Classification of Maximal Subalgebras and Corresponding Reductive Pairs of Lie Algebra of All $2 \times 2$ Real Matrices

Shtukar U*

Math/Physics Department, North Carolina Central University, Durham, USA

### Abstract

The purpose of the article is to describe all 3-dimensional subalgebras and all corresponding reductive pairs of Lie algebra of all $2 \times 2$ real matrices. This Lie algebra is 4-dimensional as a vector space, it's not simple, and it’s not solvable. The evaluation procedure utilizes the canonical bases for subspaces that were introduced. In Part I of this article, all 3-dimensional subalgebras of the given Lie algebra $g$ are classified. All reductive pairs $(h, m)$ with 3-dimensional subalgebras $h$ are found in Part II. Surprisingly, there is only one reductive pair $(h, m)$ with special 3-dimensional subalgebra $h$ and 1-dimensional complement $m$. Finally, all reductive pairs $(h, m)$ with 1-dimensional subalgebras $h$ of algebra $g$ are classified in Part III of the article.

### Keywords:
Lie algebra; Subalgebras; Reductive pairs

### Introduction

Reductive homogeneous spaces appeared for the first time in the fundamental manuscript [1,2] of Katsumi Nomizu, in which the author investigated invariant affine connections and Riemannian metrics on them. Sagle and Winter in their article [3] analyzed algebraic structures generated by reductive pairs of simple Lie algebras. The next problem studied by some authors was classification of subalgebras of some Lie algebras. For example, Patera and Winternitz have classified all subalgebras of real Lie algebras of dimensions $d=3$ and $d=4$ in their manuscript [4]. This classification of subalgebras of low dimensional real Lie algebras was done by a representative of each conjugacy class where the conjugacy was considered under the group of inner automorphisms of Lie algebras. The articles mentioned above have stimulated this research for all subalgebras and all reductive pairs of the same Lie algebra will be done at the separate article. New time. The classification of 2-dimensional subalgebras with its reductive pairs of the given Lie algebra is utilized a different method. Our method involves canonical bases for subspaces [1] that allow us to find all 3-dimensional subalgebras and the corresponding reductive pairs of the given Lie algebra $g$. Our classification of reductive pairs is done here for the first time. The classification of 2-dimensional subalgebras with its reductive pairs of the same Lie algebra will be done at the separate article. New knowledge concerning the structure of this Lie algebra is important for Geometry, and Physics.

We start with standard definitions for the readers’ convenience.

#### Definition 1

Let $g$ be a vector space over a field $F$. Then $g$ is called a Lie algebra over $F$ if there exists a Lie bracket operation $[x, y] \in g$ for any $x, y \in g$ such that:

\[ [ax, y] = a[x, y] = [x, ay] \quad \text{for any} \quad a \in F, \]

\[ [x, y + z] = [x, y] + [x, z], \quad [x, y] = -[y, x] \quad \text{and} \]

\[ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (x, y, z \in g). \]

We call $[x, y]$ a Lie product.

#### Definition 2

Let $g$ be a Lie algebra. A subspace $h \subset g$ is called a (Lie) subalgebra of $g$, if $[h, h] \subset h$.

#### Definition 3

Let $g$ be a Lie algebra, $h$ be subalgebra of $g$. If there exists a subspace $m$ of $g$ such that $h \oplus m = g$ and $[h, m] = m$, then $(h, m)$ is called a reductive pair of $g$, and $(g, h, m)$ is called a reductive triple. We say also that subspace $m$ is a reductive complement for $h$.

Lie algebra $g$ and its standard basis: This Lie algebra contains all $2 \times 2$ matrices over the field of all real numbers. The standard basis of this algebra consists of the next four matrices:

\[ e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

It is well known that the Lie multiplication operation $[A, B]$ for any two square matrices $A$ and $B$ of the same size is defined to be $[A, B] = ABBB$. According this rule, the fundamental products of the basic vectors (matrices) $e_1, e_2, e_3, e_4$ can be computed:

\[ e_1 e_2 = -e_2 e_1, e_2 e_3 = -e_3 e_2, e_3 e_4 = -e_4 e_3, \quad \text{and} \quad e_4 e_1 = -e_1 e_4. \]

All other products of basic vectors are zeros.

