Journal of Generalized Lie Theory and Applications Vol. 2 (2008), No. 2, 65–78

Classical elliptic current algebras. I

Stanislav PAKULIAK [a], *Vladimir RUBTSOV* [b], *and Alexey SILANTYEV* [c]

[a] *Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia*
Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia
E-mail: pakuliak@theor.jinr.ru

[b] *Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia*
Départment de Mathématiques, Université d'Angers, 2 Bd. Lavoisier, 49045 Angers, France
E-mail: Volodya.Roubtsov@univ-angers.fr

[c] *Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia*
Départment de Mathématiques, Université d'Angers, 2 Bd. Lavoisier, 49045 Angers, France
E-mail: silant@tonton.univ-angers.fr

Abstract

In this paper we discuss classical elliptic current algebras and show that there are two different choices of commutative *test function algebras* related to a complex torus leading to two different elliptic current algebras. Quantization of these classical current algebras gives rise to two classes of quantized dynamical quasi-Hopf current algebras studied by Enriquez, Felder and Rubtsov and by Arnaudon, Buffenoir, Ragoucy, Roche, Jimbo, Konno, Odake and Shiraishi.

In memory of Leonid Vaksman

1 Introduction

Classical elliptic algebras are "quasi-classical limits" of quantum algebras whose structure is defined by an elliptic R-matrix. The first elliptic R-matrix appeared as a matrix of Boltzmann weights for the eight-vertex model [3]. This matrix satisfies the Yang-Baxter equation using which one proves integrability of the model. An investigation of the eight-vertex model [4] uncovered its relation to the so-called generalized ice-type model – the Solid-On-Solid (SOS) model. This is a face type model with Boltzmann weights which form a matrix satisfying a *dynamical* Yang-Baxter equation.

In this paper we restrict our attention to *classical current algebras* (algebras which can be described by a collection of currents) related to the classical r-matrices and which are quasi-classical limits of SOS-type *quantized elliptic current algebras*. The latter were introduced by Felder [13, 14] and the corresponding R-matrix is called usually a *Felder R-matrix*. In *loc. cit.* the current algebras were defined by dynamical RLL-relations. At the same time Enriquez and one of authors (V.R.) developed a theory of quantum current algebras related to arbitrary genus complex curves (in particular to an elliptic curve) as a quantization of certain (twisted) Manin pairs [9] using *Drinfeld's new realization* of quantized current algebras. Further, it was shown in [7] that the Felder algebra can be obtained by twisting of the Enriquez-Rubtsov elliptic algebra. This twisted algebra will be denoted by $E_{\tau,\eta}$ and it is a quasi-Hopf algebra.

Originally, the dynamical Yang-Baxter equation appeared in [16, 13, 14]. The fact that elliptic algebras could be obtained as quasi-Hopf deformations of Hopf algebras was noted first in a special case in [2] and was discussed in [15]. The full potential of this idea was realized in papers [1] and [17]. It was explained in these papers how to obtain the universal dynamical Yang-Baxter equation for the twisted elliptic universal R-matrix from the Yang-Baxter equation for the universal R-matrix of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$. It was also shown that the image of the twisted R-matrix in finite-dimensional representations coincides with SOS type R-matrix.

Konno proposed in [20] an RSOS type elliptic current algebra (which will be denoted by $U_{p,q}(\hat{\mathfrak{sl}}_2)$) generalizing some ideas of [19]. This algebra was studied in detail in [18] where it was shown that commutation relations for $U_{p,q}(\hat{\mathfrak{sl}}_2)$ expressed in terms of L-operators coincide with the commutation relations of the Enriquez-Felder-Rubtsov algebra up to a shift of the elliptic module by the central element. It was observed in [18] that this difference of central charges can be explained by different choices of contours on the elliptic curve entering in these extensions. In the case of the algebra $E_{\tau,\eta}$ the elliptic module is fixed, while in the case of $U_{p,q}(\hat{\mathfrak{sl}}_2)$, $p = e^{i\pi\tau}$, it turn out to be a dynamical parameter shifted by the central element. Commutation relations for these algebras coincide when the central charge is zero, but the algebras themselves are different. Furthermore, the difference between these two algebras was interpreted in [8] as a difference in definitions of half-currents (or Gauss coordinates) in L-operator representation. The roots of this difference are related to different decomposition types of so-called *Green kernels* introduced in [9] for quantization of Manin pairs: they are expanded into Taylor series in the case of the algebra $E_{\tau,\eta}$ and into Fourier series for $U_{p,q}(\hat{\mathfrak{sl}}_2)$.

Here, we continue a comparative study of different elliptic current algebras. Since the Green kernel is the same in both the classical and the quantum case we restrict ourselves only to the classical case for the sake of simplicity. The classical limits of quasi-Hopf algebras $E_{\tau,\eta}$ and $U_{p,q}(\hat{\mathfrak{sl}}_2)$ are *quasi-Lie bialgebras* denoted by $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ respectively. We will give an "analytic" description of these algebras in terms of distributions. Then, the different expansions of Green kernels will be interpreted as the action of distributions on different test function algebras. We will call them *Green distributions.* The scalar products for test function algebras which define their embedding in the corresponding space of distributions are defined by integration over different contours on the surface.

Let us describe briefly the structure of the paper. Section 2 contains some basic notions and constructions which are used throughout the paper. Here, we remind some definitions from [9]. Namely, we define test function algebras on a complex curve Σ, a continuous non-degenerate scalar product, distributions on the test functions and a generalized notion of Drinfeld currents associated with these algebras and with a (possibly infinite-dimensional) Lie algebra $\mathfrak{g}$. Hence, our currents will be certain $\mathfrak{g}$-valued distributions. Then we review the case when $\mathfrak{g}$ is a loop algebra generated by a semi-simple Lie algebra $\mathfrak{a}$. We also discuss a centrally and a co-centrally extended version of $\mathfrak{g}$ and different bialgebra structures. The latter are based on the notion of Green distributions and related half-currents.

We describe in detail two different classical elliptic current algebras which correspond to two different choices of the test function algebras (in fact they correspond to two different coverings of the underlying elliptic curve).

Section 3 is devoted to the construction and comparison of classical elliptic algebras $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. In the first two subsections we define elliptic Green distributions for both test function algebras. We pay special attention to their properties because they manifest the main differences between the corresponding elliptic algebras. Further, we describe these classical elliptic algebras in terms of the half-currents constructed using the Green distributions. We can see how the half-currents inherit the properties of Green distributions. In the last subsection we show that the half-currents describe the corresponding bialgebra structure. Namely, we recall the universal classical r-matrices for both elliptic classical algebras $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ and make explicit their relation to the L-operators. Then, the corresponding co-brackets for half-currents are expressed in a matrix form via the L-operators.

In the next paper [21] we will describe different degenerations of the classical elliptic current algebras in terms of degenerations of Green distributions. We will discuss also the inverse problem of reconstruction of the trigonometric and elliptic classical r-matrices from the rational and trigonometric r-matrices using approach of [11].

2 Currents and half-currents

Current realization of the quantum affine algebras and Yangians was introduced by Drinfeld in [5]. In these cases the currents can be understood as elements of the space $\mathcal{A}[[z,z^{-1}]]$, where $\mathcal{A}$ is a corresponding algebra. Here we introduce a more general notion of currents suitable even for the case when the currents are expressed by integrals instead of formal series.

Test function algebras. Let $\mathfrak{K}$ be a function algebra on a one-dimensional complex manifold Σ with a point-wise multiplication and a continuous invariant (non-degenerate) scalar product $\langle\cdot,\cdot\rangle\colon \mathfrak{K}\times\mathfrak{K}\to\mathbb{C}$. We shall call the pair $(\mathfrak{K},\langle\cdot,\cdot\rangle)$ *a test function algebra.* The non-degeneracy of the scalar product implies that the algebra $\mathfrak{K}$ can be extended to a space $\mathfrak{K}'$ of linear continuous functionals on $\mathfrak{K}$. We use the notation $\langle a(u),s(u)\rangle$ or $\langle a(u),s(u)\rangle_u$ for the action of the distribution $a(u)\in\mathfrak{K}'$ on a test function $s(u)\in\mathfrak{K}$. Let $\{\epsilon^i(u)\}$ and $\{\epsilon_i(u)\}$ be dual bases of $\mathfrak{K}$. A typical example of the element from $\mathfrak{K}'$ is the series $\delta(u,z)=\sum_i\epsilon^i(u)\epsilon_i(z)$. This is a delta-function distribution on $\mathfrak{K}$ because it satisfies $\langle\delta(u,z),s(u)\rangle_u=s(z)$ for any test function $s(u)\in\mathfrak{K}$.

