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Classification of 2-Dimensional Subalgebras and Corresponding Reductive Pairs of Lie Algebra of All Real 2 × 2 Matrices

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Abstract

The purpose of the article is to describe all 2-dimensional subalgebras and all corresponding reductive pairs of Lie algebra *g* of all 2 × 2 real matrices. This Lie algebra is 4-dimensional as a vector space, it's not simple, and it's not solvable. The evaluation procedure utilizes canonical bases for subspaces that were introduced. Part I of the article contains necessary basic information. In Part II, all 2-dimensional subalgebras of the given Lie algebra *g* are classified. All reductive pairs {*h*, *m*} with 2-dimensional subalgebras *h* are found in Part III. The separate article contributes classification of all 3-dimensional subalgebras and its reductive pairs. Together, both articles give the total classification of all subalgebras and all reductive pairs of Lie algebra g.

Keywords: Lie algebra; Subalgebras; Reductive pairs

Introduction

Katsumi Nomizu introduced reductive homogeneous spaces at his fundamental manuscript [1,2] where the author investigated invariant affine connections and Riemannian metrics on them. Sagle and Winter at their article [3] analyzed algebraic structures generated by reductive pairs of simple Lie algebras. One more problem that concerns to this article is classification of subalgebras of low dimensional Lie algebras. For example, Patera and Winternitz classified all subalgebras of real Lie algebras of dimensions d=3 and d=4 at the manuscript [4]. Their classification of subalgebras of real Lie algebras was done by a representative of each conjugacy class where the conjugacy was considered under the group of inner automorphisms of Lie algebras. All the articles mentioned above have stimulated this research of all 2-dimensional subalgebras and their reductive pairs of Lie algebra g of all real 2 × 2 matrices. In contrast to the article [5], the current research is utilized a different method. Our method involves canonical bases for subspaces [1] that allow us to find all 2-dimensional subalgebras and the corresponding reductive pairs of the given Lie algebra g. This article finalizes the total classification of reductive pairs of the considering Lie algebra. New knowledge concerning the structure of Lie algebra g is important for Algebra, Geometry, and Physics.

Part I. Basic information and necessary statement

We remind some information for the readers' convenience that includes the basic statement about canonical bases from the article [1].

Definition 1: Let g be Lie algebra, h be Lie subalgebra of g. If there exists a subspace m of g such that $g=h\oplus m$ and $[h, m]\subset m$, then {h, m} is called a reductive pair of g, and {g, h, m} is called a reductive triple. We say also that subspace m is a reductive complement for h at g.

Lie algebra g and its standard basis

This Lie algebra contains all 2 × 2 matrices over the field of all real numbers. The standard basis of this algebra consists of the next four matrices

$$\vec{e}_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \vec{e}_2=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ \vec{e}_3=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ \vec{e}_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

It is well known that the Lie multiplication operation [A, B] for any two square matrices A and B of the same size is defined to be [A, B]=AB–BA. According this rule, the fundamental nonzero products of the basic vectors (matrices) $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ can be computed:

$$\left[\vec{e}_1,\vec{e}_2\right]=\vec{e}_2,\ \left[\vec{e}_1,\vec{e}_3\right]=-\vec{e}_3,\ \left[\vec{e}_2,\vec{e}_3\right]=\vec{e}_1-\vec{e}_4,\ \left[\vec{e}_2,\vec{e}_4\right]=\vec{e}_2,\ \left[\vec{e}_3,\vec{e}_4\right]=-\vec{e}_3\ . \quad (*)$$

All other products of basic vectors are zeros.

Canonical bases for 2-dimensional subspaces of 4-dimensional vector space

Let $h=Span\{\vec{a},\vec{b}\}$ be any 2-dimensional subspace of a 4-dimensional vector space g generated by linearly independent vectors $\vec{a}=a_1\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3+a_4\vec{e}_4$, and $\vec{b}=b_1\vec{e}_1+b_2\vec{e}_2+b_3\vec{e}_3+b_4\vec{e}_4$. According the article [1], all canonical bases for 2-dimensional subspaces h are:

(1) $\vec{a}=\vec{e}_1+a_3\vec{e}_3+a_4\vec{e}_4,\ \vec{b}=\vec{e}_2+b_3\vec{e}_3+b_4\vec{e}_4$;

(2) $\vec{a}=\vec{e}_1+a_2\vec{e}_2+a_4\vec{e}_4\ \ \vec{b}=\vec{e}_3+b_4\vec{e}_4$;

(3) $\vec{a}=\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3,\ \vec{b}=\vec{e}_4$; (4) $\vec{a}=\vec{e}_2+a_4\vec{e}_4,\ \vec{b}=\vec{e}_3+b_4\vec{e}_4$;

(5) $\vec{a}=\vec{e}_2+a_3\vec{e}_3,\ \vec{b}=\vec{e}_4$; (6) $\vec{a}=\vec{e}_3,\ \vec{b}=\vec{e}_4$.

Bases of the type (1) form 4-dimensional manifold, bases of the type (2) form 3-dimensional manifold, bases of the type (3) form 2-dimensional manifold, bases of the type (4) form another 2-dimensional manifold, bases of the type (5) form 1-dimensional manifold, and the basis (6) is a unique one. These manifolds are interested to be studied but the main goal of this article is different. About terminology: we will say just a basis (1)–(6) instead of a manifold of bases.

Part II. 2-dimensional subalgebras of Lie algebra g

Now we start to determine when a 2-dimensional subspace

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$h = Span\{\vec{a},\vec{b}\}$ is subalgebra of Lie algebra *g*. Obviously, the 6 nonequivalent canonical bases listed above should be used to analyze any subspace *h*. The condition $[h, h]\subset h$ will be checked for each of these 6 canonical bases. The table (*) of products from Part I will be used when a product $\lfloor \vec{a},\vec{b} \rfloor$ is computed.

Let $\vec{a} = \vec{e_1} + a_3\vec{e_3} + a_4\vec{e_4}$, $\vec{b} = \vec{e_2} + b_3\vec{e_3} + b_4\vec{e_4}$ be the basis (1) for *h*. Evaluate the product $\lfloor \vec{a},\vec{b} \rfloor$. We have

$$\lfloor \vec{a},\vec{b} \rfloor = \lfloor \vec{e_1} + a_3\vec{e_3} + a_4\vec{e_4}, \vec{e_2} + b_3\vec{e_3} + b_4\vec{e_4} \rfloor = \vec{e_2} - b_3\vec{e_3} - a_3(\vec{e_1} - \vec{e_4}) - a_3b_4\vec{e_3} - a_4\vec{e_2} + a_4b_3\vec{e_3}.$$

This product $\lfloor \vec{a},\vec{b} \rfloor$ should be located into the subspace *h*, i.e. $\lfloor \vec{a},\vec{b} \rfloor = x\vec{a} + y\vec{b}$. So, we have the following conditions for *x*, *y* and for the components of $\vec{a}$ and $\vec{b}$:

$x=-a_3$, $y=1a_4$, $xa_3+yb_3=-b_3-a_3b_4+a_4b_3$, $xa_4+yb_4=a_3$.

The last 2 conditions generate the next system of 2 equations for a_3, a_4, b_3, b_4:

$$-a_3^{\,2} + (1-a_4)b_3 = -b_3 - a_3b_4 + a_4b_3,\ -a_3a_4 + (1-a_4)b_4 = a_3,$$

or $2(1-a_4)b_3 = a_3^{\,2} - a_3b_4$, $(1-a_4)b_4 = (1+a_4)a_3$.

To solve this system of equations, consider two cases: $a_4=1$, and $a_4\neq1$. If $a_4=1$, then $a_3(a_3-b_3)=0$, $2a_3=0$. So, $a_3=0$, $a_4=1$, and b_3, b_4 are any components. The first set of 2-dimensional subalgebras is:

$$h_1 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_2} + b_3\vec{e_3} + b_4\vec{e_4}\}.$$

If $a_4\neq1$, then $b_4 = \dfrac{1+a_4}{1-a_4}a_3$, $b_3 = \dfrac{-a_4a_3^{\,2}}{(1-a_4)^2}$, and a new set of 2-dimensional subalgebras is:

$$h_2 = Span\{\vec{e_1} + a_3\vec{e_3} + a_4\vec{e_4}, \vec{e_2} - \frac{a_4a_3^{\,2}}{(1-a_4)^2}\vec{e_3} + \frac{1+a_4}{1-a_4}a_3\vec{e_4}\},\ a_4\neq1.$$

Let $\vec{a} = \vec{e_1} + a_2\vec{e_2} + a_4\vec{e_4}$, $\vec{b} = \vec{e_3} + b_4\vec{e_4}$ be the basis (2) for *h*. Evaluate the product $\left[\vec{a},\vec{b}\right]$:

$$\left[\vec{a},\vec{b}\right] = \left[\vec{e_1} + a_2\vec{e_2} + a_4\vec{e_4}, \vec{e_3} + b_4\vec{e_4}\right] = -\vec{e_3} + a_2(\vec{e_1} - \vec{e_4}) + a_2b_4\vec{e_2} + a_4\vec{e_3} = x\vec{a} + y\vec{b}.$$

