Common Fixed Point Theorem in \(T_0\) Quasi Metric Space

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Abstract

In this paper, we prove fixed point theorems for generalized C-contractive and generalized S-contractive mappings in a bi-complete di-metric space. The relationship between \(q\)-spherically complete \(T_u\) Ultra-quasi-metric space and bi-complete di-metric space is pointed out in proposition 3.2. This work is motivated by Petals and Fvidalis in a \(T_0\)-ultra-quasi-metric space.

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Introduction

In Agyingi [1] proved that every generalized contractive mapping defined in a \(q\)-spherically complete \(T_u\)-ultra-quasi metric space has a unique fixed point. In Petals and Fvidalis [2] proved that every contractive mapping on a spherically complete non Archimedean normed space has a unique fixed point. Agyingi and Gega proved fixed point theorems in a \(T_0\)-ultra-quasi-metric space [3-5]. Later many authors published number of papers in this space [6-10].

In this paper we shall prove a fixed point theorem for generalized c-contractive and generalized s-contractive mappings in a bi-complete di-metric space.

If we delete, in the used definition of the pseudo metric \(d\) on the set \(X\), the symmetry condition, \(d(x, y)=d(y, x)\), whenever \(x, y \in X\) we are led to the concept of quasi-pseudo metric.

Definition 1.1: Let \((X, m)\) be a metric space. Let \(T: X \rightarrow X\) is called a C-contraction if there exist, \(0 \leq k \leq \frac{1}{2}\) such that for all \(x, y \in X\) the following inequality holds [10],

\[m(Tx, Ty) \leq km(x, Tx) + m(y, Ty)\]

Definition 2.1: Let \((X, m)\) be a metric space. A map \(T: X \rightarrow X\) is called a S-contraction if there exist \(0 \leq k < \frac{1}{3}\) such that for all \(x, y \in X\) the following inequality holds [10],

\[m(Tx, Ty) \leq km(x, Tx) + m(y, Ty) + m(x, y)\]

Preliminaries

Now we recall some elementary definitions and terminology from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1: Let \(X\) be a non empty set. A function \(d: X \times X \rightarrow [0, \infty)\) is called quasi pseudo metric on \(X\) if

\[d(x, x) = 0, \forall x \in X\]

\[d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X\]

Moreover if \(d(x, y) = 0 = d(y, x) \Rightarrow x = y\) then \(d\) is said to be a \(T_u\) quasi metric or di-metric. The latter condition is referred as the \(T_u\) condition.

Example 2.1: On \(R \times R\), we define the real valued map \(d\) given by

\[d(x, y) = |x - y| = \max \{|x - y|, 0\}\]

then \((R, d)\) is a di metric space.

Remark 2.1

Let \(d\) be quasi-pseudo metric on \(X\), then the map \(d^2\) defined by \(d^2(x, y) = d(y, x)\) whenever \(x, y \in X\) is also a quasi pseudo metric on \(X\), called the conjugate of \(d\).

It is also denoted by \(d^1\) or \(d^2\). It is easy to verify that the function \(d^2\) defined by \(d^2 = d \circ d^{-1}\)

i.e. \(d^2(x, y) = \max \{d(x, y), d(y, x)\}\)

defines a metric on \(X\) whenever \(d\) is a \(T_0\) quasi pseudo metric.

In some cases, we need to replace \([0, \infty)\) by \([0, \infty]\) (where for a \(d\) attaining the value \(\infty\), the triangle inequality is interpreted in the obvious way). In such case we speak of extended quasi-pseudo metric.

Definition 2.2: The di metric space \((X, d)\) is said to be bi complete if the metric space \((R, d)\) is complete.

Example 2.2: Let \(X = [0, \infty)\) define for each \(x, y \in X\), \(n(x, y) = x\) if \(x > y\) and \(n(y, x) = 0\) if \(x < y\). It is not difficult to check that \((X, n)\) is a \(T_u\) quasi pseudo metric space. Notice that, for \(x, y \in [0, \infty)\), we have \(n^2(x, y) = \max \{x, y\}\) if \(x > y\) and \(n^2(x, y) = 0\) if \(x = y\), the matrix \(n^2\) is complete on \((X, d)\).

Definition 2.3: Let \((X, d)\) be quasi pseudo metric space, for \(x, e, x, e \in X \& e > 0\)

\[B_d(x, e) = \{y \in X : d(x, y) < e\}\]

denotes the open \(e\) ball at \(x\). The collection of such balls is a base for a topology \(\tau(d)\) induced by \(d\) on \(X\). Similarly for \(x, e, x, e \in X \& e > 0\)

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\( C_d(x, e) = \{ y \in X : d(x, y) \leq e \} \)
denotes the closed \( e \)-ball at \( x \).

**Definition 2.4:** Let \( (X, d) \) be quasi pseudo metric space Let \( (x_i), i \in I \) be a family of points in \( X \) and let \( (r_i), i \in I \) be a family of non negative real numbers.

