example 3.4: Consider $G=SU(3)$ and $x\in T$, the maximal torus of G. Then the centralizer $Z(x)=T$. In the Lie algebra of G, select $\tilde{x}$ to be regular (in the interior of the alcove) so that $Z(\tilde{x})$ is a connected, abelian, reductive subgroup $\{H\in\mathfrak{h}:[\mathfrak{h},\tilde{x}]=1\}$. By choosing $\tilde{y}$ to be regular, $Z(x, y)=Z(x)\cap Z(y)=\mathcal{C}G$ and hence (x, y) is a c-pair of rank zero.

Remark 0.1: For $c\in\mathcal{C}G$, denote by S_c the torus in T fixed under the action of the center. The choice of roots $\Delta(c)=\{a\in\Delta|r_a\notin Z\}$ where $c=\exp(\lambda)$ for $\lambda'\sum_{a\in\Delta}r_{a'}a'\in\mathfrak{t}$ defining the fixed subtorus $S_c\subset T$ is independent of the choice of lift λ, let $\lambda'\sum_{a\in\Delta}r_{a'}a'\in\mathfrak{t}$ be such that $\exp(\lambda')=c'$. Then under exp: $\mathfrak{t}\to T$, the kernel of this map is an integral lattice defined with respect to T. Namely, $\mathrm{Ker}(exp)Q^\vee$, thus for $\lambda-\lambda'\in\mathrm{Ker}(\pi)$ this implies that $r_a-r_{a'}\in Z$ and hence $\Delta(c)=\Delta(c')$ if and only if

$r_a - r_{a'} \equiv 0(\mathrm{mod}\, Z)$ which we have since $r_a - r_{a'} \in Q^\vee$. Therefore, $\Delta(c)$ depends only on the choice $c \in \mathcal{C}G \cong P^\vee/Q^\vee$ and any two elements in the Lie algebra t differ by an element in the coroot lattice $Q^\vee$. By definition of a c– pair of rank zero, $c \in DZ(S_c)$. Thus the moduli space is precisely $\mathfrak{M}=(T\times T)/W(T,G)$. It certainly will not be true that for a general non-simply connected group that every element of a commuting n-tuple can be put inside the maximal torus.

We show that the fundamental group of the centralizer is finite cyclic by using diagram automorphisms.

Proposition 3.5: Under a cyclic permutation of the vertices in the extended root diagram of a group of type A_n where the permutation is given by the fundamental group $\pi_1 DZ(x_1)=\mathbb{Z}_k$, the quotient space has the form:

$$A_n / \mathbb{Z}_k \cong (\underset{k}{\times} A_{l-1}) \times T^{k-1} \rtimes \mathbb{Z}_k$$

Where $n+1=kl$, $\pi_1 DZ(x_1) \cong \mathbb{Z}_k$, with $1 \le k \le 6$, and $\underset{k}{\times} A_{l-1}$ is the product of k groups of type A_{l-1} or equivalently as the join of k,(l–1)-simplices.

Example 3.6: The diagram quotient space $A_5/\mathbb{Z}_2 = \{SU(3)\times SU(3)\times_{\mathbb{Z}_3} \times T^1\} \rtimes \mathbb{Z}_2$ has a nontrivial action of $\mathbb{Z}_2$ on $A_2 \times A_2$ by switching the vertices in the alcove over each A_2 and the outer automorphism acts by switching the two copies of A_2.

Proof. Since any inner automorphism of type A_n is dihedral, it is either a rotation or a reflection. Consider the cyclic permutation τ given by rotation. (note: this is an element of a group of affine automorphisms of a vector space which normalizes the alcove of a root system on that vector space. Such automorphisms are equivalent to diagram automorphisms of the extended Dynkin diagram of the root system.) If $\tau \in \mathbb{Z}_k$ has order n+1 then the fixed point set is simply the barycenter and thus $\mathfrak{h}^\tau=\{0\}$. If n is odd then τ may have order $k \mid (n+1)$. If $k = \frac{n+1}{2}$ then either the barycenter is the only fixed point or the fixed point set is the join of type A_{2k-1} or there is a rotation subgroup of τ of order exactly k which implies it is an involution of the extended diagram which fixes two vertices and thus the quotient coroot diagram is a product of type A_1. The $\mathbb{Z}_2$ action on the alcove over A_1 is simply to switch the two vertices leaving the barycenter fixed.