Currents. Consider an infinite-dimensional complex Lie algebra $\mathfrak{g}$ and an operator $\hat{x}\colon\mathfrak{K}\to\mathfrak{g}$. The expression $x(u)=\sum_i\epsilon^i(u)\hat{x}[\epsilon_i]$ does not depend on a choice of dual bases in $\mathfrak{K}$ and is called a current corresponding to the operator $\hat{x}$ ($\hat{x}[\epsilon_i]$ means an action of $\hat{x}$ on ϵ_i). We should interpret the current $x(u)$ as a $\mathfrak{g}$-valued distribution such that $\langle x(u),s(u)\rangle=\hat{x}[s]$. That is the current $x(u)$ can be regarded as a kernel of the operator $\hat{x}$ and the latter formula gives its invariant definition.

Loop algebras. Let $\{\hat{x}_k\}$, $k=1,\dots,n$ be a finite number of operators $\hat{x}_k\colon\mathfrak{K}\to\mathfrak{g}$, where $\mathfrak{g}$ is an infinite-dimensional space spanned by $\hat{x}_k[s]$, $s\in\mathfrak{K}$. Consider the corresponding currents $x_k(u)$. For these currents we impose the standard commutation relations

$$[x_k(u),x_l(v)]=\sum_{m=1}^{n}C_{kl}^m x_m(u)\delta(u,v) \tag{2.1}$$

where C_{kl}^m are structure constants of some semi-simple Lie algebra $\mathfrak{a}$, $\dim\mathfrak{a}=n$ (equality (2.1) is understood in sense of distributions). These commutation relations equip $\mathfrak{g}$ with a Lie algebra structure. The Lie algebra $\mathfrak{g}$ defined in such a way can be viewed as a Lie algebra $\mathfrak{a}\otimes\mathfrak{K}$ with the brackets $[x\otimes s(z),y\otimes t(z)]=[x,y]_{\mathfrak{a}}\otimes s(z)t(z)$, where $x,y\in\mathfrak{a}$, $s,t\in\mathfrak{K}$. This algebra possesses an invariant scalar product $\langle x\otimes s,y\otimes t\rangle=(x,y)\langle s(u),t(u)\rangle_u$, where $(\cdot,\cdot)$ is an invariant scalar product on $\mathfrak{a}$ proportional to the Killing form.

Central extension. The algebra $\mathfrak{g}=\mathfrak{a}\otimes\mathfrak{K}$ can be extended by introducing a central element c and a co-central element d. Let us consider the space $\hat{\mathfrak{g}}=(\mathfrak{a}\otimes\mathfrak{K})\oplus\mathbb{C}\oplus\mathbb{C}$ and define an algebra structure on this space. Let the element $c\equiv(0,1,0)$ commute with everything and the commutator of the element $d\equiv(0,0,1)$ with the elements $\hat{x}[s]\equiv(x\otimes s,0,0)$, $x\in\mathfrak{a}$, $s\in\mathfrak{K}$, is given by the formula $[d,\hat{x}[s]]=\hat{x}[s']$, where s' is a derivation of s. Define the Lie bracket between the elements of type $\hat{x}[s]$ requiring the scalar product defined by formulae

$$\langle\hat{x}[s],\hat{y}[t]\rangle=\langle x\otimes s,y\otimes t\rangle,\quad \langle c,\hat{x}[s]\rangle=\langle d,\hat{x}[s]\rangle=0$$

to be invariant. It gives the formula

$$[\hat{x}[s],\hat{y}[t]]=([x\otimes s,y\otimes t]_0,0,0)+c\cdot B(x_1\otimes s_1,x_2\otimes s_2) \tag{2.2}$$

where $[\cdot,\cdot]_0$ is the Lie bracket in the algebra $\mathfrak{g}=\mathfrak{a}\otimes\mathfrak{K}$ and $B(\cdot,\cdot)$ is a standard 2-cocycle: $B(x\otimes s,y\otimes t)=(x,y)\langle s'(z),t(z)\rangle_z$. The expression $\hat{x}[s]$ depends linearly on $s\in\mathfrak{K}$ and, therefore, can be regarded as an action of operator $\hat{x}\colon \mathfrak{K} \to \hat{\mathfrak{g}}$. The commutation relations for the algebra $\hat{\mathfrak{g}}$ in terms of currents $x(u)$ corresponding to these operators can be written in the standard form: $[c, x(u)] = [c, d] = 0$ and

$$[x_1(u), x_2(v)] = x_3(u)\delta(u,v) - c \cdot (x_1, x_2) d\delta(u,v)/du, \quad [d, x(u)] = -dx(u)/du \tag{2.3}$$

where $x_1, x_2 \in \mathfrak{a}$, $x_3 = [x_1, x_2]_{\mathfrak{a}}$.

Half-currents. To describe different bialgebra structures in the current algebras we have to decompose the currents in these algebras into difference of the currents which have good analytical properties in certain domains: $x(u) = x^+(u) - x^-(u)$. The $\mathfrak{g}$-valued distributions $x^+(u)$, $x^-(u)$ are called *half-currents*. To perform such a decomposition we will use so-called Green distributions [9]. Let $\Omega^+, \Omega^- \subset \Sigma \times \Sigma$ be two domains separated by a hypersurface $\bar{\Delta} \subset \Sigma \times \Sigma$ which contains the diagonal $\Delta = \{(u,u) \mid u \in \Sigma\} \subset \bar{\Delta}$. Assume that there exist distributions $G^+(u,z)$ and $G^-(u,z)$ regular in Ω^+ and Ω^- respectively such that $\delta(u,z) = G^+(u,z) - G^-(u,z)$. To define half-currents corresponding to these Green distributions we decompose them as $G^+(u,z) = \sum_i \alpha_i^+(u)\beta_i^+(z)$ and $G^-(u,z) = \sum_i \alpha_i^-(u)\beta_i^-(z)$. Then the half-currents are defined as $x^+(u) = \sum_i \alpha_i^+(u)\hat{x}[\beta_i^+]$ and $x^-(u) = \sum_i \alpha_i^-(u)\hat{x}[\beta_i^-]$. This definition does not depend on a choice of decompositions of the Green distributions. The half-currents are currents corresponding to the operators $\hat{x}^\pm = \pm\hat{x} \cdot P^\pm$, where $P^\pm[s](z) = \pm\langle G^\pm(u,z), s(u)\rangle$, $s \in \mathfrak{K}$. One can express the half-currents through the current $x(u)$, which we shall call *a total current* in contrast with the half ones:

$$x^+(u) = \langle G^+(u,z)x(z)\rangle_z, \quad x^-(u) = \langle G^-(u,z)x(z)\rangle_z \tag{2.4}$$

Here $\langle a(z)\rangle_z \equiv \langle a(z), 1\rangle_z$.

Two elliptic classical current algebras. In this paper we will consider the case when Σ is a covering of an elliptic curve and Green distributions are regularization of certain quasi-doubly periodic meromorphic functions. We will call the corresponding centrally extended algebras of currents by *elliptic classical current algebras.* The main aim of this paper is to show the following facts:

- There are two essentially different choices of the test function algebras $\mathfrak{K}$ in this case corresponding to the different covering Σ.
- The same quasi-doubly periodic meromorphic functions regularized with respect to the different test function algebras define the different quasi-Lie bialgebra structures and, therefore, the different classical elliptic current algebras.
- The internal structure of these two elliptic algebras is essentially different in spite of a similarity in the commutation relations between their half-currents.

The first choice corresponds to $\mathfrak{K} = \mathcal{K}_0$, where $\mathcal{K}_0$ consists of complex-valued one-variable functions defined in a vicinity of origin equipped with the scalar product

$$\langle s_1(u), s_2(u)\rangle = \oint_{C_0} \frac{du}{2\pi i} s_1(u)s_2(u) \tag{2.5}$$

Here C_0 is a contour encircling zero and belonging to the intersection of domains of functions $s_1(u)$, $s_2(u)$, such that the scalar product is a residue in zero. These functions can be extended up to meromorphic functions on the covering $\Sigma = \mathbb{C}$. The regularization domains Ω^+, Ω^- for Green distributions in this case consist of the pairs (u,z) such that $\min(1,|\tau|) > |u| > |z| > 0$ and $0 < |u| < |z| < \min(1,|\tau|)$, respectively, where τ is an elliptic module, and $\bar{\Delta} = \{(u,z) \mid |u| = |z|\}$.