Let $h$ be any 3-dimensional subspace of Lie algebra $g$. We can describe subspace $h$ as $h = \text{Span}(a, b, c)$ where $a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, $b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4$ and $c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4$ are 3 linearly independent vectors.

*Corresponding author: Shtukar U, Associate Professor, Math/Physics Department, North Carolina Central University, 1801 Fayetteville Street, Durham, NC 27707, USA, Tel: 919-597-0375; E-mail: vshtukar@yahoo.com

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According to the article [1], all canonical bases for 3-dimensional subspaces of 4-dimensional vector space are:

1. \( a = e_1 + a_1 e_2 + b e_3 + c e_4 \),
2. \( \bar{a} = e_1 + a_1 e_2 + \bar{b} e_3 + \bar{c} e_4 \),
3. \( \breve{a} = e_1 + \breve{a}_1 \breve{e}_2 + \breve{b} \breve{e}_3 + \breve{c} \breve{e}_4 \),
4. \( \ddot{a} = e_1 + \ddot{a}_1 \ddot{e}_2 + \ddot{b} \ddot{e}_3 + \ddot{c} \ddot{e}_4 \).

Part I. Maximal subalgebras of Lie algebra \( g \)

Now we start to determine that a 3-dimensional subspace \( h = \text{Span}(\vec{a}, \vec{b}, \vec{c}) \) is a subalgebra of Lie algebra \( g \) when vectors \( \vec{a}, \vec{b}, \vec{c} \) form one of the canonical bases (1), (2), (3), or (4) listed above. We have to check that the condition \( [h, h] \subset h \) is true for each of these bases. The necessary evaluation procedure follows.

Let \( \vec{a} = e_1 + a_1 e_2 + b e_3 + c e_4 \) be the basis (1) for \( h \).

Evaluate Lie products \([\vec{a}, \vec{b}], [\vec{a}, \vec{c}], [\vec{b}, \vec{c}]\): 

\[
[\vec{a}, \vec{b}] = \begin{bmatrix} e_1 + a_1 e_2 + b e_3 + c e_4 \end{bmatrix} \begin{bmatrix} e_5 + a_5 e_6 + b_5 e_7 + c_5 e_8 \end{bmatrix} = \begin{bmatrix} e_1 e_5 + e_2 b_5 + e_3 c_5 + e_4 b e_7 + e_4 c e_8 \end{bmatrix}
\]

So, \( x_1 = 0, y_1 = -a_1, z_1 = 0, (a_1 - 1) c_1 = 0 \) and \( x_2 = 1, y_2 = c_1, z_2 = b_1 + a_1 c_1 + b_1 c_1 = -1 \).

The system of 3 equations for \( a_1, b_1, c_1 \) is:

1. \( a_1 - 1 = 0 \),
2. \( a_1 c_1 = 0 \),
3. \( a_1 c_1 + b_1 c_1 = -1 \).

We have two different solutions:

1. \( a_1 = 1, c_1 = 0, b_1 = 0 \),
2. \( a_1 = 0, c_1 = b_1 = 1 \).

The vector \( e_1 - e_4 \) doesn’t belong to the subspace \( h = \text{Span}(\vec{a}, \vec{b}, \vec{c}) \) at this case. So, this subspace is not subalgebra of algebra \( g \).

The next statement describes all 3-dimensional subalgebras of Lie algebra \( g \).

**Theorem 1:** All different 3-dimensional subalgebras of all 2 \( \times \) 2 real matrices are listed here:

1. \( h_1 = \text{Span}(e_1, e_2, e_3) \),
2. \( h_2 = \text{Span}(e_1 - e_2, e_3, e_4) \).