This vector equality generates the next system of 2 equations for a_2, a_4, b_4:

$$a_2^{\,2} = a_2b_4,\ a_2a_4 + (a_4-1)b_4 = -a_2.$$

The 2[nd] equation gives the result $b_4 = \dfrac{1+a_4}{1-a_4}a_2$ ($a_4\neq1$), and the 1[st] equation produces $b_4=a_2$ or $a_2=0$. If $a_2=0$ then b4=0. If b4=a_2 then $a_4=0$. If $a_4=1$ then $a_2=0$. So, three new sets of 2-dimensional subalgebras are obtained:

$$h_3 = Span\{\vec{e_1} + a_4\vec{e_4}, \vec{e_3}\}\ (a_4\neq1),\quad h_4 = Span\{\vec{e_1} + a_2\vec{e_2}, \vec{e_3} + a_2\vec{e_4}\},$$

$$h_5 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_3} + b_4\vec{e_4}\}.$$

Let $\vec{a} = \vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}$, $\vec{b} = \vec{e_4}$ be the basis (3) for *h*. Compute the product $\left[\vec{a},\vec{b}\right]$:

$$\left[\vec{a},\vec{b}\right] = \left[\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}, \vec{e_4}\right] = a_2\vec{e_2} - a_3\vec{e_3} = x\vec{a} + y\vec{b}.$$

The last vector equality generates the values $x=0$, $y=0$, and we have immediately $a_2=0$, $a_3=0$. This means that the following abelian subalgebra is obtained:

$$h_6 = Span\{\vec{e_1}, \vec{e_4}\}.$$

Let $\vec{a} = \vec{e_2} + a_4\vec{e_4}$, $\vec{b} = \vec{e_3} + b_4\vec{e_4}$ be the basis (4) for *h*. Evaluate the product $\left[\vec{a},\vec{b}\right]$, and determine when it belongs to *h*. We have:

$$\left[\vec{a},\vec{b}\right] = \left[\vec{e_2} + a_4\vec{e_4}, \vec{e_3} + b_4\vec{e_4}\right] = (\vec{e_1} - \vec{e_4}) + b_4\vec{e_2} + a_4\vec{e_3} = x\vec{a} + y\vec{b}.$$

The last vector equality is impossible because basic vector $\vec{e_1}$ can't be generated by vectors $\vec{b}$ and $\vec{b}$. This means that no subalgebra exists with the basis (4).

Let $\vec{a} = \vec{e_2} + a_3\vec{e_3}$, $\vec{b} = \vec{e_4}$ be the basis (5) for *h*. Evaluate the product $\left[\vec{a},\vec{b}\right]$, and determine when it belongs to *h*. We have:

$$\left[\vec{a},\vec{b}\right] = \left[\vec{e_2} + a_3\vec{e_3}, \vec{e_4}\right] = \vec{e_2} - a_3\vec{e_3} = x\vec{a} + y\vec{b}.$$

The last vector equality is satisfied if and only if $a_3=0$. So, the following subalgebra is obtained $h_7 = Span\{\vec{e_2}, \vec{e_4}\}$.

Let $\vec{a} = \vec{e_3}$, $\vec{b} = \vec{e_4}$ be the basis (6) for *h*. Evaluate the product $\left[\vec{a},\vec{b}\right]$, and determine if it belongs to *h*. We have $\left[\vec{a},\vec{b}\right] = \left[\vec{e_3}, \vec{e_4}\right] = -\vec{e_3} = -\vec{a} \in Span\{\vec{a},\vec{b}\}$.

So, the new 2-dimensional subalgebra $h_8 = Span\{\vec{e_3}, \vec{e_4}\}$ of Lie algebra *g* is found.

The next theorem summarizes all results of Part II.

Theorem 1: Lie algebra *g* has two different 2-parameters sets of 2-dimensional subalgebras:

$$h_1 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_2} + b_3\vec{e_3} + b_4\vec{e_4}\};$$

$h_2 = Span\{\vec{e_1} + a_3\vec{e_3} + a_4\vec{e_4}, \vec{e_2} - \dfrac{a_4a_3^{\,2}}{(1-a_4)^2}\vec{e_3} + \dfrac{1+a_4}{1-a_4}a_3\vec{e_4}\}$, $a_4\neq1$; three different 1-parameter sets of 2-dimensional subalgebras.

$$h_3 = Span\{\vec{e_1} + a_4\vec{e_4}, \vec{e_3}\},\ a_4\neq1,\ h_4 = Span\{\vec{e_1} + a_2\vec{e_2}, \vec{e_3} + a_2\vec{e_4}\},\ h_5 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_3} + b_4\vec{e_4}\}$$

and three special 2-dimensional subalgebras,

$$h_6 = Span\{\vec{e_1}, \vec{e_4}\};\quad h_7 = Span\{\vec{e_2}, \vec{e_4}\};\ h_8 = Span\{\vec{e_3}, \vec{e_4}\}.$$

Part III. Reductive pairs with 2-dimensional subalgebras of Lie algebra *g*

How many of 2-dimensional subalgebras *h* form reductive pairs $\{h, m\}$ of Lie algebra *g*? To answer this question, we will find all reductive complements *m* for each subalgebra *h* such that the conditions $[h, m]\subset m$ and $g=h\oplus m$ are satisfied. A complement *m* for any 2-dimensional subalgebra *h* should be a 2-dimensional subspace, and we can describe it as $m = Span\{\vec{c},\vec{d}\}$ where vectors $\vec{c}$ and $\vec{d}$ form some canonical basis for *m*. Remind all canonical bases for 2-dimensional subspaces:

(1) $\vec{c} = \vec{e_1} + c_3\vec{e_3} + c_4\vec{e_4}$, $\vec{d} = \vec{e_2} + d_3\vec{e_3} + d_4\vec{e_4}$;

(2) $\vec{c} = \vec{e_1} + c_2\vec{e_2} + c_4\vec{e_4}$, $\vec{d} = \vec{e_3} + d_4\vec{e_4}$;

(3) $\vec{c} = \vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3}$, $\vec{d} = \vec{e_4}$; (4) $\vec{c} = \vec{e_2} + c_4\vec{e_4}$, $\vec{d} = \vec{e_3} + d_4\vec{e_4}$,;

(5) $\vec{c} = \vec{e_2} + c_3\vec{e_3}$, $\vec{d} = \vec{e_4}$,; (6) $\vec{c} = \vec{e_3}$, $\vec{d} = \vec{e_4}$,.

We start to utilize five sets of 2-dimensional subalgebras and three special 2-dimensional subalgebras listed in below Theorem 1.

Theorem 1: Subalgebra $h_1 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_2} + b_3\vec{e_3} + b_4\vec{e_4}\}$.

Find all reductive complements for h_1 if they exist.

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Let $\vec{c}=\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4},\ \vec{d}=\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ be the basis (1) for a possible reductive complement *m*. Multiply basic vectors $\vec{a}=\vec{e_1}+\vec{e_4},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$, by $\vec{c}$ and $\vec{d}$.

$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=-c_3\vec{e_3}+c_3\vec{e_3}=\vec{0}$; it is the identity.

$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_2}-d_3\vec{e_3}-\vec{e_2}+d_3\vec{e_3}=\vec{0}$; it's the identity.

$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=-\vec{e_2}+c_3(\vec{e_1}-\vec{e_4})+c_4\vec{e_2}+b_3\vec{e_3}-b_3c_4\vec{e_3}+b_4c_3\vec{e_3}=x_3\vec{c}+y_3\vec{d}.$

So, $x_3=c_3$, $y=c_4-1$, $x_3c_3+y_3d_3=b_3-b_3c_4+b_4c_3$, $x_3c_4+y_3d_4=-c_3$.

$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}-b_3(\vec{e_1}-\vec{e_4})-b_3d_4\vec{e_3}-b_4\vec{e_2}+b_4d_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$

So, $x_4=d_3-b_3$, $y_4=d_4-b_4$, $x_4c_3+y_4d_3=-b_3d_4+b_4d_3$, $x_4c_4+y_4d_4=b_3-d_3$.

The conditions above generate the following system of 4 nonlinear equations for components c_3, c_4, d_3, d_4:

$c_3^{\,2}+(c_4-1)d_3=b_3-b_3c_4+b_4c_3$, $c_3c_4+(c_4-1)d_4=-c_3$,

$(d_3-b_3)c_3+(d_4-b_4)d_3=-b_3d_4+b_4d_3$, $(d_3-b_3)c_4+(d_4-b_4)d_4=b_3-d_3$.

It makes sense to solve the system in two different cases: $c_4=1$, and $c_4\neq 1$.

If $c_4=1$, then $c_3=0$, $(d_4-b_4)d_3=b_4d_3-b_3d_4$, $2(d_3-b_3)=(b_4-d_4)d_4$. We see that $\vec{c}=\vec{e_1}+\vec{e_4}$, and $\vec{d}$ is some vector determined by the last two equations above. This pair $\{h_1, m\}$ with $m=Span\{\vec{e_1}+\vec{e_4},\vec{d}\}$ is not a reductive pair because the intersection $h_1\cap m=\{\vec{e_1}+\vec{e_4}\}\neq\vec{0}$ isn't zero vector for any $\vec{d}$.