The second choice corresponds to $\mathfrak{K} = K = K(\mathrm{Cyl})$. The algebra K consists of entire periodic functions $s(u) = s(u+1)$ on $\mathbb{C}$ decaying exponentially at $\operatorname{Im} u \to \pm\infty$ equipped with an invariant scalar product

$$\langle s(u), t(u)\rangle = \int_{-1/2}^{1/2} \frac{du}{2\pi i} s(u)t(u), \quad s, t \in K \tag{2.6}$$

These functions can be regarded as functions on cylinder $\Sigma = \mathrm{Cyl}$. The regularization domains Ω^+, Ω^- for Green distributions consist of the pairs (u, z) such that $-\operatorname{Im}\tau < \operatorname{Im}(u - z) < 0$ and $0 < \operatorname{Im}(u - z) < \operatorname{Im}\tau$ respectively and $\bar{\Delta} = \{(u, z) \mid \operatorname{Im} u = \operatorname{Im} z\}$.

Integration contour. The geometric roots of the difference between these two choices can be explained as follows. These choices of test functions on different coverings Σ of elliptic curve correspond to the homotopically different contours on the elliptic curve. Each test function can be considered as an analytical continuation of a function from this contour – a real manifold – to the corresponding covering. This covering should be chosen as a most homotopically simple covering which permits to obtain a bigger source of test functions. In the first case, this contour is a homotopically trivial and coincides with a small contour around fixed point on the torus. We can always choose a local coordinate u such that $u = 0$ in this point. This explains the notation $\mathcal{K}_0$. This contour corresponds to the covering $\Sigma = \mathbb{C}$ and it enters in the pairing (2.5). In the second case, it goes along a cycle and it can not be represented as a closed contour on $\mathbb{C}$. Hence the most simple covering in this case is a cylinder $\Sigma = \mathrm{Cyl}$ and the contour is that one in the pairing (2.6). This leads to essentially different properties of the current elliptic algebras based on the test function algebras $\mathfrak{K} = \mathcal{K}_0$ and $\mathfrak{K} = K(\mathrm{Cyl})$.

Restriction to the $\mathfrak{sl}_2$ case. To make these differences more transparent we shall consider only the simplest case of Lie algebra $\mathfrak{a} = \mathfrak{sl}_2$ defined as a three-dimensional complex Lie algebra with commutation relations $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. We denote the constructed current algebra $\hat{\mathfrak{g}}$ for the case $\mathfrak{K} = \mathcal{K}_0$ as $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and for $\mathfrak{K} = K = K(\mathrm{Cyl})$ as $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. These current algebras may be identified with classical limits of the quantized currents algebra $E_{\tau,\eta}(\mathfrak{sl}_2)$ of [7] and $U_{p,q}(\widehat{\mathfrak{sl}}_2)$ of [18] respectively. The Green distributions appear in the algebras $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ as a regularization of the same meromorphic quasi-doubly periodic functions but in different spaces: $(\mathcal{K}_0 \otimes \mathcal{K}_0)'$ and $(K \otimes K)'$ respectively. Primes mean the extension to the space of the distributions. We call them *elliptic Green distributions*. We define the algebras $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ to be *a priori* different, because the main component of our construction, elliptic Green distributions are *a priori* different being understood as distributions of different types: related to algebras $\mathcal{K}_0$ and K respectively. It means, in particular, that their quantum analogs, the algebras $E_{\tau,\eta}(\mathfrak{sl}_2)$ and $U_{p,q}(\widehat{\mathfrak{sl}}_2)$ are different.

3 Half-currents and co-structures

We start with a suitable definition of theta-functions and a conventional choice of standard bases. This choice is motivated and corresponds to definitions and notations of [8].

Theta-function. Let $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ be a module of the elliptic curve $\mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z}+\tau\mathbb{Z}$ is a period lattice. The odd theta function $\theta(u) = -\theta(-u)$ is defined as a holomorphic function on $\mathbb{C}$ with the properties

$$\theta(u + 1) = -\theta(u), \quad \theta(u + \tau) = -e^{-2\pi i u - \pi i\tau}\theta(u), \quad \theta'(0) = 1 \tag{3.1}$$

3.1 Elliptic Green distributions on $\mathcal{K}_0$

3.1.2. Dual bases. Fix a complex number λ. Consider the following bases in $\mathcal{K}_0$ $(n \geq 0)$:

$\epsilon_{n;\lambda}(u) = (-u)^n$, $\epsilon^{-n-1;\lambda}(u) = u^n$,

$$\epsilon^{n;\lambda}(u) = \frac{1}{n!}\left(\frac{\theta(u+\lambda)}{\theta(u)\theta(\lambda)}\right)^{(n)}, \quad \epsilon_{-n-1;\lambda}(u) = \frac{(-1)^n}{n!}\left(\frac{\theta(u-\lambda)}{\theta(u)\theta(-\lambda)}\right)^{(n)}$$

for $\lambda \notin \Gamma$ and the bases $\epsilon_{n;0}(u) = (-u)^n$, $\epsilon^{-n-1;0}(u) = u^n$,

$$\epsilon^{n;0}(u) = \frac{1}{n!}\left(\frac{\theta'(u)}{\theta(u)}\right)^{(n)}, \quad \epsilon_{-n-1;0}(u) = \frac{(-1)^n}{n!}\left(\frac{\theta'(u)}{\theta(u)}\right)^{(n)}$$

for $\lambda = 0$. Here $(\cdot)^{(n)}$ means n-times derivative. These bases are dual: $\langle \epsilon^{n;\lambda}(u), \epsilon_{m;\lambda}(u)\rangle = \delta^n_m$ and $\langle \epsilon^{n;0}(u), \epsilon_{m;0}(u)\rangle = \delta^n_m$ with respect to the scalar product (2.5), which means

$$\sum_{n\in\mathbb{Z}} \epsilon^{n;\lambda}(u)\epsilon_{n;\lambda}(z) = \delta(u,z), \quad \sum_{n\in\mathbb{Z}} \epsilon^{n;0}(u), \epsilon_{n;0}(z) = \delta(u,z) \tag{3.2}$$

3.1.3. Green distributions for $\mathcal{K}_0$ and the addition theorems. Here we follow the ideas of [9] and [8]. We define the following distribution

$$G^+_\lambda(u,z) = \sum_{n\geq 0} \epsilon^{n;\lambda}(u)\epsilon_{n;\lambda}(z), \quad G^-_\lambda(u,z) = -\sum_{n<0} \epsilon^{n;\lambda}(u)\epsilon_{n;\lambda}(z) \tag{3.3}$$

$$G(u,z) = \sum_{n\geq 0} \epsilon^{n;0}(u)\epsilon_{n;0}(z) = \sum_{n<0} \epsilon^{n;0}(z)\epsilon_{n;0}(u) \tag{3.4}$$

One can check that these series converge in sense of distributions and, therefore, define continuous functionals on $\mathcal{K}_0$ called Green distributions. Their action on a test function $s(u)$ reads

$$\langle G^{\pm}_\lambda(u,z), s(u)\rangle_u = \oint\limits_{\substack{|u|>|z|\\|u|<|z|}} \frac{du}{2\pi i}\frac{\theta(u-z+\lambda)}{\theta(u-z)\theta(\lambda)} s(u) \tag{3.5}$$

$$\langle G(u,z), s(u)\rangle_u = \oint\limits_{|u|>|z|} \frac{du}{2\pi i}\frac{\theta'(u-z)}{\theta(u-z)} s(u) \tag{3.6}$$

where integrations are taken over circles around zero which are small enough such that the corresponding inequality takes place.

One can define a 'rescaling' of a test function $s(u)$ as a function $s\left(\frac{u}{\alpha}\right)$, where $\alpha \in \mathbb{C}$, and therefore a 'rescaling' of distributions by the formula $\langle a(\frac{u}{\alpha}), s(u)\rangle = \langle a(u), s(\alpha u)\rangle$. On the contrary, we are unable to define a 'shift' of test functions by a standard rule, because the operator $s(u) \mapsto s(u+z)$ is not a continuous one [1]. Nevertheless we use distributions 'shifted' in some sense. Namely, we say that a two-variable distribution $a(u,z)$ (a linear continuous functional $a\colon \mathcal{K}_0 \otimes \mathcal{K}_0 \to \mathbb{C}$) is 'shifted' if it possesses the properties: (i) for any $s \in \mathcal{K}_0$ the functions $s_1(z) = \langle a(u,z), s(u)\rangle_u$ and $s_2(u) = \langle a(u,z), s(z)\rangle_z$ belong to $\mathcal{K}_0$; (ii) $\frac{\partial}{\partial u}a(u,z) = -\frac{\partial}{\partial z}a(u,z)$. Here the subscripts u and z mean the corresponding partial action, for instance, $\langle a(u,z), s(u,z)\rangle_u$ is a distribution acting on $\mathcal{K}_0$ by the formula

$$\Big\langle \langle a(u,z), s(u,z)\rangle_u, t(z)\Big\rangle = \langle a(u,z), s(u,z)t(z)\rangle$$

[1]Consider, for example, the sum $s_N(u) = \sum_{n=0}^{N}(\frac{u}{\alpha})^n$. For each z there exist α such that the sum $s_N(u+z)$ diverges, when $N \to \infty$.

