Complete Left-Invariant Affine Structures on Solvable Non-Unimodular Three-Dimensional Lie Groups

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Abstract

In this paper, we shall use a method based on the theory of extensions of left-symmetric algebras to classify complete left-invariant affine real structures on solvable non-unimodular three-dimensional Lie groups.

Keywords: Extensions of left-symmetric algebras; Left-invariant affine connections; Novikov algebras

Introduction

The notion of a left-symmetric algebra appeared for the first time in the work of Koszul [1] and Vinberg [2] concerning bounded homogeneous domains and convex homogeneous cones, respectively. Over the field of real numbers, left-symmetric algebras are of special interest because of their role in the differential geometry of affine manifolds (i.e. smooth manifolds with flat torsion-free affine connections), and in the representation theory of Lie groups [3,4]. In fact, for a given simply connected Lie group G with Lie algebra g, the left-invariant affine structures on G are in one-to-one correspondence with the left-symmetric structures on g compatible with the Lie structure [5].

On the other hand, it is well known that there is a one-to-one correspondence between left-invariant affine structures on a Lie group G and locally simply transitive affine actions of G on an n-dimensional real vector space V [5]. The classification of left-invariant affine structures on a given Lie group G is then reduced to the classification of compatible left-symmetric products on the Lie algebra g of G. It has been proved [6] that a simply connected Lie group G which acts simply transitively on R^n by affine transformations is necessarily solvable. Since a few years, there has been a growing interest in the study of simply transitive affine actions of Lie groups on R^n. This interest is mostly due to the example of Benoist [7], who constructed a simply connected nilpotent Lie group not admitting any locally simply transitive affine action on R^n. This example provided a negative answer to the following question of Milnor [3]: Does any simply connected solvable Lie group admit a simply transitive affine action on R^n?

From another point of view, there is also the question of classifying all simply transitive affine actions of a given solvable Lie group G admitting such an action. This question, even in the abelian case $G = R^n$, seems to be very hard. When G is nilpotent, the classification has been completely achieved up to dimension four [8,9].

Recently, a method based on the theory of extensions of left-symmetric algebras has been proposed [10] to classify complete left-invariant affine real structures on a given solvable Lie group of low dimension. Since the classification in the case of solvable unimodular Lie groups of dimension three was obtained [8], we will use that method to carry out in this paper the classification of complete left-invariant affine structures on three-dimensional solvable non-unimodular Lie groups.

The paper is organized as follows. In section 2, we will briefly recall some necessary definitions and basic results on left-symmetric algebras and their extensions. In section 3, using the classification of the three-dimensional complex simple left-symmetric algebras given [11] and a result [12], we shall first show that any complete real left-symmetric algebra $A_3$ of dimension 3 whose Lie algebra is solvable and non-unimodular is not simple. Therefore, we can get $A_3$ as an extension of complete left-symmetric algebras. By using the Lie group exponential maps, we shall deduce the classification of all complete left-invariant affine structures on solvable non-unimodular Lie groups of dimension 3 in terms of simply transitive actions of subgroups of the affine group $Aff(R^3) = GL(R^3) \times R^3$ (see Theorem 13).

Throughout this paper, all considered vector spaces, Lie algebras, and left-symmetric algebras are supposed to be over the field R. We shall also suppose that all considered Lie groups are simply connected.

Left-symmetric Algebras and their Extensions

Let $A$ be a finite-dimensional vector space over R. A left-symmetric product on $A$ is a bilinear product that we denote by $x \cdot y$ satisfying

\[ (x \cdot y) \cdot z - (y \cdot x) \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \]

for all $[x, y] = x \cdot y - y \cdot x$. In this case, $A$ together with a left-symmetric product is called left-symmetric algebra.

Now if $A$ is a left-symmetric algebra, then the commutator $[x, y] = x \cdot y - y \cdot x$ defines a structure of Lie algebra on $A$, called the associated Lie algebra. On the other hand, if $G$ is a Lie algebra with a left-symmetric product satisfying (2), then we say that this left-symmetric structure is compatible with the Lie structure on $G$.

Let $G$ be a simply connected Lie group with a left-invariant affine connection $\nabla$. Define a product $\cdot$ on the Lie algebra $g$ of $G$ by

\[ x \cdot y = \nabla_y x, \]

for all $x, y \in g$. Then, the flat and torsion-free conditions on $\nabla$...
correspond to conditions (1) and (2), respectively.