If $c_4\neq 1$, then $d_4=\dfrac{1+c_4}{1-c_4}c_3,\ d_3=\dfrac{c_3-b_4}{1-c_4}c_3-b_3,\ d_3=\dfrac{b_4-d_4}{c_4+1}d_4+b_3$ and $(d_3-b_3)c_3+(d_4-b_4)d_3=b_4d_3-b_3d_4$.

Utilizing the 1st formula for d_4, we compare two different values for d_3. The corresponding procedure is long, we omit details. At the end of it, we obtain the following equation for c_3:

$c_3^{\,2}-b_4(1-c_4)c_3-b_3(1-c_4)^2=0$. (**)

Substitute now formulas for d_3 and d_4 obtained above into the 3rd equation that was not used before: $(d_3-b_3)c_3+(d_4-b_4)d_3=b_4d_3-b_3d_4$. Simplifying step by step this equation and utilizing several times the equation (**), we obtain the following result: $c_3c_4=0$. So, $c_3=0$ or $c_4=0$.

If $c_3=0$ then from the equation (**) we obtain $c_4=1$ or $b_3=0$. For $c_4=1$ we have $d_3=b_3$, $d_4=0$, and the following pair appears $h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}\},\ m=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_3\vec{e_3}\}$. This pair is not reductive because the intersection $h_1\cap m=\{\vec{e_1}+\vec{e_4}\}$ is not zero vectors. For the case $b_3=0$ we have $d_3=0$, $d_4=0$. The corresponding pair for this case is $h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4}\},\ m_1=Span\{\vec{e_1}+c_4\vec{e_4},\vec{e_2}\}$. This pair is reductive if $b_4\neq 0$, $c_4\neq 1$.

If $c_4=0$ then $d_4=c_3$, $d_3=0$, and c_3 is the solution of the equation $c_3^{\,2}-b_4c_3-b_3=0$. We have the following reductive pair:

$h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}\},\ m_2=Span\{\vec{e_1}+c_3\vec{e_3},\vec{e_2}+c_3\vec{e_4}\}$, $c_3\neq b_4$, and c_3 is the solution of the equation $c_3^{\,2}-b_4c_3-b_3=0$.

2. Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4},\ \vec{d}=\vec{e_3}+d_4\vec{e_4}$, be the basis (2). Multiply vectors $\vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$,

$\vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$ by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=c_2\vec{e_2}-c_2\vec{e_2}=\vec{0}$; it is the identity.

$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_3}+\vec{e_3}=\vec{0}$; it is the identity.

$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=-\vec{e_2}+c_4\vec{e_2}+b_3\vec{e_3}-b_3c_2(\vec{e_1}-\vec{e_4})-b_3c_4\vec{e_3}-$
$-b_4c_2\vec{e_2}=x_3\vec{c}+y_3\vec{d}.$

So, $x_3=-b_3c_2$, $y_3=b_3-b_3c_4$, $x_3c_2=c_4-1-b_4c_2$, $x_3c_4+y_3d_4=b_3c_2$.

$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}-b_3d_4\vec{e_3}+b_4\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$

So, $x_4=1$, $y_4=b_4-b_3d_4$, $x_4c_2=d_4$, $x_4c_4+y_4d_4=-1$.

We obtain the next system of 4 equations for the components c_2, c_4, d_4:

$-b_3c_2^{\,2}=c_4-1-b_4c_2,\ -b_3c_2c_4+b_3(1-c_4)d_4=b_3c_2,\ c_2=b_4,\ c_4+(b_4-b_3d_4)d_4=-1\cdot$

Analyzing this system of 4 equations, we obtain the following solution of it (details are omitted): $d_4=c_2$, $c_4=0$, and c_2 is the solution of the equation $b_3c_2^{\,2}-b_4c_2-1=0$. The corresponding reductive pair is:

$h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}\},\ m_3=Span\{\vec{e_1}+c_2\vec{e_2},\vec{e_3}+c_2\vec{e_4}\}$,

where $b_3c_2^{\,2}-b_4c_2-1=0$.

3. Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_3\vec{e_3},\ \vec{d}=\vec{e_4}$, be the basis (3). Multiply vectors $\vec{a}=\vec{e_1}+\vec{e_4},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$, by $\vec{c}$ and $\vec{d}$. We obtain the following products:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_3\vec{e_3}\right]=c_2\vec{e_2}-c_3\vec{e_3}-c_2\vec{e_2}+c_3\vec{e_3}=\vec{0}$; it is the identity.

$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_4}\right]=\vec{0}$; it is the identity.

$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=-\vec{e_2}+c_4\vec{e_2}+b_3\vec{e_3}-b_3c_2(\vec{e_1}-\vec{e_4})-b_3c_4\vec{e_3}-$
$b_4c_2\vec{e_2}=x_3\vec{c}+y_3\vec{d}.$

$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_4}\right]=\vec{e_2}-b_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$

The third and fourth equalities generate the following conditions:

$x_3=c_3-b_3c_2$, $y_3=b_3c_2-c_3$, $x_3c_2=-1-b_4c_2$, $x_3c_3=b_3+b_4c_3$ $x_4=0$, $y_4=0$, $x_4c_2=1$, $x_4c_3=-b_3$.

These conditions produce the obvious contradiction: 0=1. This means that no reductive pair for h_1 exists when the complement has the basis (3).

Let $\vec{c}=\vec{e_2}+c_4\vec{e_4},\ \vec{d}=\vec{e_3}+d_4\vec{e_4}$ be the basis (4). Multiply vectors $\vec{a}=\vec{e_1}+\vec{e_4},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$ by $\vec{c}$ and $\vec{d}$. We have two identities $\lfloor\vec{a},\vec{c}\rfloor=\vec{0},\ \lfloor\vec{a},\vec{d}\rfloor=\vec{0}$, and

$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_2}+c_4\vec{e_4}\right]=c_4\vec{e_2}-b_3(\vec{e_1}-\vec{e_4})-b_3c_4\vec{e_3}-b_4\vec{e_2}=x_3\vec{c}+y_3\vec{d},$

$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}-b_3d_4\vec{e_3}+b_4\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$

The product $\vec{c},\vec{d}$ can't be generated by vectors $\vec{c},\vec{d}$. So, no reductive complement with basis (4) exists for subalgebra h_1.

Let $\vec{c}=\vec{e_2}+c_3\vec{e_3},\ \vec{d}=\vec{e_4}$ be the basis (5). Multiply vectors $\vec{a}=\vec{e_1}+\vec{e_4},\vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$, by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_2}+c_3\vec{e_3}\right]=\vec{e_2}-c_3\vec{e_3}-\vec{e_2}+c_3\vec{e_3}=\vec{0}$; it is the identity.

$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+\vec{e_4},\vec{e_4}\right]=\vec{0}$; it is the identity.

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_2}+c_3\vec{e_3}\right]=c_3(\vec{e_1}-\vec{e_4})-b_3(\vec{e_1}-\vec{e_4})+b_3c_3\vec{e_3}-b_4\vec{e_2}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_4}\right]=\vec{e_2}-b_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

The solution for those vector equations is $c_3=0$, $b_3=0$. As the result, the following pair $h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4}\},\ m=Span\{\vec{e_2},\vec{e_4}\}$ is obtained but it is not a reductive pair because $\vec{c}=\vec{e_3},\ \vec{d}=\vec{e_4}$

6. Let $\vec{c}=\vec{e_3},\vec{d}=\vec{e_4}$ be the basis (6). Multiply vectors $\vec{a}=\vec{e_1}+\vec{e_4},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}$ by $\vec{c}$ and $\vec{d}$. We have two identities $\left[\vec{a},\vec{c}\right]=\vec{0}$, $\left[\vec{a},\vec{d}\right]=\vec{0}$, and

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_3}\right]=(\vec{e_1}-\vec{e_4})+b_4\vec{e_3}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\vec{e_4}\right]=\vec{e_2}-b_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

The products $\left[\vec{b},\vec{c}\right]$ and $\left[\vec{b},\vec{d}\right]$ are not generated by vectors $\vec{c},\vec{d}$. This means that no reductive complement with the basis (6) exists for subalgebra h_1.

Sub algebra $h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4}\}$, $a_4\neq 1$.

Find reductive complements for h_2 if they exist.

Let $\vec{c}=\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4},\ \vec{d}=\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ be the basis (1) for a possible reductive complement m. Multiply basic vectors $\vec{a}$, $\vec{b}$ by $\vec{c}$, $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=-c_3\vec{e_3}+a_3\vec{e_3}-a_3c_4\vec{e_3}+a_4c_3\vec{e_3}=x_1\vec{c}+y_1\vec{d}.$$

So, $x_1=0$, $y=0$, $x_1c_3+y_1d_3=a_3-c_3-a_3c_4+a_4c_3$, $x_1c_4+y_1d_4=0$.

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_2}-d_3\vec{e_3}-a_3(\vec{e_1}-\vec{e_4})-a_3d_4\vec{e_3}-a_4\vec{e_2}+a_4d_3\vec{e_3}=x_2\vec{c}+y_2\vec{d}.$$

So, $x_2=-a_3$, $y_2=a_4$, $x_2c_3+y_2d_3=-d_3-a_3d_4+a_4d_3$, $x_2c_4+y_2d_4=a_3$.