The condition (ii) means the equality $\langle a(u,z), s'(u)t(z)\rangle = -\langle a(u,z), s(u)t'(z)\rangle$. The condition (i) implies that for any $s \in \mathcal{K}_0 \otimes \mathcal{K}_0$ the expression

$$\big\langle a(u,z), s(u,z)\big\rangle_u = \sum_i \big\langle a(u,z), p_i(u)\big\rangle_u q_i(z) \tag{3.7}$$

where $s(u,z) = \sum_i p_i(u)q_i(z)$, belongs to $\mathcal{K}_0$ (as a function of z).

The Green distributions (3.3) and (3.4) are examples of the 'shifted' distributions. The formula (3.2) implies that

$$G^+_\lambda(u,z) - G^-_\lambda(u,z) = \delta(u,z), \quad G(u,z) + G(z,u) = \delta(u,z) \tag{3.8}$$

The last formulae can be also obtained from (3.5), (3.6) taking into account that the function $s(u)$ has poles only in the points $u = 0$. As it is seen from (3.5), the oddness of function $\theta(u)$ leads to the following connection between the λ-depending Green distributions: $G^+_\lambda(u,z) = -G^-_{-\lambda}(z,u)$.

Now we define a *semidirect product of two 'shifted' distributions* $a(u,z)$ and $b(v,z)$ as a linear continuous functional $a(u,z)b(v,z)$ acting on $s \in \mathcal{K}_0 \otimes \mathcal{K}_0 \otimes \mathcal{K}_0$ by the rule

$$\big\langle a(u,z)b(v,z), s(u,v,z)\big\rangle = \Big\langle a(u,z), \big\langle b(v,z), s(u,v,z)\big\rangle_v \Big\rangle_{u,z}$$

Proposition 3.1. *The* semi-direct products *of Green distributions are related by the following addition formulae*

$$G^+_\lambda(u,z)G^-_\lambda(z,v) = G^+_\lambda(u,v)G(u,z) - G^+_\lambda(u,v)G(v,z) - \frac{\partial}{\partial\lambda}G^+_\lambda(u,v) \tag{3.9}$$

$$G^+_\lambda(u,z)G^+_\lambda(z,v) = G^+_\lambda(u,v)G(u,z) + G^+_\lambda(u,v)G(z,v) - \frac{\partial}{\partial\lambda}G^+_\lambda(u,v) \tag{3.10}$$

$$G^-_\lambda(u,z)G^-_\lambda(z,v) = -G^-_\lambda(u,v)G(z,u) - G^-_\lambda(u,v)G(v,z) - \frac{\partial}{\partial\lambda}G^+_\lambda(u,v) \tag{3.11}$$

$$G^-_\lambda(u,z)G^+_\lambda(z,v) = -G^+_\lambda(u,v)G(z,u) + G^+_\lambda(u,v)G(z,v) - \frac{\partial}{\partial\lambda}G^+_\lambda(u,v) \tag{3.12}$$

Proof. The actions of both hand sides of (3.9), for example, can be reduced to the integration over the same contours with some kernels. One can check the equality of these kernels using the degenerated Fay's identity [12]

$$\frac{\theta(u-z+\lambda)}{\theta(u-z)\theta(\lambda)}\frac{\theta(z+\lambda)}{\theta(z)\theta(\lambda)} = \frac{\theta(u+\lambda)}{\theta(u)\theta(\lambda)}\frac{\theta'(u-z)}{\theta(u-z)} + \frac{\theta(u+\lambda)}{\theta(u)\theta(\lambda)}\frac{\theta'(z)}{\theta(z)} - \frac{\partial}{\partial\lambda}\frac{\theta(u+\lambda)}{\theta(u)\theta(\lambda)} \tag{3.13}$$

The other formulae can be proved in the same way. □

3.1.4. Projections. Let us notice that the vectors $\epsilon_{n;\lambda}(u)$ and $\epsilon^{-n-1;\lambda}(u)$ span two complementary subspaces of $\mathcal{K}_0$. The formulae (3.3) mean that the distributions $G^+_\lambda(u,z)$ and $G^-_\lambda(u,z)$ define orthogonal projections P^+_λ and P^-_λ onto these subspaces. They act as $P^+_\lambda[s](z) = \big\langle G^+_\lambda(u,z), s(u)\big\rangle_u$ and $P^-_\lambda[s](z) = -\big\langle G^-_\lambda(u,z), s(u)\big\rangle_u$. Similarly, the operators

$$P^+[s](z) = \big\langle G(u,z), s(u)\big\rangle_u, \quad P^-[s](z) = \big\langle G(z,u), s(u)\big\rangle_u$$

are projections onto the Lagrangian (involutive) subspaces spanned by the vectors $\epsilon_{n;0}(u)$ and $\epsilon_{-n-1;0}(u)$, respectively. The fact that the corresponding spaces are complementary to each other is encoded in the formulae (3.8), which can be rewritten as $P^+_\lambda + P^-_\lambda = \mathrm{id}$, $P^+ + P^- = \mathrm{id}$. The idempotent properties and orthogonality of these projections

$$P^\pm_\lambda \cdot P^\pm_\lambda = P^\pm_\lambda, \quad P^\pm \cdot P^\pm = P^\pm, \quad P^+_\lambda \cdot P^-_\lambda = P^-_\lambda \cdot P^+_\lambda = 0, \quad P^+ \cdot P^- = P^- \cdot P^+ = 0$$

are encoded in the formulae

$$\big\langle G^+_\lambda(u,z)G^+_\lambda(z,v)\big\rangle_z = G^+_\lambda(u,v)\,, \qquad \big\langle G^+_\lambda(u,z)G^-_\lambda(z,v)\big\rangle_z = 0 \tag{3.14}$$

$$\big\langle G^-_\lambda(u,z)G^-_\lambda(z,v)\big\rangle_z = -G^-_\lambda(u,v)\,, \qquad \big\langle G^-_\lambda(u,z)G^+_\lambda(z,v)\big\rangle_z = 0 \tag{3.15}$$

$$\big\langle G(u,z)G(z,v)\big\rangle_z = G(u,v)\,, \qquad \big\langle G(u,z)G(v,z)\big\rangle_z = 0 \tag{3.16}$$

which immediately follow from (3.3) and also can be obtained from the relations (3.9) – (3.12) if one takes into account $\big\langle G(u,z)\big\rangle_z = 0$, $\big\langle G(z,u)\big\rangle_z = 1$.

3.2 Elliptic Green distributions on K

3.2.1. Green distributions and dual bases for K. The analogs of the Green distributions $G^+_\lambda(u,z)$, $G^-_\lambda(u,z)$ are defined in this case by the following action on the space K

$$\big\langle \mathcal{G}^\pm_\lambda(u-z), s(u)\big\rangle_u = \int\limits_{\substack{-\operatorname{Im}\tau<\operatorname{Im}(u-z)<0\\ 0<\operatorname{Im}(u-z)<\operatorname{Im}\tau}} \frac{du}{2\pi i}\frac{\theta(u-z+\lambda)}{\theta(u-z)\theta(\lambda)}s(u) \tag{3.17}$$

$$\big\langle \mathcal{G}(u-z), s(u)\big\rangle_u = \int\limits_{-\operatorname{Im}\tau<\operatorname{Im}(u-z)<0} \frac{du}{2\pi i}\frac{\theta'(u-z)}{\theta(u-z)}s(u) \tag{3.18}$$

where we integrate over line segments of unit length (cycles of cylinder) such that the corresponding inequality takes place. The role of dual bases in the algebra K is played by $\{j_n(u) = e^{2\pi i n u}\}_{n\in\mathbb{Z}}$ and $\{j^n(u) = 2\pi i e^{-2\pi i n u}\}_{n\in\mathbb{Z}}$, a decomposition with respect to these bases is the usual Fourier expansion. The Fourier expansions for the Green distributions are [2]

$$\mathcal{G}^\pm_\lambda(u-z) = \pm 2\pi i\sum_{n\in\mathbb{Z}}\frac{e^{-2\pi i n(u-z)}}{1-e^{\pm 2\pi i(n\tau-\lambda)}} \tag{3.19}$$

$$\mathcal{G}(u-z) = \pi i + 2\pi i\sum_{n\neq 0}\frac{e^{-2\pi i n(u-z)}}{1-e^{2\pi i n\tau}} \tag{3.20}$$