**Corollary:** The subalgebras above are maximal for the given Lie algebra.

Part II. Reductive pairs with 3-dimensional subalgebras \( h \) of Lie algebra \( g \)

How many of 3-dimensional subalgebras \( h \) form reductive pairs \( [h, m] \) in this Lie algebra? To answer this question, we will use the conditions from the Definition 3, i.e. \([h, m] = 0 = h \otimes m \) where \( m \) is an appropriate 1-dimensional reductive complement for a given 3-dimensional subalgebra \( h \). The list of all 3-dimensional subalgebras from Theorem 2 will be used to find all possible reductive complements. Let \( m = \text{Span}(d, e_1, e_2, e_3, e_4) \) be a 1-dimensional complement. To simplify our evaluation, we consider 2 possible cases for the generating vector \( d \):

1. \( d = e_2 + d e_1 + d e_3 + d e_4 (d \neq 0) \),
2. \( d = d e_1 + d e_3 + d e_4 (d = 0) \).

**Subalgebra** \( h_1 \) **Case 1:** Multiply basic vectors \( a = e_1 + e_2, b = e_3, c = e_4 - \frac{1}{b_1} e_1 \) from \( h_1 \) by vector \( d = e_2 + d e_1 + d e_3 + d e_4 \).

We have:

\[
[\ddot{e}_1, e_2 + d_1 e_1 + d_3 e_3 + d_4 e_4] = \ddot{e}_1 e_2 + d_1 \ddot{e}_1 e_1 + d_3 \ddot{e}_1 e_3 + d_4 \ddot{e}_1 e_4 = \ddot{e}_1 e_2 + d_1 \ddot{e}_1 e_1 + d_3 \ddot{e}_1 e_3 + d_4 \ddot{e}_1 e_4 = \ddot{e}_1 e_2 + d_1 \ddot{e}_1 e_1 + d_3 \ddot{e}_1 e_3 + d_4 \ddot{e}_1 e_4
\]

(it’s an identity).

From the vector equalities above, we obtain the following system of conditions for the components \( d_2, d_3, d_4 \) and coefficients \( y, z \): \( y = d_2, yd_1 = -d_3 + d_4, yd_2 = d_3 - d_4, yd_3 = d_4 - d_3, yd_4 = d_2 - d_4 \). These equalities produce the system of 6 equations for \( d_2, d_3, d_4 \):

\[
\begin{align*}
&d_2 y = -d_3 + d_4, &d_3 y = d_4 - d_3, &d_4 y = d_2 - d_4, \\
&d_2 d_1 = -d_3 + d_4, &d_3 d_1 = d_4 - d_3, &d_4 d_1 = d_2 - d_4.
\end{align*}
\]
From the equations $d_1 d_4 = -d_3$, $-d_2 d_3 = d_4$, we receive $d_1 = -1$, $d_2 = 0$, $d_3 = 0$ or $d_4 = 0$.

If $d_1 = -1$, then $d_2 = -\frac{1}{b_1}$, and $d_3 = b_1$. The first solution is the vector $d = e_1 - \frac{1}{b_1} e_2 + b_1 e_3 - e_4$, and the subspace $m = \text{Span} \{d = e_1 - \frac{1}{b_1} e_2 + b_1 e_3 - e_4\}$ is a possible reductive complement for $h_1$. Unfortunately, the condition $g = h_1 \oplus m$ is not satisfied because $d = a - \frac{1}{b_1} h_1 + c$ is not a subspace of $h_1 = m \oplus h_1$. So, $m$ is not reducible and does not exist for $h_1$.

If $d_2 = 0$, $d_3 = 0$, and $d_4 = 1$ then vector $d = e_1 + e_3$ is the consequent result. In this case, $m \oplus h_1$ again. This means that a reductive complement for subalgebra $h_1$ doesn’t exist for this case.