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=-\vec{e_2}+c_3(\vec{e_1}-\vec{e_4})+c_4\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{a_4a_3^2}{(1-a_4)^2}c_4\vec{e_3}+\frac{1+a_4}{1-a_4}a_3c_3\vec{e_3}=x_3\vec{c}+y_3\vec{d}.$$

So, $x_3=c_3$, $y_3=c_4-1$, $x_3c_3+y_3d_3=\frac{a_4a_3^2}{(1-a_4)^2}(c_4-1)+\frac{1+a_4}{1-a_4}a_3c_3$, $x_3c_4+y_3d_4=-c_3$.

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}+\frac{a_4a_3^2}{(1-a_4)^2}d_4\vec{e_3}+\frac{a_4a_3^2}{(1-a_4)^2}(\vec{e_1}-\vec{e_4})-\frac{1+a_4}{1-a_4}a_3\vec{e_2}+\frac{1+a_4}{1-a_4}a_3d_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

So, $x_4=d_3+\frac{a_4a_3^2}{(1-a_4)^2}$, $y_4=d_4-\frac{1+a_4}{1-a_4}a_3$, $x_4c_3+y_4d_3=\frac{a_4a_3^2d_4}{(1-a_4)^2}+\frac{1+a_4}{1-a_4}a_3d_3$,

$x_4c_4+y_4d_4=-d_3-\frac{a_4a_3^2}{(1-a_4)^2}$

From the equalities above, we obtain the following system of 7 nonlinear equations for components c_3, c_4, d_3, d_4:

$a_3-c_3-a_3c_4+a_4c_3=0$, $-a_3c_3+(1-a_4)d_3=-d_3-a_3d_4+a_4d_3$, $-a_3c_4+(1-a_4)d_4=a_3$,

$c_3c_4+(c_4-1)d_4=-c_3$, $c_3c_3+(c_4-1)d_3=\frac{a_4a_3^2}{(1-a_4)^2}(c_4-1)+\frac{1+a_4}{1-a_4}a_3c_3$,

$$[d_3+\frac{a_4a_3^2}{(1-a_4)^2}]c_3+(d_4-\frac{1+a_4}{1-a_4}a_3)d_3=\frac{a_4a_3^2}{(1-a_4)^2}d_4+\frac{1+a_4}{1-a_4}a_3d_3,$$

$$[d_3+\frac{a_4a_3^2}{(1-a_4)^2}]c_4+(d_4-\frac{1+a_4}{1-a_4}a_3)d_4=-d_3-\frac{a_4a_3^2}{(1-a_4)^2}.$$

The 1st equation gives us $c_3=\frac{1-c_4}{1-a_4}a_3$, the 2nd equation gives $d_3=\frac{a_3(c_3-d_4)}{2(1-a_4)}$, the 3rd equation gives $d_4=\frac{1+c_4}{1-a_4}a_3$, and the 4th equation gives $d_4=\frac{1+c_4}{1-c_4}c_3$. Substitute the values found for c_3, d_3, d_4 into the 5th equation, and simplify the corresponding expression. We obtain the identity $1-c_4=1-c_4$. Substituting the values found for c_3, d_3, d_4 into the 6th equation, we obtain the identity 0=0 as well. Substitute the same values for c_3, d_3, d_4 into the 7th equation. Simplifying the corresponding expression, we obtain the equation $2c_4^2+c_4(1-a_4)=0$. This equation has two solutions for c_4: $c_4=0$ or $c_4=\frac{a_4-1}{2}$. So, we obtain two pairs for subalgebra h_2. The first pair is $\{h_2, m_1\}$ where:

$$h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4}\},\ a_4\neq 1,$$

$m_1=Span\{\vec{e_1}+\frac{a_3}{1-a_4}\vec{e_3},\vec{e_2}+\frac{a_3}{1-a_4}\vec{e_4}\}$. This pair is reductive if $a_3\neq 0$, $a_4\neq 0$.

The second pair is $\{h_2, m_2\}$ where:

$$h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4}\},\ a_4\neq 1,$$

$$m_2=Span\{\vec{e_1}+\frac{3-a_4}{2(1-a_4)}a_3\vec{e_3}+\frac{a_4-1}{2}\vec{e_4},\vec{e_2}+\frac{a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{2(1-a_4)}a_3\vec{e_4}\}.$$

This pair is reductive if $a3\neq 0$, $a4\neq -1$.

2. Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4},\ \vec{d}=\vec{e_3}+d_4\vec{e_4}$ be the basis (2) for a possible reductive complement m. Multiply basic vectors $\vec{a}$, $\vec{b}$ from h_2 by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=c_2\vec{e_2}+a_3\vec{e_3}-a_3c_2(\vec{e_1}-\vec{e_4})-a_3c_4\vec{e_3}-a_4c_2\vec{e_2}=x_1\vec{c}+y_1\vec{d}.$$

So, $x_1=-a_3c_2$, $y_1=a_3-a_3c_4$, $x_1c_2=c_2-a_4c_2$, $x_1c_4+y_1d_4=a_3c_2$.

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_3}-a_3d_4\vec{e_3}+a_4\vec{e_3}=x_2\vec{c}+y_2\vec{d}.$$

So, $x_2=0$, $y_2=a_4-a_3d_4-1$, $x_2c_2=0$, $x_2c_4+y_2d_4=0$.

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=-\vec{e_2}+c_4\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+c_2\frac{a_4a_3^2}{(1-a_4)^2}(\vec{e_1}-\vec{e_4})+c_4\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}-\frac{1-a_4}{1-a_4}a_3c_2\vec{e_2}=x_3\vec{c}+y_3\vec{d}.$$

So, $x_3=\frac{a_4a_3^2}{(1-a_4)^2}c_2,\ y_3=\frac{a_4a_3^2}{(1-a_4)^2}(c_4-1),\ x_3c_2=c_4-1-\frac{1+a_4}{1-a_4}a_3c_2,$
$x_3c_4+y_3d_4=-\frac{a_4a_3^2}{(1-a_4)^2}c_2$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}+ d_4\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

So, $x_4=1$, $y_4=\frac{a_4a_3^2}{(1-a_4)^2}d_4+\frac{1+a_4}{1-a_4}a_3$, $x_4c_2=d_4$, $x_4c_4+y_4d_4=-1$.

These equalities produce the system of 7 nonlinear equations for components c_2, c_4, d_4:

$-a_3c_2^2=c_2(1-a_4)$, $-a_3c_2c_4+(a_3-a_3c_4)d_4=a_3c_2$, $(a_4-a_3d_4-1)d_4=0$,

$\frac{a_4a_3^2}{(1-a_4)^2}c_2^2=c_4-1-\frac{1+a_4}{1-a_4}a_3c_2$,

$\frac{a_4a_3^2}{(1-a_4)^2}c_2c_4+(c_4-1)\frac{a_4a_3^2}{(1-a_4)^2}d_4=-\frac{a_4a_3^2}{(1-a_4)^2}c_2$,

$d_4=c_2$, $c_4+\frac{a_4a_3^2}{(1-a_4)^2}d_4^2+\frac{1+a_4}{1-a_4}a_3d_4=-1$.

Consider two cases for the system. If $c_2=0$ then $c_4=1$, $d_4=0$, and from the last equation $c_4=-1$. We obtain a contradiction, so the system has no solution at this case.

Let $c_2\neq 0$. Then from the 1st equation we have $c_2=\frac{a_4-1}{a_3}$, and from the 2nd equation we obtain $a_3c_4=0$. The last equation gives two possible results: $a_3=0$ or $c_4=0$. If $a_3=0$ then from the 4th and 7th equations we obtain the contradiction $c_4=1$, $c_4=-1$ again. So, $c_4=0$. Substitute the values of c_2, d_4, c_4 into the 4th equation, the 5th equation, and the 7th equation. We obtain the identities at all these cases. So, the following reductive pair appears:

$h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4}\}$, $a_4\neq 1$,

$m_3=Span\{\vec{e_1}+\frac{a_4-1}{a_3}\vec{e_2},\vec{e_3}+\frac{a_4-1}{a_3}\vec{e_4}\}$, $a_3\neq 0$.

3. Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_3\vec{e_3}$, $\vec{d}=\vec{e_4}$ be the basis (3). Multiply basic vectors $\vec{a}$, $\vec{b}$ by $\vec{c}$ and $\vec{d}$.

One product is very important for this case. Compute $\left[\vec{b},\vec{d}\right]$:

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_4}\right]=\vec{e_2}+\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

So, $x_4=0$, $y_4=0$, $x_4c_2=1$, $x_4c_3=\frac{a_4a_3^2}{(1-a_4)^2}$.

This product $\left[\vec{b},\vec{d}\right]$ produces the obvious contradiction $x_4=0$, $x_4c_2=1$. So, subalgebra h_2 has no reductive complement with basis (3).

4. Let $\vec{c}=\vec{e_2}+c_4\vec{e_4}$, $\vec{d}=\vec{e_3}+d_4\vec{e_4}$ be the basis (4). Multiply vector $\vec{b}$ from h_2 and vector $\vec{d}$. We have:

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}+\frac{a_4a_3^2}{(1-a_4)^2}d_4\vec{e_3}+ \frac{1+a_4}{1-a_4}a_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

This product $\left[\vec{b},\vec{d}\right]$ contains vector $\vec{e_1}$ which can't be generated by vectors $\vec{c}$ and $\vec{d}$. This means that h_2 has no reductive complement with basis (4).