These expansions are in accordance with formulae

$$\mathcal{G}^+_\lambda(u-z) - \mathcal{G}^-_\lambda(u-z) = \delta(u-z) \tag{3.21}$$

$$\mathcal{G}(u-z) + \mathcal{G}(z-u) = \delta(u-z) \tag{3.22}$$

where $\delta(u-z)$ is a delta-function on K, given by the expansion

$$\delta(u-z) = \sum_{n\in\mathbb{Z}} j^n(u)j_n(z) = 2\pi i\sum_{n\in\mathbb{Z}} e^{-2\pi i n(u-z)} \tag{3.23}$$

3.2.2. Addition theorems. Now we obtain some properties of these Green distributions and compare them with the properties of their analogs $G^+_\lambda(u,z)$, $G^-_\lambda(u,z)$, $G(u,z)$ described in subsection 3.1. In particular, we shall see that some properties are essentially different. Let us start with the properties of Green distribution which are similar to the case of algebra $\mathcal{K}_0$. They satisfy the same addition theorems that were described in the subsection 3.1.

Proposition 3.2. *The semi-direct product of Green distributions for algebra K is related by the formulae (3.9)–(3.12) with the distributions $\mathcal{G}^\pm_\lambda(u-z)$, $\mathcal{G}(u-z)$ instead of $G^\pm_\lambda(u-z)$, $G(u-z)$ respectively.*

[2] Fourier expansions presented in this subsection are obtained considering integration around boundary of fundamental domain.

Proof. The kernels of these distributions are the same and therefore the addition formula in this case is also based on the Fay's identity (3.13). □

3.2.3. Analogs of projections. The Green distributions define the operators on K:

$$\mathcal{P}^+_\lambda[s](z) = \big\langle \mathcal{G}^+_\lambda(u-z), s(u)\big\rangle_u, \qquad \mathcal{P}^-_\lambda[s](z) = \big\langle \mathcal{G}^-_\lambda(u-z), s(u)\big\rangle_u$$
$$\mathcal{P}^+[s](z) = \big\langle \mathcal{G}(u-z), s(u)\big\rangle_u, \qquad \mathcal{P}^-[s](z) = \big\langle \mathcal{G}(z-u), s(u)\big\rangle_u$$

which are similar to their analogs $P^\pm_\lambda$, $P^\pm$ and satisfy $\mathcal{P}^+_\lambda + \mathcal{P}^-_\lambda = \mathrm{id}$, $\mathcal{P}^+ + \mathcal{P}^- = \mathrm{id}$ (due to (3.21)), but they are not projections. This fact is reflected in the following relations:

$$\big\langle \mathcal{G}^+_\lambda(u-z)\mathcal{G}^+_\lambda(z-v)\big\rangle_z = \mathcal{G}^+_\lambda(u-v) - \frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\mathcal{G}^+_\lambda(u-v) \tag{3.24}$$
$$\big\langle \mathcal{G}^+_\lambda(u-z)\mathcal{G}^-_\lambda(z-v)\big\rangle_z = -\frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\mathcal{G}^+_\lambda(u-v) \tag{3.25}$$
$$\big\langle \mathcal{G}^-_\lambda(u-z)\mathcal{G}^+_\lambda(z-v)\big\rangle_z = -\frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\mathcal{G}^+_\lambda(u-v) \tag{3.26}$$
$$\big\langle \mathcal{G}^-_\lambda(u-z)\mathcal{G}^-_\lambda(z-v)\big\rangle_z = -\mathcal{G}^-_\lambda(u-v) - \frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\mathcal{G}^+_\lambda(u-v) \tag{3.27}$$
$$\frac{\partial}{\partial\lambda}\mathcal{G}^+_\lambda(u-v)\big\langle \mathcal{G}(u-z)\mathcal{G}(z-v)\big\rangle_z = \mathcal{G}(u-v) - \frac{1}{4\pi i}\gamma(u-v) \tag{3.28}$$
$$\big\langle \mathcal{G}(u-z)\mathcal{G}(v-z)\big\rangle_z = \big\langle \mathcal{G}(z-u)\mathcal{G}(z-v)\big\rangle_z = \frac{1}{4\pi i}\gamma(u-v) \tag{3.29}$$

and $\gamma(u-z)$ is a distribution which has the following action and expansion

$$\big\langle \gamma(u-z), s(u)\big\rangle = -\frac{\theta'''(0)+4\pi^2}{3} + \int\limits_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{du}{2\pi i}\frac{\theta''(u-z)}{\theta(u-z)}s(u)$$
$$\gamma(u-z) = -2\pi^2 + 8\pi^2\sum_{n\neq 0}\frac{e^{-2\pi i n(u-z)+2\pi i n\tau}}{(1-e^{2\pi i n\tau})^2}$$

3.2.4. Comparison of the Green distributions. Contrary to (3.14)–(3.16) the formulae (3.24)–(3.29) contain some additional terms in the right hand sides obstructing the operators $\mathcal{P}^\pm_\lambda$, $\mathcal{P}^\pm$ to be projections. They do not decompose the space $K(\mathrm{Cyl})$ in a direct sum of subspaces as it would be in the case of projections $P^\pm_\lambda$, $P^\pm$ acting on $\mathcal{K}_0$. Moreover, as one can see from the Fourier expansions (3.19), (3.20) of Green distributions the images of the operators coincide with whole algebra K: $\mathcal{P}^\pm_\lambda\big(K(\mathrm{Cyl})\big) = K(\mathrm{Cyl})$, $\mathcal{P}^\pm\big(K(\mathrm{Cyl})\big) = K(\mathrm{Cyl})$. As we shall see this fact has a deep consequence for the half-currents of the corresponding Lie algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. As soon as we are aware that the positive operators $\mathcal{P}^+_\lambda$, $\mathcal{P}^+$ as well as negative ones $\mathcal{P}^-_\lambda$, $\mathcal{P}^-$ transform the algebra K to itself, we can surmise that they can be related to each other. This is actually true. ¿From formulae (3.19), (3.20) we conclude that

$$\mathcal{G}^+_\lambda(u-z-\tau) = e^{2\pi i\lambda}\mathcal{G}^-_\lambda(u-z), \quad \mathcal{G}(u-z-\tau) = 2\pi i - \mathcal{G}(z-u) \tag{3.30}$$

In terms of operator's composition these properties look as

$$\mathcal{T}_\tau\circ\mathcal{P}^+_\lambda = \mathcal{P}^+_\lambda\circ\mathcal{T}_\tau = -e^{2\pi i\lambda}\mathcal{P}^-_\lambda, \quad \mathcal{T}_\tau\circ\mathcal{P}^+ = \mathcal{P}^+\circ\mathcal{T}_\tau = 2\pi i\mathcal{I} - \mathcal{P}^- \tag{3.31}$$

where $\mathcal{T}_t$ is a shift operator: $\mathcal{T}_t[s](z) = s(z+t)$, and $\mathcal{I}$ is an integration operator: $\mathcal{I}[s](z) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{du}{2\pi i}s(u)$. This property is no longer true for the case of Green distributions from section 3.1.

3.3 Elliptic half-currents

3.3.5. Tensor subscripts. First introduce the following notation. Let $\mathbf{U} = \mathcal{U}(\mathfrak{g})$ be a universal enveloping algebra of the considered Lie algebra $\mathfrak{g}$ and V be a $\mathbf{U}$-module. For an element

$$t = \sum_k a_1^k \otimes \ldots \otimes a_n^k \otimes u_1^k \otimes \ldots \otimes u_m^k \quad \in \operatorname{End} V^{\otimes n} \otimes \mathbf{U}^{\otimes m}$$

where $n, m \geq 0$, $a_1^k, \ldots, a_n^k \in \operatorname{End} V$, $u_1^k, \ldots, u_m^k \in \mathbf{U}$ we shall use the following notation for an element of $\operatorname{End} V^{\otimes N} \otimes \mathbf{U}^{\otimes M}$, $N \geq n, M \geq m$:

$$t_{i_1,\ldots,i_n,\mathbf{j_1},\ldots,\mathbf{j_m}} = \sum_k \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V \otimes a_1^k \otimes \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V \otimes a_n^k \otimes \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V \otimes$$
$$\otimes 1 \otimes \ldots \otimes 1 \otimes u_1^k \otimes 1 \otimes \ldots\ldots \otimes u_m^k \otimes 1 \otimes \ldots \otimes 1$$

where a_s^k stays in the i_s-th position in the tensor product and u_s^k stays in the j_s-th position.