Specifically, if n +1=kl then every node in the extended diagram included in this k-orbit is nonzero which leaves the quotient coroot diagram as the join of k,(l–1)-simplicies with the barycenter (since the barycenter is the only fixed space under the action of the full center) times the remaining torus and semidirect product with rotation group. In terms of extended roots in the diagram, if $\tilde{\Delta} = \{\tilde{a}, a_1, \ldots, a_n\}$ is the set of simple roots for A_n then the quotient space $A_n/\mathbb{Z}_k$ is defined by the elements in the orbit, $\tilde{\Delta}/\mathbb{Z}_k = \{\tilde{a}, a_l, a_{2l}, \ldots, a_{(k-1)l}\}$. Thus the gaps between the nodes are of length (l–1). Therefore, the fixed space will be given by $A_n/\mathbb{Z}_k = (\underset{k}{\times} A_{l-1}) \times T^{k-1} \rtimes \mathbb{Z}_k$. What we have shown is that in A_n the $\mathrm{Stab}_\tau(\underset{k}{\times} A_{l-1}) = \mathbb{Z}_k$. The fact that $\pi_1 DZ(x_1) \cong \mathbb{Z}_k$ where $1 \le k \le 6$, follows directly from looking at the coroot integers for all the extended Dynkin diagrams.

When the centralizer of an element is in a group other than G, we will denote the group in the centralizer notation.

Proposition 3.7: Let G be a simple group of dimension n. The centralizer $Z(A_k \times_F T^{n-k}) = Z_{A_k} \times_F T^{n-k}$.

Proof. Given $A_k \times_F T^{n-k}$ and $\zeta \in F$, for any element $[A, t]=[A\zeta, \zeta^{-1}A] \in A_k \times_F T^{n-k}$ its centralizer is $Z([A,t])=\{[B,s]:[A,t][B,s]=[B,s][A,t]\}$. This implies that $[AB,ts]=[BA,st]$. But since $st=ta \in T$, $AB=BA$. Therefore, $Z([A,t]) = Z_{A_k}(A) \times_F T^{n-k}$ which is connected and thus Proposition 3.5 applies. The conclusion follows because the components in the almost direct product are simply connected.

Proposition 3.8: Given $G_1 \times_F G_2$ where G_1, G_2 are subgroups of G, $F \subseteq \mathcal{C}G_1$ and $F \subseteq \mathcal{C}G_2$ and $F \in (DG_1 \cap DG_2)$. Then for $[a, b] \in G_1 \times_F G_2$,

$$\{1\} \to F \to Z([a,b]) \xrightarrow{\pi} Z_{G_1}(a) \times_F Z_{G_2}(b) \to F.$$

Proof. Consider the map $G_1 \times_F G_2 \xrightarrow{\pi} G_1/F \times G_2/F$. The kernel is ker($\pi$)=\{[$c$, d]: c,$d \in F$\}. Thus we have the injective map $\{1\} \to G_1 \times_F G_2 \xrightarrow{\pi} G_1/F \times G_2/F$. By definition of the centralizer of an element in $G_1 \times_F G_2$

$$Z([a,b])=\{[c, d] : [ac, bd]=[ca, db]\}$$
$$= \{\exists f_1, f_2 \in F, [acf_1, bd] \sim [ac, f_1^{-1}bd] = [caf_2, db] \sim [ca, f_2^{-1}db]\}$$
$$= \{[c,d] : [a,c] = f[b,d] = f^{-1}, f = f_1 f_2^{-1}, [a,c][b,d] = 1\}$$

This demonstrates that the coker of π is F and that π is not surjective. Therefore, $\frac{\pi^{-1}(Z([a]) \times Z([b]))}{F} = Z([a,b])$. Note also that by the definition of the centralizer of $[a, b] \in G_1 \times_F G_2$, that the generalized Stiefel-Whitney class [7] is $w_2(a, c)$=– $w_2(b,d) \in F$. Hence $w_2 : H^2(T^*) \to Z_n$ defines an obstruction.