### Subalgebra $h_1$, Case 2:

**Multiplying basic vectors from $h_1$ by vector $d$:** $d = e_1 + e_3$, $d = e_2 + e_3$, and $d = e_1 + e_2 + e_3$. We have:

$$[e_1 + de_1, e_2 + de_2, e_3 + de_3, e_4 + de_4] = d_1(e_1 - e_2 + b_1 d_3 e_3 + d_4 e_4) + y d,$$

$$(e_1 + de_1, e_2 + de_2, e_3 + de_3, e_4 + de_4) = d_1(e_1 - e_2 + b_1 d_3 e_3 + d_4 e_4) + y d.$$ 

The last system of equations has just the zero solution for $d_2$, $d_3$, $d_4 = 0$, $d_1 = 0$. The zero vectors $d = 0$ is the only solution for the system. This means that a nonzero reductive complement for $h_1$ doesn’t exist.

### Subalgebra $h_2$, Case 1:

**Multiplying basic vectors from $h_2$ by vector $d$:** $d = e_1 + e_3$, $d = e_2 + e_3$, and $d = e_1 + e_2$. We have:

$$[e_1 + de_1, e_2 + de_2, e_3 + de_3, e_4 + de_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d,$$

$$[e_1 + de_1, e_2 + de_2, e_3 + de_3, e_4 + de_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d.$$ 

From the vector equalities above, we obtain the following system of equations for the components $d_1, d_2, d_3, d_4$, and coefficients $y, z$:

$$d_1 = 0, \quad y d_1 = d_2 - b_1 d_3, \quad y d_2 = d_3 - d_4, \quad b_1 d_3 = 1 - d_2 - d_4, \quad z d = d_2.$$ 

The last system of equations has just the zero solution for $d_2$, $d_3, d_4 = 0$, $d_1 = 0$. The zero vectors $d = 0$ is the only solution for the system. This means that a nonzero reducible complement for $h_1$ doesn’t exist.

### Subalgebra $h_2$, Case 2:

**Multiplying basic vectors from $h_2$ by vector $d$:** $d = e_1 + e_4$, $d = e_2 + e_4$, $d = e_3 + e_4$. We have:

$$[e_1 + de_1, e_2 + de_2, e_3 + de_3, e_4 + de_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d,$$

$$[e_1 + de_1, e_2 + de_2, e_3 + de_3, e_4 + de_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d.$$ 

The last system of vector equalities, we obtain a system of equations for $d_1, d_2, d_3$, that has just one solution $d_1 = 0, d_2 = 0, d_3 = 1$. The subspace $m = \text{Span} \{e_1 + e_4, e_2 + e_4, e_3 + e_4\}$ generated by vector $d = e_1 + e_4$ satisfies the conditions $[h_2, m] \subset m$ and $g = h_2 \oplus m$, so $[h_2, m]$ is a reductive pair for this case where $h_2 = \text{Span} \{e_1 - e_2, e_2, e_3\}$, $m = \text{Span} \{e_1 + e_4\}$.

### Subalgebra $h_1$, Case 2:

**Multiplying basic vectors from $h_1$ by vector $d$:** $d = e_1 + e_3$, $d = e_2 + e_3$, and $d = e_1 + e_2$. We have:

$$[e_1 - e_2, e_2 + e_3, e_3 + e_4, e_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d,$$

$$[e_1 - e_2, e_2 + e_3, e_3 + e_4, e_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d.$$ 

Transforming this system of vector equalities into a system of equations for the components $d_1, d_2, d_3$, and solving the system, the only zero solution $d_1 = 0, d_2 = 0, d_3 = 0$ is obtained. So, a nonzero reducible complement for $h_1$ doesn’t exist for this case.

Subalgebras $h_1$ and $h_2$ don’t produce any reducible pair. The details are similar for the cases of subalgebras $h_1$ and $h_2$, therefore they are omitted.

The total analysis conducted in this Part III establishes the following statement.