3.3.6. Half-currents. The total currents $h(u)$, $e(u)$ and $f(u)$ of the algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ can be divided into half-currents using the Green distributions $G(u,z)$, $-G(z,u)$ for $h(u)$; $G^+_\lambda(u,z)$, $G^-_\lambda(u,z)$ for $e(u)$; and $G^+_{-\lambda}(u,z) = -G^-_\lambda(z,u)$, $G^-_{-\lambda}(u,z) = -G^+_\lambda(z,u)$. The relations of type (2.4), then, look as

$$h^+(u) = \big\langle G(u,v)h(v)\big\rangle_v, \qquad h^-(u) = -\big\langle G(v,u)h(v)\big\rangle_v \tag{3.32}$$
$$e^+_\lambda(u) = \big\langle G^+_\lambda(u,v)e(v)\big\rangle_v, \qquad e^-_\lambda(u) = \big\langle G^-_\lambda(u,v)e(v)\big\rangle_v \tag{3.33}$$
$$f^+_\lambda(u) = \big\langle G^+_{-\lambda}(u,v)f(v)\big\rangle_v, \qquad f^-_\lambda(u) = \big\langle G^-_{-\lambda}(u,v)f(v)\big\rangle_v \tag{3.34}$$

so that $h(u) = h^+(u) - h^-(u)$, $e(u) = e^+_\lambda(u) - e^-_\lambda(u)$, $f(u) = f^+_\lambda(u) - f^-_\lambda(u)$.

3.3.7. rLL**-relations for** $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$**.** The commutation relations between the half-currents can be written in a matrix form. Let us introduce the matrices of *L-operators*:

$$L^\pm_\lambda(u) = \begin{pmatrix} \frac{1}{2}h^\pm(u) & f^\pm_\lambda(u) \\ e^\pm_\lambda(u) & -\frac{1}{2}h^\pm(u) \end{pmatrix} \tag{3.35}$$

as well as the r-matrices:

$$r^+_\lambda(u,v) = \begin{pmatrix} \frac{1}{2}G(u,v) & 0 & 0 & 0 \\ 0 & -\frac{1}{2}G(u,v) & G^+_{-\lambda}(u,v) & 0 \\ 0 & G^+_\lambda(u,v) & -\frac{1}{2}G(u,v) & 0 \\ 0 & 0 & 0 & \frac{1}{2}G(u,v) \end{pmatrix} \tag{3.36}$$

Proposition 3.3. *The commutation relations of the algebra* $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ *in terms of half-currents can be written in the form:*

$$[d, L^\pm_\lambda(u)] = -\frac{\partial}{\partial u}L^\pm_\lambda(u) \tag{3.37}$$

$$[L^\pm_{\lambda,1}(u), L^\pm_{\lambda,2}(v)] = [L^\pm_{\lambda,1}(u) + L^\pm_{\lambda,2}(v), r^+_\lambda(u-v)] + H_1\frac{\partial}{\partial\lambda}L^\pm_{\lambda,2}(v) - $$
$$- H_2\frac{\partial}{\partial\lambda}L^\pm_{\lambda,1}(u) + h\frac{\partial}{\partial\lambda}r^+_\lambda(u-v) \tag{3.38}$$

$$[L^+_{\lambda,1}(u), L^-_{\lambda,2}(v)] = [L^+_{\lambda,1}(u) + L^-_{\lambda,2}(v), r^+_\lambda(u-v)] + H_1\frac{\partial}{\partial\lambda}L^-_{\lambda,2}(v) - H_2\frac{\partial}{\partial\lambda}L^+_{\lambda,1}(u)$$
$$+ h\frac{\partial}{\partial\lambda}r^+_\lambda(u-v) + c\cdot\frac{\partial}{\partial u}r^+_\lambda(u-v) \tag{3.39}$$

where $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *and* $h = \hat{h}[\epsilon_{0;0}]$*. The L-operators satisfy an important relation*

$$[H + h, L^\pm(u)] = 0 \tag{3.40}$$

Proof. Using the formulae (3.14) – (3.16) we calculate the scalar products on the half-currents: $\langle L^{\pm}_{\lambda,1}(u), L^{\pm}_{\lambda,2}(v)\rangle = 0$, $\langle L^{+}_{\lambda,1}(u), L^{-}_{\lambda,2}(v)\rangle = -r^{+}_{\lambda}(u,v)$. Differentiating these formulae by u we can obtain the values of the standard co-cycle on the half-currents: $B\big(L^{\pm}_{\lambda,1}(u), L^{\pm}_{\lambda,2}(v)\big) = 0$, $B\big(L^{+}_{\lambda,1}(u), L^{-}_{\lambda,2}(v)\big) = \frac{\partial}{\partial u} r^{+}_{\lambda}(u,v)$. Using the formulae (3.9)–(3.12) one can calculate the brackets $[\cdot,\cdot]_0$ on the half-currents. Representing them in the matrix form and adding the co-cycle term one can derive the relations (3.38), (3.39). Using the formulae $[h, L^{\pm}_{\lambda}(v)] = \mathrm{tr}_1 \langle H_1 [L^{+}_{\lambda,1}(u), L^{\pm}_{\lambda,2}(v)]\rangle_u$, $\mathrm{tr}_1 \langle H_1 r^{+}_{\lambda}(u,v)\rangle_u = H$, $\mathrm{tr}_1 \langle [H_1, L^{+}_{\lambda,1}(u)] r^{+}_{\lambda}(u,v)\rangle_u = 0$ we obtain the relation (3.40) from (3.38), (3.39). □

3.3.8. rLL-relations for $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. Now consider the case of the algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. The half-currents, L-operators $\mathcal{L}^{\pm}_{\lambda}(u)$ and r-matrix $\mathfrak{r}^{+}_{\lambda}(u-v)$ are defined by the same formulas as above with distributions $G(u,v)$ and $G^{\pm}_{\lambda}(u,v)$ replaced everywhere by the distributions $\mathcal{G}(u,v)$ and $\mathcal{G}^{\pm}_{\lambda}(u,v)$. We have

Proposition 3.4. *The commutation relations of algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ in terms of half-currents can be written in the form:*

$$\begin{aligned}[]
[\mathcal{L}^{\pm}_{\lambda,1}(u), \mathcal{L}^{\pm}_{\lambda,2}(v)] &= [\mathcal{L}^{\pm}_{\lambda,1}(u) + \mathcal{L}^{\pm}_{\lambda,2}(v), \mathfrak{r}^{+}_{\lambda}(u-v)] + H_1 \frac{\partial}{\partial\lambda}\mathcal{L}^{\pm}_{\lambda,2}(v) \\
&\quad - H_2 \frac{\partial}{\partial\lambda}\mathcal{L}^{\pm}_{\lambda,1}(u) + h\frac{\partial}{\partial\lambda}\mathfrak{r}^{+}_{\lambda}(u-v) - c\cdot\frac{\partial}{\partial\tau}\mathfrak{r}^{+}_{\lambda}(u-v) \qquad (3.41)\\
[\mathcal{L}^{+}_{\lambda,1}(u), \mathcal{L}^{-}_{\lambda,2}(v)] &= [\mathcal{L}^{+}_{\lambda,1}(u) + \mathcal{L}^{-}_{\lambda,2}(v), \mathfrak{r}^{+}_{\lambda}(u-v)] + H_1 \frac{\partial}{\partial\lambda}\mathcal{L}^{-}_{\lambda,2}(v) \\
&\quad - H_2 \frac{\partial}{\partial\lambda}\mathcal{L}^{+}_{\lambda,1}(u) + h\frac{\partial}{\partial\lambda}\mathfrak{r}^{+}_{\lambda}(u-v) + c\cdot\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial\tau}\right)\mathfrak{r}^{+}_{\lambda}(u-v) \qquad (3.42)
\end{aligned}$$

where $h = \hat{h}[j_0]$. We also have in this case the relation

$$[H + h, \mathcal{L}^{\pm}(u)] = 0 \qquad (3.43)$$

Proof. To express the standard co-cycle on the half currents through the derivatives of the r-matrix we need the following formulae

$$\begin{aligned}
\frac{1}{2\pi i}\frac{\partial}{\partial u}\frac{\partial}{\partial\lambda}\mathcal{G}^{+}_{\lambda}(u-v) &= \frac{\partial}{\partial\tau}\mathcal{G}^{+}_{\lambda}(u-v)\\
\frac{1}{2\pi i}\frac{\partial}{\partial u}\frac{\partial}{\partial\lambda}\mathcal{G}^{-}_{\lambda}(u-v) &= \frac{\partial}{\partial\tau}\mathcal{G}^{-}_{\lambda}(u-v) = \frac{\partial}{\partial\tau}\mathcal{G}^{+}_{\lambda}(u-v)\\
\frac{1}{4\pi i}\frac{\partial}{\partial u}\gamma(u-v) &= \frac{\partial}{\partial\tau}\mathcal{G}(u-v)
\end{aligned}$$