Conversely, if $G$ is a simply connected Lie group with Lie algebra $\mathcal{G}$ and $x \ast y$ denotes a left-symmetric product on $\mathcal{G}$ compatible with the Lie bracket, then the left-invariant connection given by $\nabla_{x}y = x \ast y$ defines a left-invariant affine structure $V$ on $G$. We deduce that if $G$ is a simply connected Lie group with Lie algebra $\mathcal{G}$, then the study of left-invariant affine structures on $G$ is equivalent to the study of left-symmetric structures on $G$ compatible with the Lie structure.

Let $A$ be a left-symmetric algebra whose associated Lie algebra is $\mathcal{G}$, and let $L_{x}$ and $R_{x}$ denote the left and right multiplications, respectively i.e. $L_{x}y = x \ast y$ and $R_{y}x = y \ast x$. The identity (1) is now equivalent to the formula

$$[L_{x}, L_{y}] = L_{[x, y]}$$

or, in other words, the linear map $L : \mathcal{G} \rightarrow \text{End}(A)$ is a representation of Lie algebras.

If a left-symmetric algebra $A$ has no proper two-sided ideal and it is not the zero algebra of dimension 1, then $A$ is called simple. $A$ is called semi simple, if it is a direct sum of simple left-symmetric algebras.

We say that $A$ is complete if $R_{x}$ is a nilpotent operator for all $x \in A$. It turns out that, for a given simply connected Lie group $G$ with Lie algebra $\mathcal{G}$, the complete left-invariant affine structures on $G$ are in one-to-one correspondence with the complete left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure. It is also known that an $n$-dimensional simply connected Lie group admits a complete left-invariant affine structure if and only if it acts simply transitively on $\mathbb{R}^{n}$ by affine transformations [9]. A simply connected Lie group is acting simply transitively on $\mathbb{R}^{n}$ by affine transformations must be solvable according to [6]. It is well known that not every solvable (even nilpotent) Lie group can admit an affine structure [7].

We say that $A$ is Novikov algebra if it satisfies the identity

$$(x \ast y) \circ z = (x \circ z) \ast y,$$

for all $x, y, z \in A$. (3)

In terms of left and right multiplications, (3) is equivalent to the formula

$$[R_{x}, R_{y}] = 0,$$

for all $x, y \in A$.

The left-symmetric algebra $A$ is called derivation algebra if it satisfies the identity

$$(x \ast y) \circ z = (y \ast z) \circ x,$$

or, equivalently, all left and right multiplications $L_{x}$ and $R_{x}$ are derivations of $g$.

Recall that a Lie algebra $\mathcal{G}$ is an extension of the Lie algebra $\mathcal{G}$ by the Lie algebra $A$ if there exists a short exact sequence of Lie algebras

$$0 \rightarrow A \rightarrow \mathcal{G} \rightarrow \mathcal{G} / A \rightarrow 0.$$

In other words, $A$ is an ideal of $\mathcal{G}$ such that $\mathcal{G} / A \cong \mathcal{G}$.

For $(x, a)$ and $(y, b)$ in $\mathcal{G} \cong \mathcal{G} \oplus A$, the extended Lie bracket is given by

$$[(x, a), (y, b)] = [(x, y), [a, b]] + \phi(x)b - \phi(y)a + a(\omega(x, y)),$$

where $\phi : \mathcal{G} \rightarrow \text{Der}(A)$ is a linear map and $w : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{A}$ is an alternating bilinear map such that

$$\phi(xz) = \omega(x, z) + \phi(x)a(z) - \phi(z)x.$$
δh(x,y) = ρx(δ(h(x))) + λy(δ(h(y))) − h(x,y), \tag{6}

and for a bilinear map g ∈ L(K,V) we set

\[ \delta g(x,y) = g(x,y) − g(x)+λ(y)(g(x)+λ(x)) − g(λ(x)+λ(y)) − g(λ(x)+λ(y)) \tag{7} \]

where λ and ρ are linear maps λ,ρ:K → End(V).

It is straightforward to check that \( \delta \circ \delta = 0 \). Therefore, if we set \( Z^1_{(K,V)} = \ker \delta \) and \( B^1_{(K,V)} : \text{Im } \delta \), we can define a notion of second co-homology for the actions λ and ρ by simply setting \( H^1_{(K,V)} = Z^1_{(K,V)} / B^1_{(K,V)} \). As in the Lie algebras, we can prove the following [9].

**Proposition 2:** For given linear maps λ, ρ:K → End(V), the equivalent classes of extensions

\[ 0 \rightarrow V \rightarrow A \rightarrow K \rightarrow 0 \]

of K by V are in one-to-one correspondence with the elements of the second co-homology group \( H^1_{(K,V)} \).