**Theorem 2:** The only one reducible pair with 3-dimensional subalgebra exists for Lie algebra $g$ of all real $2 \times 2$ matrices; it is $[h, m]$, where, $h = \text{Span} \{e_1 - e_2, e_2, e_3\}$, $m = \text{Span} \{e_1 + e_4\}$.

**Corollary:** The subspace $m = \text{Span} \{e_1 + e_4\}$ is a 1-dimensional ideal of Lie algebra $g$. Moreover, the subalgebra $h = \text{Span} \{e_1 - e_2, e_2, e_3\}$ is a 3-dimensional ideal of Lie algebra $g$.

### Part III. Reductive pairs $[h, m]$ with 1-dimensional subalgebras $h$ of Lie algebra $g$

It is well known that each 1-dimensional subspace $h$ of algebra $g$ is an abelian subalgebra of $g$. The corresponding reductive complements $m$ for each $h$ should be 3-dimensional subspaces $m$ such that $[h, m] \subset m$ and $g = h \oplus m$. Therefore, the canonical bases for 3-dimensional subspaces that are found in the Part I can be utilized for $m$. The list of all canonical bases contains the next 4 bases:

1. $a = e_1 + a e_4, b = e_2 + b e_4, c = e_3 + c e_4$.
2. $a = e_1 + a e_4, b = e_2 + b e_4, c = e_3$.
3. $a = e_1 + a e_4, b = e_2, c = e_3$.
4. $a = e_1, b = e_2, c = e_3$.

Let $h = \text{Span} \{d_1, d_2, d_3, d_4\}$ be 1-dimensional subalgebra in algebra $g$. We will consider two cases for the generating vector $d$:

(a) $d = d_1 + d_2 + d_3 + d_4$ if $d \neq 0$; (b) $d = d_1 + d_3 + d_4$ if $d = 0$.

To determine if a subalgebra $h$ forms a reducible pair with some complement $m$, we will check that the conditions $[h, m] \subset m$ and $g = h \oplus m$ are satisfied.

1a. Consider the basis (1) for a complement $m$ and case (a) for $d$.

**Multiplying vectors $\vec{a}, \vec{b}, \vec{c}$ by vector $\vec{d}$:** We have:

$$[\vec{a}, \vec{d}] = [\vec{a}, \vec{e}_1 + \vec{e}_4, \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4] =$$

$$d_2 e_1 - d_1 e_2 - a d_3 e_3 + a d_4 e_4 = x a + y b + z c$$

So, $x = 0, y = d_2 - d_1, z = a - d_3$. Let $d_4 = (a - d_3) b + (a - d_3) c = 0$.

$$[\vec{b}, \vec{d}] = [\vec{b}, \vec{e}_1 + \vec{e}_4, \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4] =$$

$$e_1 - e_2, e_2 + e_3, e_3 + e_4, e_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d,$$

$$[\vec{c}, \vec{d}] = [\vec{c}, \vec{e}_1 + \vec{e}_4, \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4] =$$

$$e_1 - e_2, e_2 + e_3, e_3 + e_4, e_4] = d_1(e_1 - e_2 - 2d_3 e_3 - 2d_4 e_4) + y d.$$
From the 2nd and 3rd equations above, we can find the 4th component $h$

These solutions produce the following two reductive pairs:

For the condition $a_4+1+2b_4≠0$, we have $d_4=1+b_4d_2+c_4d_4$, and the corresponding reductive pair is:

Consider the special case when $c_4=0$. Then we have the following system of equations for $d_4$, $d_2$, $d_1$: $d_4b_4≠0$, $d_2(a_1+1)+b_1(b_1+d_2)=d_1(a_1+1)=0$. If $a_4≠1$, then $d_2=0$, and $d_1=\frac{b_4}{a_4+1}(1−d_2)$, with any $b_4$. As the result, the following new reductive pair is obtained:

Comparing these two equalities for $d_4$, we obtain $b_4x=−1$ or $c_4d_4=−b_4d_2$. Using these conditions, we obtain for this case $a_4=1$ the following solutions:

These two reductive pairs are particular cases of the pairs $[h_1, m_1]$, $[h_2, m_2]$ obtained in the next step 2.