Using these formulae we obtain

$$B\big(\mathcal{L}^{\pm}_{\lambda,1}(u), \mathcal{L}^{\pm}_{\lambda,2}(v)\big) = -\frac{\partial}{\partial\tau}\mathfrak{r}^{+}_{\lambda}(u-v), \quad B\big(\mathcal{L}^{+}_{\lambda,1}(u), \mathcal{L}^{-}_{\lambda,2}(v)\big) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial\tau}\right)\mathfrak{r}^{+}_{\lambda}(u-v)$$

Using the formulae

$$\begin{aligned}[]
&[h, \mathcal{L}^{\pm}_{\lambda}(v)] = 2\,\mathrm{tr}_1 \langle H_1[\mathcal{L}^{+}_{\lambda,1}(u), L^{\pm}_{\lambda,2}(v)]\rangle_u\\
&\mathrm{tr}_1 \langle H_1 r^{+}_{\lambda}(u,v)\rangle_u = H/2\\
&\mathrm{tr}_1 \langle [H_1, L^{+}_{\lambda,1}(u)] r^{+}_{\lambda}(u,v)\rangle_u = \frac{i}{\pi}\frac{\partial}{\partial\lambda}\mathcal{L}^{\pm}_{\lambda}(v)
\end{aligned}$$

we get the relation (3.43) from (3.41), (3.42). □

3.3.9. Peculiarities of half-currents for $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. To conclude this subsection we discuss the implication of the properties of Green distributions described in the end of the previous section to the Lie algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. The fact that the images of the operators $\mathcal{P}^{\pm}_{\lambda}$, $\mathcal{P}^{\pm}$ coincide with all the space K means that the commutation relations between the positive (or negative) half-currents are sufficient to describe all the Lie algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. This is a consequence of construction of the Lie algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ as the central extension of $\mathfrak{sl}_2\otimes K$. To obtain all commutation relations given in Proposition 3.4 from relations between only positive (or negative) half-currents one can use, firstly, the connection between positive and negative ones:

$$h^+(u-\tau)=2\pi i h+h^-(u),\quad e^+(u-\tau)=e^{2\pi i\lambda}e^-(u),\quad f^+(u-\tau)=e^{-2\pi i\lambda}f^-(u)$$

which follows from the properties of Green distributions expressed in formulae (3.30); secondly, relations (3.43), which also follow from the relations between only positive (respectively negative) half-currents; and finally, one needs to use the equality

$$\frac{\partial}{\partial\tau}\mathcal{G}^{\pm}_{\lambda}(u-z-\tau)=e^{2\pi i\lambda}(-\frac{\partial}{\partial u}+\frac{\partial}{\partial\tau})\mathcal{G}^{-}_{\lambda}(u-z)$$

At this point we see the essential difference of the Lie algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ with the Lie algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$.

3.4 Coalgebra structures of $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$

We describe here the structure of *quasi-Lie bialgebras* for our Lie algebras $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$. We will start with an explicit expression for universal (dynamical) r-matrices for both Lie algebras.

Proposition 3.5. *The universal r-matrix for the Lie algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ defined as*

$$r_\lambda=\frac{1}{2}\sum_{n\ge0}\hat h[\epsilon^{n;0}]\otimes\hat h[\epsilon_{n;0}]+\sum_{n\ge0}\hat f[\epsilon^{n;\lambda}]\otimes\hat e[\epsilon_{n;\lambda}]+\sum_{n<0}\hat e[\epsilon_{n;\lambda}]\otimes\hat f[\epsilon^{n;\lambda}]+c\otimes d$$

satisfies the Classical Dynamical Yang-Baxter Equation (CDYBE)

$$[r_{\lambda,\mathbf{12}},r_{\lambda,\mathbf{13}}]+[r_{\lambda,\mathbf{12}},r_{\lambda,\mathbf{23}}]+[r_{\lambda,\mathbf{13}},r_{\lambda,\mathbf{23}}]=h_{\mathbf{1}}\frac{\partial}{\partial\lambda}r_{\lambda,\mathbf{23}}-h_{\mathbf{2}}\frac{\partial}{\partial\lambda}r_{\lambda,\mathbf{13}}+h_{\mathbf{3}}\frac{\partial}{\partial\lambda}r_{\lambda,\mathbf{12}}\tag{3.44}$$

Denote by Π_u the evaluation representation $\Pi_u\colon\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)\to\operatorname{End}V_u$, where $V_u=\mathbb{C}^2\otimes\mathcal{K}_0$ and the subscript u means the argument of the functions belonging to $\mathcal{K}_0$:

$$\Pi_u\colon\hat h[s]\mapsto s(u)H,\quad\Pi_u\colon\hat e[s]\mapsto s(u)E,\quad\Pi_u\colon\hat f[s]\mapsto s(u)F\tag{3.45}$$

and $\Pi_u\colon c\mapsto 0$, $\Pi_u\colon d\mapsto\frac{\partial}{\partial u}$, where $H=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $E=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $F=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $s\in\mathcal{K}_0$. The relations between L-operators and the universal r-matrix are given by the formulae

$$L^+_\lambda(u)=(\Pi_u\otimes\mathrm{id})r_\lambda,\quad L^-_\lambda(u)-c\frac{\partial}{\partial u}=-(\Pi_u\otimes\mathrm{id})r_{\lambda,\mathbf{21}}\tag{3.46}$$

and $r^+_\lambda(u-v)=(\Pi_u\otimes\Pi_v)r_\lambda$. Taking into account these formulae and applying $(\Pi_u\otimes\Pi_v\otimes\mathrm{id})$, $(\mathrm{id}\otimes\Pi_u\otimes\Pi_v)$, $(\Pi_u\otimes\mathrm{id}\otimes\Pi_v)$ to the equation (3.44) we derive the relation (3.38) with the signs '+', the relation (3.38) with the signs '−' and the relation (3.39) respectively. Applying $(\Pi_u\otimes\mathrm{id})$ or $(\mathrm{id}\otimes\Pi_u)$ to the identity $[\Delta h,\mathfrak{r}_\lambda]=0$ we derive the relation (3.40).

The co-bracket $\delta\colon\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)\to\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)\wedge\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and an element $\varphi\in\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)\wedge\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)\wedge\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ are defined as $\delta x=[\Delta x,r_\lambda]=[x\otimes1+1\otimes x,r_\lambda]$ for $x\in\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and

$$\varphi=-[r_{\lambda,\mathbf{12}},r_{\lambda,\mathbf{13}}]-[r_{\lambda,\mathbf{12}},r_{\mathbf{23}}]-[r_{\lambda,\mathbf{13}},r_{\lambda,\mathbf{23}}]=-h_{\mathbf{1}}\frac{\partial}{\partial\lambda}r_{\lambda,\mathbf{23}}+h_{\mathbf{2}}\frac{\partial}{\partial\lambda}r_{\lambda,\mathbf{13}}-h_{\mathbf{3}}\frac{\partial}{\partial\lambda}r_{\lambda,\mathbf{12}}$$

They equip the Lie algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ with a structure of a quasi-Lie bialgebra [6]. This fact follows from the equality $r_{\mathbf{12}} + r_{\mathbf{21}} = \Omega$, where Ω is a tensor Casimir element of algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$. To calculate this co-bracket on the half-currents in the matrix form we apply $(\Pi_u \otimes \mathrm{id} \otimes \mathrm{id})$, $(\mathrm{id} \otimes \mathrm{id} \otimes \Pi_u)$ to the equation (3.44) and derive

$$\delta L^+_\lambda(u) = -[L^+_{\lambda,\mathbf{1}}(u), L^+_{\lambda,\mathbf{2}}(u)] + H\frac{\partial}{\partial\lambda} r_\lambda - h \wedge \frac{\partial}{\partial\lambda} L^+_\lambda(u)$$
$$\delta L^-_\lambda(u) = -[L^-_{\lambda,\mathbf{1}}(u), L^-_{\lambda,\mathbf{2}}(u)] + H\frac{\partial}{\partial\lambda} r_\lambda - h \wedge \frac{\partial}{\partial\lambda} L^-_\lambda(u) - c \wedge \frac{\partial}{\partial u} L^-_\lambda(u)$$

We can see also that $\delta h = 0$, $\delta c = 0$, $\delta d = 0$.