Corollary 3.9: Following propsition 3.8, if G_1=T for some torus and G_2 is of type A_r then:

$$F \to Z_{A_r}(\tilde{A}) \xrightarrow{\pi} Z_{A_r/F}([A]) \to F \to \{1\}.$$

Proof. Given a sequence $\{1\} \to T^k \to T^k \times_F A_r \xrightarrow{\pi} A_r/F$ inside $Z_G([t, A])$ we have that $[t, A][s, B]=[ts,AB]=[st,BA]=[s, B][t,A]$ and in $Z_{A_r}([A]) = \{[B] : [A,B] = [B,A]\}$ which implies that $Z_G([t,A]) \to \pi^{-1}(Z_{A_r}([A]))$ so that $AB=BA\zeta$ for $\zeta \in F$. Thus they are equal up to an element in the finite group. Therefore we have $Z_G([t,A]) \to \pi^{-1}(Z_{A_r}([A])) \to F$. Suppose that $[A] \in A_r/F$ and consider its lift $\tilde{A} \in A_r$ arbitrary. Then:

$$F \to Z_{A_r}(\tilde{A}) \xrightarrow{\pi} Z_{A_r/F}([A]) \to F \to \{1\}$$

because the kernel is Ker(π)=F and from what we have already deduced, $AB=BA\zeta$ for $\zeta \in F$. Hence:

$$Z_G([t,A]) = \frac{\pi^{-1}(Z_{A_r}([A]))}{F}.$$

We used the simply connected component as follows. If we consider $[B]$ such that there exists a $\tilde{B}$ with $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, then mulitiplication of the equivalence classes is $[s,\tilde{B}][t,\tilde{A}] = [st, \tilde{B}\tilde{A} = [ts, \tilde{A}\tilde{B}]$. Since $\pi_1(G)$=\{1\} when we lift to the universal covering we can say that for $[t,A] \in \tilde{G} = T^k \times_F G$ then $Z_{\tilde{G}}([t,A]) = T^k \times_F Z(A)$ and more importantly that $Z_{\tilde{G}}([t,A])$ is connected.

Corollary 3.10: Consider a subgroup in G of the form $A_k \times_F A_r$, for r + k=n+1, then the centralizer of an element $[a, b] \in A_k \times_F A_r$, for r + k=n +1, is $Z_G([a,b]) = \frac{(Z_{A_k}(a) \times_{F'} Z_{A_r}(b))}{F}$, where $F' = \mathcal{C}DZ_{A_k} \cap \mathcal{C}DZ_{A_r} \supseteq F$.

Corollary 3.11: Consider a subgroup in G of the form $A_k \times_F D_{n-k}$. then the centralizer of an element $[a,b] \in A_k \times_F D_{n-k}$ is of the form:

$$Z_G([a,d]) = \frac{(Z_{A_k}(a) \times_{F'} Z_{D_{n-k}}(d))}{F},$$

where $F' = \mathcal{C}DZ_{A_k} \cap \mathcal{C}DZ_{D_{n-k}} \supseteq F$.

It does not necessarily follow that $\pi_0 Z(x_1,\ldots,x_n) \subset \pi_1 DZ(x_1)$ because $DZ(x_1)$ is not necessarily connected. The fact that for G of type D_n that $\pi_1 D_n = \mathcal{C}\, D_n \cong \mathbb{Z}\times\mathbb{Z}$ and that the characteristic class for a principal G-bundle over T^n lies in $H^2(T^n;\pi_1(G)) \cong \mathbb{Z}\times\mathbb{Z}$ means that there is a possibility that the component group of the centralizer of an n-tuple inside a group of type D_n will not be finite cyclic.

Conclusion

We have shown that for an arbitrary compact, connected simple group G and a commuting n-tuple, that the component group of the centralizer can be defined in term of the roots and therefore, we may use diagram automorphisms to define the moduli space of commuting elements. We have also seen that the centralizer of a commuting n-tuple is determined by the order and choice of elements. For example, we can generate a commuting n-tuples of rank zero of arbitrary length by finding a nontrivial triple, say, and then adding arbitrarily many elements from the torus, thereby not altering the centralizer.

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