5. Let $\vec{c}=\vec{e_2}+c_3\vec{e_3}$, $\vec{d}=\vec{e_4}$ be the basis (5). Multiply basic vectors $\vec{b}$, $\vec{b}$ by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}+c_3\vec{e_3}\right]=\vec{e_2}-c_3\vec{e_3}-a_3(\vec{e_1}-\vec{e_4})-a_4\vec{e_2}+a_4c_3\vec{e_3}=x_1\vec{c}+y_1\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_4}\right]=-a_3\vec{e_3}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_2}+c_3\vec{e_3}\right]=c_3(\vec{e_1}-\vec{e_4})+\frac{a_4a_3^2}{(1-a_4)^2}(\vec{e_1}-\vec{e_4})- \frac{a_3(1+a_4)}{1-a_4}\vec{e_2}+\frac{1+a_4}{1-a_4}a_3c_3\vec{e_3}.$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_4}\right]=\vec{e_2}+\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

Vector equalities $\left[\vec{a},\vec{c}\right]$ and $\left[\vec{b},\vec{c}\right]$ give immediately $a_3=0$, $c_3=0$. So, we obtain the following pair $h_2=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_2}\}$, $m=Span\{\vec{e_2},\vec{e_4}\}$ which is not reductive because $h_2\cap m\neq\vec{0}$.

6. Let $\vec{c}=\vec{e_3}$, $\vec{d}=\vec{e_4}$ be the basis (6). Multiply vectors $\vec{a}$, $\vec{b}$ by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_3}\right]=-\vec{e_3}+a_4\vec{e_3}=x_1\vec{c}+y_1\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_4}\right]=-a_3\vec{e_3}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_2}-\frac{a_4a_3^2}{(1-a_4)^2}\vec{e_3}+\frac{1+a_4}{1-a_4}a_3\vec{e_4},\vec{e_3}\right]=(\vec{e_1}-\vec{e_4})+\frac{1+a_4}{1-a_4}a_3\vec{e_3}=x_3\vec{c}+y_3\vec{d}.$$

The product $\left[\vec{b},\vec{c}\right]$ contains vector $\vec{c}$ which can't be generated by vectors $\vec{c}$ and $\vec{d}$. This means that h_2 has no reductive complement with basis (6).

Subalgebra $h_3=Span\{\vec{e_1}+a_4\vec{e_4},\vec{e_3}\}$. This subalgebra has no reductive complement at all. The corresponding evaluation procedure is long as that for subalgebra h_1, therefore it's omitted.

Subalgebra $h_4=Span\{\vec{e_1}+a_2\vec{e_2},\vec{e_3}+a_2\vec{e_4}\}$.

Find reductive complements for h_4 if they exist.

1. Let $\vec{c}=\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}$, $\vec{d}=\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ be the basis (1) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e_1}+a_2\vec{e_2}$, $\vec{b}=\vec{e_3}+a_2\vec{e_4}$ by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1}+a_2\vec{e_2},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=-c_3\vec{e_3}-a_2\vec{e_2}+a_2c_3(\vec{e_1}-\vec{e_4})+a_2c_4\vec{e_2}=x_1\vec{c}+y_1\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1}+a_2\vec{e_2},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_2}-d_3\vec{e_3}+a_2d_3(\vec{e_1}-\vec{e_4})+a_2d_4\vec{e_2}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_3}+a_2\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=\vec{e_3}-c_4\vec{e_3}+a_2c_3\vec{e_3}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_3}+a_2\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}-a_2\vec{e_2}+a_2d_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

These vector equalities produce the following system of equations:

$a_2c_2c_3+a_2d_3(c_4-1)=-c_3$, $a_2c_3c_4+a_2d_4(c_4-1)=-a_2c_3$, $a_2c_3d_3+(1+a_2d_4)d_3=-d_3$,

$a_2d_3c_4+(1+a_2d_4)d_4=-a_2d_3$, $c_4=1+a_2c_3$, $-a_2d_3c_3=a_2d_3-d_4$, $c_4=-1-a_2d_4$.

This system of equations has the solution: $c_4=1+a_2c_3$ $d_3=-\frac{a_2c_3+1}{a_2^2}$, $d_4=-\frac{a_2c_3+2}{a_2}$, $a_2\neq 0$. If $a_2=0$ then the system of equations generates the contradiction $c_4=1$ and $c_4=-1$. So, we obtain one new reductive pair:

$$h_4=Span\{\vec e_1+a_2\vec e_2,\vec e_3+a_2\vec e_4\},$$

$$m_1=Span\{\vec e_1+c_3\vec e_3+(1+a_2c_3)\vec e_4,\vec e_2-\frac{a_2c_3+1}{a_2^2}\vec e_3-\frac{a_2c_3+2}{a_2}\vec e_4\}.$$

2. Let $\vec c=\vec e_1+c_2\vec e_2+c_4\vec e_4$, $\vec d=\vec e_3+d_4\vec e_4$ be the basis (2) for a possible reductive complement m. Multiply basic vectors $\vec a=\vec e_1+a_2\vec e_2$, $\vec b=\vec e_3+a_2\vec e_4$ of h_4 by $\vec c$ and $\vec d$. We have:

$$\left[\vec a,\vec c\right]=\left[\vec e_1+a_2\vec e_2,\vec e_1+c_2\vec e_2+c_4\vec e_4\right]=c_2\vec e_2-a_2\vec e_2+a_2c_4\vec e_2=x_1\vec c+y_1\vec d,$$

$$\left[\vec a,\vec d\right]=\left[\vec e_1+a_2\vec e_2,\vec e_3+d_4\vec e_4\right]=-\vec e_3+a_2(\vec e_1-\vec e_4)+a_2d_4\vec e_2=x_2\vec c+y_2\vec d,$$

$$\left[\vec b,\vec c\right]=\left[\vec e_3+a_2\vec e_4,\vec e_1+c_2\vec e_2+c_4\vec e_4\right]=\vec e_3-c_2(\vec e_1-\vec e_4)-c_4\vec e_3-a_2c_2\vec e_3=x_3\vec c+y_3\vec d,$$

$$\left[\vec b,\vec d\right]=\left[\vec e_3+a_2\vec e_4,\vec e_3+d_4\vec e_4\right]=-d_4\vec e_3+a_2\vec e_3=x_4\vec c+y_4\vec d.$$

The system of equations has two solutions: $a_2=0$, $c_2=0$, $d_4=0$, and $a_2\neq 0$, $c_2=0$, $d_4=0$, $c_4=1$. For the first case, the corresponding pair $h_4=Span\{\vec e_1,\vec e_3\}$,

$m=Span\{\vec e_1+c_4\vec e_4,\vec e_3\}$ is not reductive because $h_4\cap m\neq\vec 0$. For the second case, the corresponding pair $h_4=Span\{\vec e_1+a_2\vec e_2,\vec e_3+a_2\vec e_4\}$, $\vec c=\vec e_1+c_2\vec e_2+c_3\vec e_3$ is reductive if $a_2\neq 0$.

3. Let $\vec c=\vec e_1+c_2\vec e_2+c_3\vec e_3$, $\vec d=\vec e_4$ be the basis (3) for a possible reductive complement m. Multiply basic vectors $\vec a=\vec e_1+a_2\vec e_2$, $\vec b=\vec e_3+a_2\vec e_4$ of h_4 by $\vec c$ and $\vec d$. We have:

$$\left[\vec a,\vec c\right]=\left[\vec e_1+a_2\vec e_2,\vec e_1+c_2\vec e_2+c_3\vec e_3\right]=c_2\vec e_2-c_3\vec e_3-a_2\vec e_2+a_2c_3(\vec e_1-\vec e_4)=x_1\vec c+y_1\vec d,$$

$$\left[\vec a,\vec d\right]=\left[\vec e_1+a_2\vec e_2,\vec e_4\right]=a_2\vec e_2=x_2\vec c+y_2\vec d,$$

$$\left[\vec b,\vec c\right]=\left[\vec e_3+a_2\vec e_4,\vec e_1+c_2\vec e_2+c_3\vec e_3\right]=\vec e_3-c_2(\vec e_1-\vec e_4)+a_2c_3\vec e_3-a_2c_2\vec e_2=x_3\vec c+y_3\vec d,$$

$\left[\vec b,\vec d\right]=\left[\vec e_3+a_2\vec e_4,\vec e_4\right]=-\vec e_3=x_4\vec c+y_4\vec d$, where $x_4=0$, $y_4=0$.

The last vector equality generates a contradiction $\vec e_3=\vec 0$, so subalgebra h_4 has no reductive complement with basis (3).