We say that \( \left( C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i) \right), i \in I \) has the mixed binary intersection property provided that \( \left( C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i) \right), i \in I \) is non empty for all \( i, j \in I \).

**Definition 2.5:** Let \( (X, d) \) be quasi pseudo metric space we say that \( (X, d) \) is Isbell complete provided that each family \( \left( C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i) \right), i \in I \) has the mixed binary intersection property such that

\[
\bigcap_{i \in I} \left( C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \right) \neq \emptyset
\]

**Proposition 2.1:** If \( (X, d) \) is an extended Isbell-complete quasi pseudo metric space then \( (X, d') \) is hyper complete. An interesting class of quasi pseudo metric space, for which, we investing a type of completeness are the ultra quasi pseudo metric.

**Definition 2.6:** Let \( X \) be a set & \( d : X \times X \rightarrow [0, \infty) \) be function a function mapping into the set \( [0, \infty) \) of non negative real’s then \( d \) is ultra quasi pseudo metric on \( X \) if

\[
d(x, x) = 0 \quad \text{for all} \quad x \in X
\]

\[
d(x, z) \leq \max \{ d(x, y), d(y, z) \} \quad \text{whenever} \quad x, y, z \in X
\]

The conjugate \( d^{-1} \) of \( d \) where \( d^{-1}(x, y) = d(y, x) \) whenever \( x, y \in X \) is also an ultra quasi pseudo metric on \( X \).

If \( d \) also satisfies the \( T_0 \) – condition, then \( d \) is called a \( T_0 \)- ultra quasi metric on \( X \). Notice that \( d' = \sup \{ d, d' \} = d \lor d' \) is an ultra-metric on \( X \) whenever \( d \) is a \( T_0 \)- ultra quasi metric.

In a literature, \( T_0 \) - ultra quasi metric spaces are also known as non Archimedean \( T_0 \) – quasi metric.

**q-spherically Completeness**

In this section we shall recall some results about q- spherical completeness belonging mainly to [8].

**Definition 3.1:** Let \( (X, d) \) be an ultra – quasi pseudo metric space Let \( (x_i), i \in I \) be a family of points in \( X \) and let \( (r_i), i \in I \) be a family of non negative real numbers we say that \( (X, d) \) is q- spherical completeness provided that each family [2]

\[
\left( C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i) \right), i \in I
\]

Satisfying \( d(x_i, s_i) \leq \max \{ r_i, s_i \} \), whenever \( i, j \in I \) is such that

\[
\bigcap_{i \in I} \left( C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \right) \neq \emptyset
\]

**Proposition 3.2:** Each q- spherically complete \( T_0 \), ultra quasi metric space \( (X, d) \) is bi-complete.[8].

**Main Results**

We recall the following interesting results respectively due to Chatterji [10] and to Shukla [11]

**Theorem 4.1a** A C- contraction on a complete metric space has a unique fixed point.

**Theorem 4.1b** A S- contraction on a complete metric space has a unique fixed point.

Following results generalizes the above theorem to setting of a bi-complete di-metric space.

**Definition 4.1:** Let \( (X, d) \) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a c-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{2} \) such that for all \( x, y \in X \) the following inequality holds.

\[
d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y) + d(x, y)]
\]

**Definition 4.2:** Let \( (X, d) \) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a S-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{3} \) such that for all \( x, y \in X \) the following inequality holds.

\[
d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y) + d(x, y)]
\]

Now we define following definitions

**Definition 4.3:** Let \( (X, d) \) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a generalized c-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{4} \) such that for all \( x, y \in X \) the following inequality holds.

\[
d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y) + d(x, y)]
\]

**Definition 4.4:** Let \( (X, d) \) be a quasi pseudo metric space. A map \( T : X \rightarrow X \) is called a generalized S-pseudo contraction if there exist \( k, 0 \leq k < \frac{1}{8} \) such that for all \( x, y \in X \) the following inequality holds.

\[
d(Tx, Ty) \leq k[d(x, y)]
\]

**Theorem 4.1:** Let \( (X, d) \) be a bi complete di metric space and let \( T : X \rightarrow X \) be a generalized c- pseudo contraction then \( T \) has a unique fixed point.

**Proof:** Since \( T : X \rightarrow X \) is a generalized c-pseudo contraction then there exist \( k, 0 \leq k < 1 \) such that for all \( x, y \in X \) the following inequality holds:

\[
d(Tx, Ty) \leq k[d(x, y)]
\]

We shall first show that \( T : (X, d') \rightarrow (X, d') \) is a generalized c-contraction.