A left-symmetric algebra extension

\[ 0 \rightarrow E \rightarrow A \rightarrow K \rightarrow 0 \]

called central if and only if \( \delta E \subseteq C(A) \) where

\[ C(A) = \left\{ x \in A : x \cdot y = y \cdot x = 0 \right\} \]

is the center of A. In particular, the extension is central whenever \( E \) is a trivial A-bimodule (i.e., \( λ = ρ = 0 \)). We say that the extension is exact if and only if \( \delta E = C(A) \). It is easy to verify [9] that the extension is exact if and only if \( I_{\mu\nu} \neq 0 \), where

\[ I_{\mu\nu} = \left\{ x \in A : x \cdot y = y \cdot x = 0 \text{ and } g(x,y) = g(y,x) \text{ for all } y \in A \right\} \]

We observe that \( I_{\mu\nu} \) is depends only on the co-homology class of g, that is \( I_{\mu\nu} \) is well defined. In case E is a trivial A-bimodule, we denote the central extension corresponding to the class \([g] \in H^1(A,E)\) by \( \tilde{A}[g] \).

Let \( \tilde{A}[g] \) and \( \tilde{A}[g'] \) be two central extensions of A by E, \( μ ∈ \text{Aut}(E) = GL(E) \) and \( η ∈ \text{Aut}(A) \), where \( \text{Aut}(E) \) and \( \text{Aut}(A) \) are the groups of left-symmetric automorphisms of E and K, respectively. It is clear that if \( h ∈ L(E,A) \), then the linear mapping \( \psi : \tilde{A}[g] \rightarrow \tilde{A}[g'] \) defined by

\[ \psi(x,a) = (\eta(x),μ(a)+h(x)) \]

is an isomorphism provided

\[ g'(η(x),η(y)) = μ(g(x,y)) + δh(x,y) \]

for all \( x,y \in A \). That is \( I_{\mu\nu} \) is well defined.

Denoting the set of all exact central extensions of A by E by

\[ H^1_{(A,E)} = \left\{ [g] ∈ H^1(A,E) : I_{\mu\nu} = 0 \right\} \]

and the orbit of [g] by \( G[g] \), it turns out that the following result is valid [9].

**Proposition 3:** Let \([g] \) and \([g'] \) be two classes in \( H^1_{(A,E)} \). Then, the central extensions \( \tilde{A}[g] \) and \( \tilde{A}[g'] \) are isomorphic if and only if \( G[g] = G[g'] \). In other words, the classification of the exact central extensions of A by E is, up to left-symmetric isomorphism the orbit space of \( H^1_{(A,E)} \) under the natural action of \( G = \text{Aut}(E) \times \text{Aut}(A) \).

We close this section by the following important result [15].

**Proposition 4:** Let \( 0 \rightarrow I \rightarrow A \rightarrow K \rightarrow 0 \) be an exact sequence of left-symmetric algebras such that A is complete then I and J are complete.

**Proof:** Let A be a complete left-symmetric algebra. Then R is nilpotent for all \( x \in A \). Since J is an ideal of A, then R is nilpotent for all \( x \in I \), that is I is complete. On the other hand, Since \( J \cong A/I \), we can define for \( x \in A \), \( R, I,J : J \rightarrow J, \) by \( R, I,J (x) = y + I \) for all \( y \in A \), \( \tau = y + I. \) Since for all \( y_1, y_2 \in A \) such that \( y_1 + I = y_2 + I \) there exists \( z \in I \) so that \( y_1 = y_2 + z \),

\[ R_1(y_1 + I) = R(y_1 + I) \]
\[ = R_1(y_1 + z + I) \]
\[ = R(y_1 + z + I) \]
\[ = R(y_1 + I) \]

then, \( R_1 \) is well defined. We also have, for all \( x, y \in A \), that

\[ R_1 \tau = (y + I)(x + I) \]
\[ = y \cdot x + I \]
\[ = R_1 \tau \]

Thus, to prove that J is complete, it is enough to prove that \( R_1 \) is nilpotent for all \( x \in A \). Since \( R_1 \) is nilpotent, then \( R_1 = 0 \) for some \( k \in N \). This implies that

\[ R_1(y + I) = I \]

for all \( y \in A \). Hence, \( R_1 \tau = 0 \) for all \( \tau \in J \), that is \( R_1 \) is nilpotent for all \( x \in A \), and hence J is complete.

**Complete Left-Symmetric Structures on Solvable Non-Unimodular Lie Algebras of Dimension 3**

Recall that a lie algebra \( \mathcal{G} \) is unimodular if and only if \( \text{tr}(ad_x) = 0 \) for all \( x \in G \). The classification of solvable non unimodular Lie algebras of dimension 3 can be found [16].