1. If $b_4d_2=−c_4d_4$, we can suppose that $a_4=0$. Like in the previous step 1, we obtain the following system of three equations:

From the 2nd and 3rd equations above, we can find the 4th component $d_4$:

Comparing the last two equalities for the same component $d_4$, we have $1+b_4d_2−d_1(\frac{a_4+1}{b_4}+c_4)=1+c_4d_2−d_1(\frac{a_4+1}{c_4}+b_4)$, and

The last equality produces two results: $d_4=\frac{b_4}{c_4}d_2$ (for $c_4≠0$), or $a_4+1+2b_4=0$. Compute the component $d_4$ in terms of $a_4$, $b_4$, $c_4$, $d_2$ when $d_4=\frac{b_4}{c_4}d_2$. We have $d_4=1−\frac{a_4+1}{c_4}d_2$. So, we obtain the following reductive pair:

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Comparing two different expressions for the same component \(d_i\), we find \(d_i = \frac{b_{i1}}{c_i} d_j\), and \(d_i = -\frac{d_j}{c_i}\), if \(b \neq 0\) or we obtain \(b \neq 0\) and \(c_i \neq 0\).

\[ d_j = -\frac{d_j}{c_i} \]

This means that two new reductive pairs are found:

\[ h = \text{Span}\{c_1, c_2, c_3, c_4\}, \quad m = \text{Span}\{c_1, c_2, c_3, c_4\} \]

For the last equation, we have \(d_i = -b d_j\). Substitute this value of \(d_i\) into the first and the second equations:

\[ a_i^2 d_i + b_i d_i = a_i (1 - d_i) - d_i, \quad a_i (d_i - b_i d_i) + b_i (d_i - 1) = b_i (1 - d_i) - b i d_i \]

From the last equation, we have \(d_i = -b d_j\). Substitute this value of \(d_i\) into the first and the second equations:

\[ a_i^2 d_i + b_i d_i = a_i (1 - d_i) - d_i \]

If \(a_i^2 \neq 0\), then the following reductive pair is obtained:

\[ h_i = \text{Span}\{c_1, c_2, c_3, c_4\}, \quad m_i = \text{Span}\{c_1, c_2, c_3, c_4\} \]

This pair is not reducible because \(h_i \subset m_i\).

2. Consider the basis (2) for the complement \(m\) and the case (b) for vector \(\vec{d}\) into the first and the second equations:

\[ \vec{x} = \vec{x} + \vec{y} + \vec{z} \]

These conditions generate the next system of equations for unknown components \(d_j, d_k, d_l\):

\[ \vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{d} \]

By vector \(\vec{b}\), we have:

\[ \vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{d} \]

If \(a_i^2 \neq 0\), then \(d_i = -b d_j\). Substitute this value of \(d_i\) into the first and the second equations:

\[ a_i^2 d_i + b_i d_i = a_i (1 - d_i) - d_i \]

Simplifying this system of equations, we obtain \(d_i = -b d_j\). Substitute this value of \(d_i\) into the first and the second equations:

\[ a_i^2 d_i + b_i d_i = a_i (1 - d_i) - d_i \]

If \(a_i^2 \neq 0\), then the following reductive pair is obtained:

\[ h_i = \text{Span}\{c_1, c_2, c_3, c_4\}, \quad m_i = \text{Span}\{c_1, c_2, c_3, c_4\} \]

This pair is not reducible because \(h_i \subset m_i\).