4. Let $\vec c=\vec e_2+c_4\vec e_4$, $\vec d=\vec e_3+d_4\vec e_4$ be the basis (4) for a possible reductive complement m. Multiply basic vectors $\vec a=\vec e_1+a_2\vec e_2$, $\vec b=\vec e_3+a_2\vec e_4$ of h_4 by $\vec c$ and $\vec d$. We have:

$$\left[\vec a,\vec c\right]=\left[\vec e_1+a_2\vec e_2,\vec e_2+c_4\vec e_4\right]=\vec e_2+a_2c_4\vec e_2=x_1\vec c+y_1\vec d,$$

$$\left[\vec a,\vec d\right]=\left[\vec e_1+a_2\vec e_2,\vec e_3+d_4\vec e_4\right]=-\vec e_3+a_2(\vec e_1-\vec e_4)+a_2d_4\vec e_2=x_2\vec c+y_2\vec d,$$

$$\left[\vec b,\vec c\right]=\left[\vec e_3+a_2\vec e_4,\vec e_2+c_4\vec e_4\right]=-(\vec e_1-\vec e_4)-c_4\vec e_3-a_2\vec e_2=x_3\vec c+y_3\vec d,$$

$$\left[\vec b,\vec d\right]=\left[\vec e_3+a_2\vec e_4,\vec e_3+d_4\vec e_4\right]=-d_4\vec e_3+a_2\vec e_4=x_4\vec c+y_4\vec d.$$

The product $\left[\vec b\ \vec c\right]$ contains vector $\vec e_1$ that can't be generated by $\vec c$ and $\vec d$, so subalgebra h_4 has no reductive complement with basis (4).

5. Let $\vec c=\vec e_2+c_3\vec e_3$, $\vec d=\vec e_4$ be the basis (5) for a possible reductive complement m. Multiply basic vectors $\vec a=\vec e_1+a_2\vec e_2$, $\vec b=\vec e_3+a_2\vec e_4$ of h_4 by $\vec c$ and $\vec d$. We have:

$$\left[\vec a,\vec c\right]=\left[\vec e_1+a_2\vec e_2,\vec e_2+c_3\vec e_3\right]=\vec e_2-c_3\vec e_3+a_2c_3(\vec e_1-\vec e_4)=x_1\vec c+y_1\vec d,$$

$$\left[\vec a,\vec d\right]=\left[\vec e_1+a_2\vec e_2,\vec e_4\right]=a_2\vec e_2=x_2\vec c+y_2\vec d,$$

$$\left[\vec b,\vec c\right]=\left[\vec e_3+a_2\vec e_4,\vec e_2+c_3\vec e_3\right]=-(\vec e_1-\vec e_4)+a_2c_3\vec e_3-a_2\vec e_2=x_3\vec c+y_3\vec d,$$

$$\left[\vec b,\vec d\right]=\left[\vec e_3+a_2\vec e_4,\vec e_4\right]=-\vec e_3=x_4\vec c+y_4\vec d.$$

The product $\left[\vec b,\vec c\right]$ contains vector $\vec e_1$ that can't be generated by vectors $\vec c,\vec d$, so subalgebra h_4 has no reductive complement with basis (5).

6. Let $\vec c=\vec e_3$, $\vec d=\vec e_4$ be the basis (6) for a possible reductive complement m. Multiply basic vectors $\vec a=\vec e_1+a_2\vec e_2$, $\vec b=\vec e_3+a_2\vec e_4$ of h_4 by $\vec c$ and $\vec d$. We have:

$$\left[\vec a,\vec c\right]=\left[\vec e_1+a_2\vec e_2,\vec e_3\right]=-\vec e_3+a_2(\vec e_1-\vec e_4)=x_1\vec c+y_1\vec d,$$

$$\left[\vec a,\vec d\right]=\left[\vec e_1+a_2\vec e_2,\vec e_4\right]=a_2\vec e_2=x_2\vec c+y_2\vec d,$$

$$\left[\vec b,\vec c\right]=\left[\vec e_3+a_2\vec e_4,\vec e_3\right]=a_2\vec e_3=x_3\vec c+y_3\vec d,$$

$$\left[\vec b,\vec d\right]=\left[\vec e_3+a_2\vec e_4,\vec e_4\right]=-\vec e_3=x_4\vec c+y_4\vec d.$$

The system of vector equations has the solution $a_2=0$. The corresponding pair $h_4=Span\{\vec e_1,\vec e_3\}$, $m=Span\{\vec e_3,\vec e_4\}$ is not reductive because $h_4\cap m\neq\vec 0$.

Subalgebra $h_5=Span\{\vec e_1+\vec e_4,\vec e_3+b_4\vec e_4\}$.

Find reductive complements for h_5 if they exist.

Let $\vec c=\vec e_1+c_3\vec e_3+c_4\vec e_4$, $\vec d=\vec e_2+d_3\vec e_3+d_4\vec e_4$ be the basis (1) for a possible reductive complement m. Multiply basic vectors $\vec a=\vec e_1+\vec e_4$ and $\vec b=\vec e_3+b_4\vec e_4$ of h_5 by $\vec c$ and $\vec d$. We have:

$\left[\vec a,\vec c\right]=\left[\vec e_1+\vec e_4,\vec e_1+c_3\vec e_3+c_4\vec e_4\right]=-c_3\vec e_3+c_3\vec e_3=\vec 0$, it is the identity;

$\left[\vec b,\vec c\right]=\left[\vec e_3+b_4\vec e_4,\vec e_1+c_3\vec e_3+c_4\vec e_4\right]=\vec e_3-c_4\vec e_3+b_4c_3\vec e_3=x_3\vec c+y_3\vec d$, it is the identity;

$\left[\vec b,\vec c\right]=\left[\vec e_3+b_4\vec e_4,\vec e_1+c_3\vec e_3+c_4\vec e_4\right]=\vec e_3-c_4\vec e_3+b_4c_3\vec e_3=x_3\vec c+y_3\vec d$, $x_3=0$, $y_3=0$;

$$\left[\vec b,\vec d\right]=\left[\vec e_3+b_4\vec e_4,\vec e_2+d_3\vec e_3+d_4\vec e_4\right]=-(\vec e_1-\vec e_4)-d_4\vec e_3-b_4\vec e_2+b_4d_3\vec e_3=x_4\vec c+y_4\vec d.$$

The solution for the system of vector equalities is $c_4=1+b_4c_3$, $d_3=-\frac{1+b_4c_3}{b_4^2}$, $d_4=-\frac{2+b_4c_3}{b_4}$, $b_4\neq 0$. This solution produces the pair $h_5=Span\{\vec e_1+\vec e_4,\vec e_3+b_4\vec e_4\}$, but it is not reductive because $h_5\cap m=\{\vec e_1+c_3\vec e_3+(1+b_4c_3)\vec e_4\}\neq\vec 0$. If $b_4=0$ then the system of equalities has no solution because the contradiction $c_4=1$, $c_4=-1$ appears.

Let $\vec c=\vec e_1+c_2\vec e_2+c_4\vec e_4$, $\vec a=\vec e_1+\vec e_4$ be the basis (2) for a possible

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reductive complement m. Multiply basic vectors $\vec{a}=\vec{e}_1+\vec{e}_4$ and $\vec{b}=\vec{e}_3+b_4\vec{e}_4$ of h_5 by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_1+c_2\vec{e}_2+c_4\vec{e}_4\right]=c_2\vec{e}_2-c_2\vec{e}_2=\vec{0}$ - it is the identity;

$\left[\vec{a},\vec{d}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_3+d_4\vec{e}_4\right]=-\vec{e}_3+\vec{e}_3=\vec{0}$ - it is the identity;

$\left[\vec{b},\vec{c}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_1+c_2\vec{e}_2+c_4\vec{e}_4\right]=\vec{e}_3-c_2(\vec{e}_1-\vec{e}_4)-c_4\vec{e}_3-b_4c_2\vec{e}_2=x_3\vec{c}+y_3\vec{d}$;

$\left[\vec{b},\vec{d}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_3+d_4\vec{e}_4\right]=-d_4\vec{e}_3+b_4\vec{e}_3=x_4\vec{c}+y_4\vec{d}$.

These vector equalities generate the following system of equations:

$c_2(b_4-c_2)=0$, $(1-c_4)d_4=(1+c_4)c_2$, $d_4(b_4-d_4)=0$.

The system has four different solutions: $c_2=0$, $d_4=0$; $c_2=0$, $c_4=1$, $d_4=b_4$;

$c_2=b_4$, $c_4=1$, $d_4=0$; $c_2=b_4$, $c_4=0$, $d_4=b_4$.

These solutions produce the following pairs: $h_5=Span\{\vec{e}_1+\vec{e}_4,\vec{e}_3+b_4\vec{e}_4\}$, $m_1=Span\{\vec{e}_1+c_4\vec{e}_4,\vec{e}_3\}$; $h_5=Span\{\vec{e}_1+\vec{e}_4,\vec{e}_3+b_4\vec{e}_4\}$,

$m=Span\{\vec{e}_1+\vec{e}_4,\vec{e}_3+b_4\vec{e}_4\}$; $h_5=Span\{\vec{e}_1+\vec{e}_4,\vec{e}_3+b_4\vec{e}_4\}$,

$m_2=Span\{\vec{e}_1+b_4\vec{e}_2-\vec{e}_4,\vec{e}_3\}$;

$h_5=Span\{\vec{e}_1+\vec{e}_4,\vec{e}_3+b_4\vec{e}_4\}$, $m=Span\{\vec{e}_1+b_4\vec{e}_2,\vec{e}_3+b_4\vec{e}_4\}$.

Two pairs $\{h_5, m_1\}$ and $\{h_5, m_2\}$ are reductive if $b_4\neq 0$, other pairs are not reductive.