Proposition 3.6. *The universal r-matrix for the Lie algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ defined by formula*

$$\mathfrak{r}_\lambda = \frac{1}{4}\hat{h}[j^0] \otimes \hat{h}[j_0] + \frac{1}{2}\sum_{n\neq 0} \frac{\hat{h}[j^n] \otimes \hat{h}[j_n]}{1 - e^{2\pi i n\tau}} + \sum_{n\in\mathbb{Z}} \frac{\hat{e}[j^n] \otimes \hat{f}[j_n]}{1 - e^{2\pi i(n\tau+\lambda)}} + \sum_{n\in\mathbb{Z}} \frac{\hat{f}[j^n] \otimes \hat{e}[j_n]}{1 - e^{2\pi i(n\tau-\lambda)}} + c \otimes d$$

satisfies the equation

$$[\mathfrak{r}_{\lambda,\mathbf{12}}, \mathfrak{r}_{\lambda,\mathbf{13}}] + [\mathfrak{r}_{\lambda,\mathbf{12}}, \mathfrak{r}_{\lambda,\mathbf{23}}] + [\mathfrak{r}_{\lambda,\mathbf{13}}, \mathfrak{r}_{\lambda,\mathbf{23}}] =$$
$$= h_{\mathbf{1}}\frac{\partial}{\partial\lambda}\mathfrak{r}_{\lambda,\mathbf{23}} - h_{\mathbf{2}}\frac{\partial}{\partial\lambda}\mathfrak{r}_{\lambda,\mathbf{13}} + h_{\mathbf{3}}\frac{\partial}{\partial\lambda}\mathfrak{r}_{\lambda,\mathbf{12}} - c_{\mathbf{1}}\frac{\partial}{\partial\tau}\mathfrak{r}_{\lambda,\mathbf{23}} + c_{\mathbf{2}}\frac{\partial}{\partial\tau}\mathfrak{r}_{\lambda,\mathbf{13}} - c_{\mathbf{3}}\frac{\partial}{\partial\tau}\mathfrak{r}_{\lambda,\mathbf{12}}$$

The relations between the universal matrix $\mathfrak{r}_\lambda$ and L-operators of the algebra $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ are the same as for the algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ with a proper modification of the evaluation representation $\Pi_u\colon \mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2) \to \mathrm{End}\,\mathcal{V}_u$, $\mathcal{V}_u = \mathbb{C}^2 \otimes K$ defined by the same formulas (3.45) as above for $s \in K$.

The bialgebra structure of $\mathfrak{u}_\tau(\widehat{\mathfrak{sl}}_2)$ is defined in analogous way as for the algebra $\mathfrak{e}_\tau(\widehat{\mathfrak{sl}}_2)$ and can be presented in the form

$$\delta\mathcal{L}^+_\lambda(u) = -[\mathcal{L}^+_{\lambda,\mathbf{1}}(u), \mathcal{L}^+_{\lambda,\mathbf{2}}(u)] + H\frac{\partial}{\partial\lambda}\mathfrak{r}_\lambda - h \wedge \frac{\partial}{\partial\lambda}\mathcal{L}^+_\lambda(u) + c \wedge \frac{\partial}{\partial\tau}\mathcal{L}^+_\lambda(u)$$
$$\delta\mathcal{L}^-_\lambda(u) = -[\mathcal{L}^-_{\lambda,\mathbf{1}}(u), \mathcal{L}^-_{\lambda,\mathbf{2}}(u)] + H\frac{\partial}{\partial\lambda}\mathfrak{r}_\lambda - h \wedge \frac{\partial}{\partial\lambda}\mathcal{L}^-_\lambda(u) - c \wedge \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial\tau}\right)\mathcal{L}^-_\lambda(u)$$

Acknowledgements

This paper is a part of PhD thesis of A. S. which he has prepared under co-supervision of S. P. and V. R. in the Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna and in LAREMA, Départment de Mathématics, Université d'Angers. He is grateful to the CNRS-Russia exchange program on mathematical physics and personally to J.-M. Maillet for financial and general support of this thesis project. V. R. is thankful to B. Enriquez for discussions. He had used during the project a partial financial support by ANR GIMP, Grant for support of scientific schools NSh-8065.2006.2 and a support of INFN-RFBR "Einstein" grant (Italy-Russia). He acknowledges a warm hospitality of Erwin Schrödinger Institute for Mathematical Physics and the Program "Poisson Sigma Models, Lie Algebroids, deformations and higher analogues" where this paper was finished. S.P. was supported in part by RFBR grant 06-02-17383.

Leonid Vaksman, an excellent mathematician, one of Quantum Group "pioneers", patient teacher and a bright person, had passed away after coward disease when this paper was finished. We dedicate it to his memory with sadness and sorrow.

References

[1] D. Arnaudon, E. Buffenoir, E. Ragoucy, and P. Roche. Universal solutions of quantum dynamical Yang-Baxter equations. Lett. Math. Phys. **44**, (1998), 201–214.

[2] O. Babelon, D. Bernard, and E. Billey. A Quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations. Phys. Lett. B, **375** (1996), 89–97.

[3] R. J. Baxter. Partition function of the eight-vertex lattice model. Ann. Phys. **70** (1972), 193–228.

[4] R. J. Baxter. Eight-vertex model in lattice statistics and one dimensional anisotropic Heisenberg chain II: Equivalence to a generalized ice-type lattice model. Ann. Phys. **76** (1973), 25–47.

[5] V. Drinfeld. New realization of Yangians and quantum affine algebras. Soviet Math. Dokl. **36** (1988), 212–216.

[6] V. Drinfeld. Quasi-Hopf algebras. Leningrad Math. J. **1** (1990), 1419–1457.

[7] B. Enriquez and G. Felder. Elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ and quasi-Hopf algebras. Comm. Math. Phys. **195** (1998), 651–689.

[8] B. Enriquez, S. Pakuliak, and V. Rubtsov. Basic representations of quantum current algebras in higher genus. Contemp. Math. **433** (2007), 177–190.

[9] B. Enriquez and V. Rubtsov. Quantum groups in higher genus and Drinfeld's new realizations method ($\mathfrak{sl}_2$ case). Ann. Sci. École Norm. Sup. (4), **30** (1997), 821-846.

[10] B. Enriquez and V. Rubtsov. Quasi-Hopf algebras associated with $\mathfrak{sl}_2$ and complex curves. Israel J. Math. **112** (1999), 61–108.

[11] L. D. Faddeev and N. Yu. Reshetikhin. Hamiltonian structures for integrable models of field theory. Theoret. and Math. Phys. **56** (1983), 847–862.

[12] J. D. Fay. *Theta functions on Riemann Surfaces.* Springer-Verlag, Berlin–New York, 1973.

[13] G. Felder. Conformal field theory and integrable systems associated to elliptic curves. Proc. ICM Zürich 1994, Birkhäuser, 1994, 1247–1255, .

[14] G. Felder. Elliptic quantum groups. Proc. ICMP Paris 1994, International Press (1995), 211-218.

[15] C. Frønsdal. Quasi-Hopf deformations of quantum groups. Lett. Math. Phys. **40** (1997), 117–134.

[16] J.-L. Gervais and A. Neveu. Novel triangle relation and absence of tachyons in Liouville string field theory. Nucl. Phys. B, **238** (1984), 125–141.

[17] M. Jimbo, H. Konno, S. Odake, and J. Shiraishi. Quasi-Hopf twistors for elliptic quantum groups. Transform. Groups, **4** (1999), 303–327.

[18] M. Jimbo, H. Konno, S. Odake, and J. Shiraishi. Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$: Drinfeld currents and vertex operators. Comm. Math. Phys. **199** (1999), 605–647.

[19] S. Khoroshkin, D. Lebedev, and S. Pakuliak. Elliptic algebra in the scaling limit. Comm. Math. Phys. **190** (1998), 597–627.

[20] H. Konno. An elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the fusion RSOS model. Comm. Math. Phys. **195** (1998), 373–403.

[21] S. Pakuliak, V. Rubtsov, and A. Silantyev. Classical elliptic current algebras. II. J. Gen. Lie Theory Appl. **2** (2008), 79–93.

Received September 28, 2007
Revised December 09, 2007