2a. Consider the basis (2) for the complement \(m\) and the case (a) for vector \(\vec{d}\) into the first and the second equations:

\[ \vec{x} = \vec{x} + \vec{y} + \vec{z} \]

These conditions generate the next system of equations for unknown components \(d_j, d_k, d_l\):

\[ \vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{d} \]

By vector \(\vec{b}\), we have:

\[ \vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{d} \]

If \(a_i^2 \neq 0\), then \(d_i = -b d_j\). Substitute this value of \(d_i\) into the first and the second equations:

\[ a_i^2 d_i + b_i d_i = a_i (1 - d_i) - d_i \]

Simplifying this system of equations, we obtain \(d_i = -b d_j\). Substitute this value of \(d_i\) into the first and the second equations:

\[ a_i^2 d_i + b_i d_i = a_i (1 - d_i) - d_i \]

If \(a_i^2 \neq 0\), then the following reductive pair is obtained:

\[ h_i = \text{Span}\{c_1, c_2, c_3, c_4\}, \quad m_i = \text{Span}\{c_1, c_2, c_3, c_4\} \]

This pair is not reducible because \(h_i \subset m_i\).

3a. Consider the basis (3) and the case (a) for the vector \(\vec{d}\) into the first and the second equations:

\[ \vec{x} = \vec{x} + \vec{y} + \vec{z} \]

These conditions generate the next system of equations for unknown components \(d_j, d_k, d_l\):

\[ \vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{d} \]

By vector \(\vec{b}\), we have:

\[ \vec{a} \cdot \vec{d} = \vec{b} \cdot \vec{d} \]
\[
\begin{align*}
\left[ a, d \right] &= \left[ e_1 + a_1 e_2, e_3 + d_1 e_1 + d_2 e_2 + d_3 e_3, e_4 + d_1 e_1 + d_2 e_2 + d_3 e_3 \right] = d_1 e_2 - d_2 e_3 - d_3 e_1 - a_1 e_2,
\end{align*}
\]

If \(a \neq 0\), then \(d_1 = 1 + a_1 d_1\), and the following reductive pair is obtained

\[
h_3 = \text{Span}\{e_1 + d_1 e_1, e_1, e_1, e_1\} \quad m_3 = \text{Span}\{e_1 + e_2 e_1, e_1, e_1\}.
\]

3b. Consider the basis (3) and the case (b) for the vector \(\tilde{d} = d_1 e_1 + d_2 e_2 + d_3 e_3\). Multiply vectors \(\tilde{a}, \tilde{b}, \tilde{c}\) by vector \(\tilde{d}\). We have

\[
\begin{align*}
\left[ a, d \right] &= \left[ e_1 + a_1 e_2, e_3 + d_1 e_1 + d_2 e_2 + d_3 e_3, e_4 + d_1 e_1 + d_2 e_2 + d_3 e_3 \right] = d_1 e_2 - d_2 e_3 - d_3 e_1 - a_1 e_2,
\end{align*}
\]

If \(a = 0\), then \(d_1 = 1 + a_1 d_1\), and the following reductive pair is obtained

\[
h_4 = \text{Span}\{e_1 + d_1 e_1, e_1, e_1\} \quad m_4 = \text{Span}\{e_1 + e_2 e_1, e_1, e_1\}.
\]

4b. Consider the basis (4) and the case (b) for vector \(\tilde{d} = d_1 e_1 + d_2 e_2 + d_3 e_3\). Multiply vectors \(\tilde{a}, \tilde{b}, \tilde{c}\) by vector \(\tilde{d}\). We have

\[
\begin{align*}
\left[ a, d \right] &= \left[ e_1 + a_1 e_2, e_3 + d_1 e_1 + d_2 e_2 + d_3 e_3, e_4 + d_1 e_1 + d_2 e_2 + d_3 e_3 \right] = d_1 e_2 - d_2 e_3 - d_3 e_1 - a_1 e_2,
\end{align*}
\]

If \(a = 0\), then \(d_1 = 1 + a_1 d_1\), and the following reductive pair is obtained

\[
h_5 = \text{Span}\{e_1 + d_1 e_1, e_1, e_1\} \quad m_5 = \text{Span}\{e_1 + e_2 e_1, e_1, e_1\}.
\]
are equivalent with respect to inner automorphisms of the given Lie algebra.

References