Let $\vec{c}=\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3$, $\vec{d}=\vec{e}_4$ be the basis (3) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e}_1+\vec{e}_4$ and $\vec{b}=\vec{e}_3+b_4\vec{e}_4$ of h_5 by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3\right]=c_2\vec{e}_2-c_2\vec{e}_2-c_3\vec{e}_3+c_3\vec{e}_3=\vec{0}$ - it is the identity;

$\left[\vec{a},\vec{d}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_4\right]=\vec{0}$ - it is the identity;

$\left[\vec{b},\vec{c}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3\right]=\vec{e}_3-c_2(\vec{e}_1-\vec{e}_4)-b_4c_2\vec{e}_2+b_4c_3\vec{e}_3=x_3\vec{c}+y_3\vec{d}$,

$\left[\vec{b},\vec{d}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_4\right]=-\vec{e}_3=x_4\vec{c}+y_4\vec{d}$.

The product $\left[\vec{b},\vec{d}\right]$ generates a contradiction $\vec{e}_3=\vec{0}$. So, subalgebra h_5 has no reductive complement with basis (3).

Let $\vec{c}=\vec{e}_2+c_4\vec{e}_4$, $\vec{d}=\vec{e}_3+d_4\vec{e}_4$ be the basis (4) for a possible reductive complement m.

Multiply basic vectors $\vec{a}=\vec{e}_1+\vec{e}_4$ and $\vec{b}=\vec{e}_3+b_4\vec{e}_4$ of h_5 by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_2+c_4\vec{e}_4\right]=\vec{e}_2-\vec{e}_2=\vec{0}$ - it is the identity;

$\left[\vec{a},\vec{d}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_3+d_4\vec{e}_4\right]=-\vec{e}_3+\vec{e}_3=\vec{0}$ - it is the identity;

$\left[\vec{b},\vec{c}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_2+c_4\vec{e}_4\right]=(\vec{e}_1-\vec{e}_4)-c_4\vec{e}_3-b_4\vec{e}_2=x_3\vec{c}+y_3\vec{d}$,

$\left[\vec{b},\vec{d}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_3+d_4\vec{e}_4\right]=-d_4\vec{e}_3+b_4\vec{e}_3=x_4\vec{c}+y_4\vec{d}$.

The product $\left[\vec{b},\vec{c}\right]$ contains vector $\vec{e}_1$ that can't be generated by vectors $\vec{c},\vec{d}$. This means that subalgebra h_5 has no reductive complement with basis (4).

5. Let $\vec{c}=\vec{e}_2+c_3\vec{e}_3$, $\vec{d}=\vec{e}_4$ be the basis (5) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e}_1+\vec{e}_4$ and $\vec{b}=\vec{e}_3+b_4\vec{e}_4$ of h_5 by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_2+c_3\vec{e}_3\right]=\vec{e}_2-c_3\vec{e}_3-\vec{e}_2+c_3\vec{e}_3=\vec{0}$ - it is the identity;

$\left[\vec{a},\vec{d}\right]=\left[\vec{e}_1+\vec{e}_4,\vec{e}_4\right]=\vec{0}$ - it is the identity;

$\left[\vec{b},\vec{c}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_2+c_3\vec{e}_3\right]=-(\vec{e}_1-\vec{e}_4)-b_4\vec{e}_2+b_4c_3\vec{e}_3=x_3\vec{c}+y_3\vec{d}$,

$\left[\vec{b},\vec{d}\right]=\left[\vec{e}_3+b_4\vec{e}_4,\vec{e}_4\right]=-\vec{e}_3=x_4\vec{c}+y_4\vec{d}$.

The product $\left[\vec{b},\vec{c}\right]$ contains vector $\vec{e}_1$ that can't be generated by vectors $\vec{c},\vec{d}$. This means that subalgebra h_5 has no reductive complement with basis (5).

6. Let $\vec{c}=\vec{e}_3$, $\vec{d}=\vec{e}_4$ be the basis (6) for a possible reductive complement m. Compare $m=Span\{\vec{e}_3,\vec{e}_4\}$ and $h_5=Span\{\vec{e}_1+\vec{e}_4,\vec{e}_3+b_4\vec{e}_4\}$. It's obvious that $m\cap h_5=\{\vec{e}_3+b_4\vec{e}_4\}$. So, this pair s not reductive.

Subalgebra $h_6=Span\{\vec{e}_1,\vec{e}_4\}$

Let $\vec{c}=\vec{e}_1+c_3\vec{e}_3+c_4\vec{e}_4$, $\vec{d}=\vec{e}_2+d_3\vec{e}_3+d_4\vec{e}_4$ be the basis (1) for a possible reductive complement m. Multiply basic vectors $\vec{b}=\vec{e}_4$ and $\vec{b}=\vec{e}_4$ of h_6 by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e}_1,\vec{e}_1+c_3\vec{e}_3+c_4\vec{e}_4\right]=-c_3\vec{e}_3=x_1\vec{c}+y_1\vec{d}$,

$\left[\vec{a},\vec{d}\right]=\left[\vec{e}_1,\vec{e}_2+d_3\vec{e}_3+d_4\vec{e}_4\right]=\vec{e}_2-d_3\vec{e}_3=x_2\vec{c}+y_2\vec{d}$,

$\left[\vec{b},\vec{c}\right]=\left[\vec{e}_4,\vec{e}_1+c_3\vec{e}_3+c_4\vec{e}_4\right]=c_3\vec{e}_3=x_3\vec{c}+y_3\vec{d}$,

$\left[\vec{b},\vec{d}\right]=\left[\vec{e}_4,\vec{e}_2+d_3\vec{e}_3+d_4\vec{e}_4\right]=-\vec{e}_2+d_3\vec{e}_3=x_4\vec{c}+y_4\vec{d}$.

The system of the vector equations has the solution $c_3=0$, $d_3=0$, $d_4=0$. The corresponding subspace $m=Span\{\vec{e}_1+c_4\vec{e}_4,\vec{e}_2\}$ is not a reductive complement for h_6 because $h_6\cap m=\{\vec{e}_1+c_4\vec{e}_4\}\neq\vec{0}$.

Let $\vec{c}=\vec{e}_1+c_2\vec{e}_2+c_4\vec{e}_4$, $\vec{d}=\vec{e}_3+d_4\vec{e}_4$ be the basis (2) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e}_1$ and $\vec{b}=\vec{e}_4$ of h_6 by $\vec{c}$ and $\vec{d}$. We have:

$\left[\vec{a},\vec{c}\right]=\left[\vec{e}_1,\vec{e}_1+c_2\vec{e}_2+c_4\vec{e}_4\right]=c_2\vec{e}_2=x_1\vec{c}+y_1\vec{d}$,

$\left[\vec{a},\vec{d}\right]=\left[\vec{e}_1,\vec{e}_3+d_4\vec{e}_4\right]=-\vec{e}_3=x_2\vec{c}+y_2\vec{d}$,

$\left[\vec{b},\vec{c}\right]=\left[\vec{e}_4,\vec{e}_1+c_2\vec{e}_2+c_4\vec{e}_4\right]=-c_2\vec{e}_2=x_3\vec{c}+y_3\vec{d}$,

$\left[\vec{b},\vec{d}\right]=\left[\vec{e}_4,\vec{e}_3+d_4\vec{e}_4\right]=\vec{e}_3=x_4\vec{c}+y_4\vec{d}$.

The system of the vector equalities has the solution $c_2=0$, $d_4=0$. The corresponding subspace $m=Span(\vec{e}_1+c_4\vec{e}_4,\vec{e}_3\}$ is not a reductive complement for h_6 because. $h_6\cap m=\{\vec{e}_1+c_4\vec{e}_4\}\neq\vec{0}$

Let $\vec{c}=\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3$, $\vec{d}=\vec{e}_4$ be the basis (3) for a possible reductive complement m. It is obvious that $h_6\cap m=\{\vec{e}_4\}\neq\vec{0}$, so subalgebra h_6 has no reductive complement with basis (3).

Let $\vec{c}=\vec{e_2}+c_4\vec{e_4}$, $\vec{d}=\vec{e_3}+d_4\vec{e_4}$ be the basis (4) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e_1}$ and $\vec{b}=\vec{e_4}$ of h_6 by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_1},\vec{e_2}+c_4\vec{e_4}\right]=\vec{e_2}=x_1\vec{c}+y_1\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_1},\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_3}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_4},\vec{e_2}+c_4\vec{e_4}\right]=\vec{e_2}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

The system of the vector equalities has the solution $c_4=0$, $d_4=0$. We obtain the new reductive pair $h_6=Span\{\vec{e_1},\vec{e_4}\}$, $m_1=Span\{\vec{e_2},\vec{e_3}\}$.

Let $\vec{c}=\vec{e_2}+c_3\vec{e_3}$, $\vec{d}=\vec{e_4}$ be the basis (5) for a possible reductive complement m. It is obvious that $h_6\cap m=\{\vec{e_4}\}\neq\vec{0}$ in this case. So, subalgebra h_6 has no reductive complement with basis (5).

Let $\vec{c}=\vec{e_3}$, $\vec{d}=\vec{e_4}$ be the basis (6) for a possible reductive complement m. Consider this pair $m=Span\{\vec{e_3},\vec{e_4}\}$, $m=Span\{\vec{e_3},\vec{e_4}\}$. It's obvious that $h_6\cap m\neq\vec{0}$. So, subalgebra h_6 has no reductive complement with basis (6).