Since for any \( x, y \in X \) we have

\[
d^{-1}(Tx, Ty) = d(Ty, Tx)
\]

\[
\leq k[d(Tx, y) + d(Tx, Tx) + d(y, Tx)]
\]

\[
\leq k[d^{-1}(y, Ty) + d^{-1}(Tx, x) + d^{-1}(x, Ty) + d^{-1}(Tx, y)]
\]

\[
d^{-1}(Tx, Ty) \leq k[d^{-1}(y, Ty) + d^{-1}(Tx, x) + d^{-1}(x, Ty) + d^{-1}(Tx, y)]
\]

We see that \( T : (X, d^{-1}) \rightarrow (X, d^{-1}) \) is a generalized C-pseudo contraction therefore

\[
d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y) + d(x, y)]
\]

\[
\leq k[d(x, y) + d'(x, y)]
\]

and

\[
d^{-1}(Tx, Ty) \leq k[d^{-1}(y, Ty) + d^{-1}(Tx, x) + d^{-1}(x, Ty) + d^{-1}(Tx, y)]
\]
\[ \text{Corollary 4.2:} \text{ Let } (X,d) \text{ be a T}_0 \text{-Isbell-Complete quasi pseudo metric spaces and } T : X \to X \text{ be a generalized c – pseudo contraction then } T \text{ has a unique fixed point.} \]

The proof follows from the proposition 3.1.

**Theorem 4.2:**

Let \((X,d)\) be a bi complete di metric space and let \(T : X \to X\) be an generalized S pseudo contraction then \(T\) has a unique fixed point.

**Proof:** As in the previous proof it is enough to prove that \(T : (X,d') \to (X,d')\) is a generalised S -contraction.

Since \(T : X \to X\) be a \(S\) -pseudo contraction then there exist \(k, 0 \leq k < 1\) such that for all \(x, y \in X\) the following inequality holds:

\[ d(Tx, Ty) \leq k \left[ d(x, Tx) + d(x, Ty) + d(Ty, x) + d(Tx, y) + d(Tx, y) + d(Ty, x) + d(Tx, y) + d(Ty, x) \right] \]

We shall first show that \(T : (X,d') \to (X,d')\) is a generalized C-contraction.

Since for any \(x, y \in X\) we have

\[ d^{-1}(Tx, Ty) = d(Ty, Tx) \]

\[ d(Tx, Ty) \leq k \left[ d(x, Tx) + d(x, Ty) + d(Ty, x) + d(Tx, y) + d(Tx, y) + d(Ty, x) + d(Tx, y) + d(Ty, x) \right] \]

\[ \leq k \left[ d^{-1}(Ty, y) + d^{-1}(Tx, x) + d^{-1}(Ty, x) + d^{-1}(Tx, x) \right] \]

\[ + d^{-1}(y, Ty) + d^{-1}(y, Ty) + d^{-1}(x, Ty) + d^{-1}(x, Tx) \]

\[ = d^{-1}(Tx, Ty) \leq k \left[ d^{-1}(x, Tx) + d^{-1}(x, Ty) + d^{-1}(y, Ty) + d^{-1}(y, Ty) \right] \]

\[ + d^{-1}(x, Tx) + d^{-1}(x, Ty) + d^{-1}(y, Ty) + d^{-1}(y, Ty) \]

We see that \(T : (X,d') \to (X,d')\) is a pseudo contraction.

Therefore

\[ d(Tx, Ty) \leq k \left[ d(x, Tx) + d(x, Ty) + d(Ty, x) + d(Tx, y) + d(Tx, y) + d(Ty, x) + d(Tx, y) + d(Ty, x) \right] \]

\[ d(Tx, Ty) \leq k \left[ d(x, Tx) + d(x, Ty) + d(y, Ty) + d(Ty, x) + d(Tx, y) + d(Tx, y) \right] \]

and

\[ d^{-1}(Tx, Ty) \leq k \left[ d^{-1}(x, Tx) + d^{-1}(x, Ty) + d^{-1}(y, Ty) + d^{-1}(y, Ty) \right] \]

\[ + d^{-1}(x, Tx) + d^{-1}(x, Ty) + d^{-1}(y, Ty) + d^{-1}(y, Ty) \]

\[ \leq k \left[ d^m(x, Tx) + d^m(Ty, x) + d^m(x, Ty) + d^m(Ty, y) \right] \]

For all \(x, y \in X\) Hence

\[ d^m(Tx, Ty) \leq k \left[ d^m(x, Tx) + d^m(Ty, Tx) + d^m(x, Ty) + d^m(Ty, y) \right] \]

\[ \text{for all } x, y \in X \]

By assumption \((X,d')\) is a bi complete. Hence \((X,d')\) is complete. There fore by theorem (4a) \(T\) has a unique fixed point. This completes the proof.

**Corollary 4.3:** Let \((X,d)\) be a \(T_\sigma\text{-Isbell-Complete quasi pseudo metric spaces and } T : X \to X\) be a pseudo contraction then \(T\) has a unique fixed point.

The proof follows from the proposition 2.1

**Corollary 4.4:** Any \(s\)-pseudo contraction on a \(q\)-spherically complete \(T_\sigma\) ultra quasi metric space has a unique fixed point.

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