Subalgebra $h_7=Span\{\vec{e_2},\vec{e_4}\}$

Let $\vec{c}=\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}$, $\vec{d}=\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ be the basis (1) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e_2}$ and $\vec{b}=\vec{e_4}$ of h_7 by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_2},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_2},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=d_3(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=c_3\vec{e_3}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_2}+d_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

The solution of the system of vector equations is $c_3=0$, $d_3=0$, $d_4=0$. This solution produces the following pair $h_7=Span\{\vec{e_2},\vec{e_4}\}$, $m=Span\{\vec{e_1}+c_4\vec{e_4},\vec{e_2}\}$ that is not reductive because $h_7\cap m=\{\vec{e_2}\}\neq\vec{0}$.

Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}$, $\vec{d}=\vec{e_3}+d_4\vec{e_4}$ be the basis (2) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e_2}$ and $\vec{b}=\vec{e_4}$ of h_7 by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_2},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=-\vec{e_2}+c_4\vec{e_2}=x_1\vec{c}+y_1\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_2},\vec{e_3}+d_4\vec{e_4}\right]=(\vec{e_1}-\vec{e_4})+d_4\vec{e_2}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_4},\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}\right]=-c_2\vec{e_2}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_4},\vec{e_3}+d_4\vec{e_4}\right]=\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

This system of vector equations has no solution because of the contradiction $c_4=1$ from the product $\left[\vec{a},\vec{c}\right]$) and $c_4=-1$ (from the product $\left[\vec{a},\vec{d}\right]$). This means that subalgebra h_7 has no reductive complement with basis (2).

Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_3\vec{e_3}$, $\vec{d}=\vec{e_4}$ be the basis (3) for a possible reductive complement m. Subalgebra h_7 has the basis $\vec{a}=\vec{e_2}$, $\vec{b}=\vec{e_4}$. It's obvious that $h_7\cap m=\{\vec{e_4}\}\neq\vec{0}$, so subalgebra h_7 has no reductive complement with basis (3).

For the bases (4), (5), and (6) we have the similar cases as for the basis (3). Subalgebra h_7 has no reductive complements at all.

Subalgebra $h_8=Span\{\vec{e_3},\vec{e_4}\}$

Let $\vec{c}=\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}$, $\vec{d}=\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}$ be the basis (1) for a possible reductive complement m. Multiply basic vectors $\vec{a}=\vec{e_3}$ and $\vec{b}=\vec{e_4}$ of h_8 by $\vec{c}$ and $\vec{d}$. We have:

$$\left[\vec{a},\vec{c}\right]=\left[\vec{e_3},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=\vec{e_3}-c_4\vec{e_3}=x_1\vec{c}+y_1\vec{d},$$

$$\left[\vec{a},\vec{d}\right]=\left[\vec{e_3},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-(\vec{e_1}-\vec{e_4})-d_4\vec{e_3}=x_2\vec{c}+y_2\vec{d},$$

$$\left[\vec{b},\vec{c}\right]=\left[\vec{e_4},\vec{e_1}+c_3\vec{e_3}+c_4\vec{e_4}\right]=c_3\vec{e_3}=x_3\vec{c}+y_3\vec{d},$$

$$\left[\vec{b},\vec{d}\right]=\left[\vec{e_4},\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}\right]=-\vec{e_2}+d_3\vec{e_3}=x_4\vec{c}+y_4\vec{d}.$$

The system of vector equations has no solution because the contradiction $c_4=1$, $c_4=-1$ appears. This means that subalgebra h_8 has no reductive complement with basis (1).

Let $\vec{c}=\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4}$, $\vec{d}=\vec{e_3}+d_4\vec{e_4}$ be the basis (2) for a possible reductive complement m. It is obvious that the corresponding pair $h_8=Span\{\vec{e_3},\vec{e_4}\}$, $m=Span\{\vec{e_1}+c_2\vec{e_2}+c_4\vec{e_4},\vec{e_3}+d_4\vec{e_4}\}$ is not reductive because $h_8\cap m=\{\vec{e_3}+d_4\vec{e_4}\}\neq\vec{0}$.

For the bases (3), (4), (5), and (6) we have the similar situations like for the bases (1) and (2). The subalgebra h_8 has no reductive complements with all these bases.

The next theorem describes all different reductive pairs that were found.

Theorem 2: Each reductive pair $\{h,m\}$ with 2-dimensional subalgebra h and 2-dimensional complement m of Lie algebra g is equal to one and only one pair from the next list of them:

1. $h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_4\vec{e_4}\}$, $m_1=Span\{\vec{e_1}+c_4\vec{e_4},\vec{e_2}\}$, $b_4\neq 0$, $c_4\neq 1$;

2. $h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}\}$, $m_2=Span\{\vec{e_1}+c_3\vec{e_3},\vec{e_2}+c_3\vec{e_4}\}$, where $c_3\neq b_4$, and c_3 is any real solution of the equation $c_3^{\,2}-b_4c_3-b_3=0$;

3. $h_1=Span\{\vec{e_1}+\vec{e_4},\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}\}$, $m_3=Span\{\vec{e_1}+c_2\vec{e_2},\vec{e_3}+c_2\vec{e_4}\}$, where c_2 is any real solution of the equation $b_3c_2^{\,2}-b_4c_2-1=0$;

4. $h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\dfrac{a_4a_3^{\,2}}{(1-a_4)^2}\vec{e_3}+\dfrac{1+a_4}{1-a_4}a_3\vec{e_4}\}$, $a_4\neq 1$,

$m_1=Span\{\vec{e_1}+\dfrac{a_3}{1-a_4}\vec{e_3},\vec{e_2}+\dfrac{a_3}{1-a_4}\vec{e_4}\}$ where $a_3\neq 0$, $a_4\neq 0$.

5. $h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\dfrac{a_4a_3^{\,2}}{(1-a_4)^2}\vec{e_3}+\dfrac{1+a_4}{1-a_4}a_3\vec{e_4}\}$, $a_4\neq 1$, $a_3\neq 0$, $a_4\neq -1$,

$m_2=Span\{\vec{e_1}+\dfrac{3-a_4}{2(1-a_4)}a_3\vec{e_3}+\dfrac{a_4-1}{2}\vec{e_4},\vec{e_2}+\dfrac{a_3^{\,2}}{(1-a_4)^2}\vec{e_3}+\dfrac{1+a_4}{2(1-a_4)}a_3\vec{e_4}\}$.

6. $h_2=Span\{\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\vec{e_2}-\dfrac{a_4a_3^{\,2}}{(1-a_4)^2}\vec{e_3}+\dfrac{1+a_4}{1-a_4}a_3\vec{e_4}\}$, $a_4\neq 1$,

$m_3 = Span\{\vec{e_1} + \frac{a_4 - 1}{a_3}\vec{e_2}, \vec{e_3} + \frac{a_4 - 1}{a_3}\vec{e_4}\}$, $a_3 \neq 0$.

7. $h_4 = Span\{\vec{e_1} + a_2\vec{e_2}, \vec{e_3} + a_2\vec{e_4}\}$,

$m_1 = Span\{\vec{e_1} + c_3\vec{e_3} + (1 + a_2c_3)\vec{e_4}, \vec{e_2} - \frac{a_2c_3 + 1}{a_2^{\ 2}}\vec{e_3} - \frac{a_2c_3 + 2}{a_2}\vec{e_4}\}$, $a_2 \neq 0$.

8. $h_4 = Span\{\vec{e_1} + a_2\vec{e_2}, \vec{e_3} + a_2\vec{e_4}\}$, $m_2 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_3}\}$, $a_2 \neq 0$.

9. $h_5 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_3} + b_4\vec{e_4}\}$, $m_1 = Span\{\vec{e_1} + c_4\vec{e_4}, \vec{e_3}\}$, $b_4 \neq 0$.

10. $h_5 = Span\{\vec{e_1} + \vec{e_4}, \vec{e_3} + b_4\vec{e_4}\}$, $m_2 = Span\{\vec{e_1} + b_4\vec{e_2} - \vec{e_4}, \vec{e_3}\}$, $b_4 \neq 0$.

11. $h_6 = Span\{\vec{e_1}, \vec{e_4}\}$, $m_1 = Span\{\vec{e_2}, \vec{e_3}\}$.

Remark: It is unknown yet which reductive pairs from Theorem 2 are equivalent with respect to the inner automorphisms of the given Lie algebra.

References

1. Shtukar U (2016) Classification of Canonical Bases for (*n*–2)-dimensional Subspaces of *n*-dimensional Vector Space. Journal of Generalized Lie Theory and Applications 10(1).
2. Shtukar U (2017) Classification of Maximal Subalgebras and Corresponding Reductive Pairs of Lie Algebra of All 2×2 Real Matrices. Journal of Generalized Lie Theory and Applications (Submitted).
3. Nomizu K (1954) Invariant Affine Connections on Homogeneous Spaces. American Journal of Mathematics 76: 33-65.
4. Patera J, Winternitz P (1977) Subalgebras of real three- and four-dimensional Lie algebras. Journal of Mathematical Physics 18: 1449-1455.
5. Sagle A, Winter DJ (1967) On Homogeneous Spaces and Reductive Subalgebras of Simple Lie Algebras. Transactions of the American Mathematics Society 128: 142-147.