**Lemma 5:** Let g be solvable non-unimodular Lie algebra of dimension 3. Then there is a basis \( \{e_1, e_2, e_3\} \) of \( G \) so that

\[ [e_i, e_j] = \alpha_{ij} e_i + \beta_{ij} e_j \]
\[ [e_i, e_j] = \gamma_{ij} e_i + (2 - \alpha_{ij} e_j \]

If we exclude the case where D is the identity matrix then the determinant \( \text{det} D = \alpha(2-\alpha) - \beta \) provides a complete isomorphism invariant for this Lie algebra.

According to this result, we can, by simple computations, find that there are five possibilities for D:

\[ D \cong \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D \cong \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D \cong \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D \cong \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

This implies that any solvable non-unimodular Lie algebra of
dimension 3 is isomorphic to one and only one of the following Lie algebras:

\[ G_{1,1} : [e_1, e_1] = e_2, \]
\[ G_{1,2} : [e_1, e_2] = e_1, [e_1, e_1] = e_3, \]
\[ G_{1,3} : [e_1, e_1] = e_2 + e_3, [e_2, e_3] = e_1, \]
\[ G_{2,1} : [e_2, e_1] = e_2, [e_1, e_1] = e_3, \]
\[ G_{2,2} : [e_1, e_2] = e_1 + e_3, [e_2, e_3] = -\zeta e_2 + e_1, \]
\[ G_{2,3} : [e_1, e_3] = e_1, [e_2, e_3] = e_2. \]

Now let \( G \) be real solvable non-unimodular Lie algebra of dimension 3. Let \( A \) be a complete left-symmetric algebra whose associated Lie algebra is \( G \).

We shall first recall the following result from [12].

**Lemma 6**: Only the complex simple left-symmetric algebras and even-dimensional complex semisimple left-symmetric algebras may have real forms, where a real form of a complex left-symmetric algebra \( A \) is a subalgebra \( A_0 \) of \( A \) such that \( A_0 \) is a real left-symmetric algebra.

Now, we can prove the following

**Proposition 7**: \( A \) is not simple. In other words, any complete left-symmetric structure on a solvable non-unimodular Lie algebra of dimension 3 is not simple.

**Proof**: Assume to the contrary that \( A \) is simple. Then, Lemma 6 shows that complexification \( A^c \) of \( A \) is simple as the dimension of \( A^c \) is odd. We can now apply Corollary 4.2 in [11] to deduce that \( A^c \) is isomorphic to the complex left-symmetric algebra \( A^c \) having a basis \( \{e_1, e_2, e_3\} \) such that the only non-trivial products are

\[ e_1 \cdot e_2 = e_3, \]
\[ e_1 \cdot e_3 = -e_2, \]
\[ e_2 \cdot e_3 = e_1. \]

Thus, the complex Lie algebra \( G^c \) associated to \( A^c \cong A^{c1} \) is unimodular and hence \( G \) must be unimodular. This contradiction shows that \( A \) is not simple.

Before returning to the left-symmetric algebra \( A \), we need to state the following facts without proofs.

**Lemma 8**: Let \( A \) be a left-symmetric algebra with associated Lie algebra \( G \) and \( R \) a two-sided ideal in \( A \). Then the Lie algebra \( R \) associated to \( R \) is an ideal in \( G \).

**Lemma 9**: Let \( G \) be solvable non-unimodular Lie algebra of dimension 3 and let \( I \) be a proper ideal of \( G \). Then \( I \) is isomorphic to \( \mathbb{R}^2 \), \( \text{aff}(\mathbb{R}) = \langle e_1, e_2, [e_1, e_2] = e_3 \rangle \).

By Proposition 7, \( A \) is not simple and hence it has a proper two-sided ideal \( I \), so we get a short exact sequence of left-symmetric algebras

\[ 0 \rightarrow I \rightarrow A \rightarrow J \rightarrow 0 \]  \hspace{1cm} (8)

If \( I \) is the Lie subalgebra associated to \( I \) then, by Lemma 8, \( I \) is an ideal in \( G \). From Lemma 9 it follows that there are three cases to be considered according to weather \( I \) is isomorphic to \( \mathbb{R}, \mathbb{R}^2 \), or \( \text{aff}(\mathbb{R}) \).

Case 1: \( I \cong \mathbb{R} \).

In this case, the short exact sequence (8) becomes

\[ 0 \rightarrow \mathbb{R}_x \rightarrow A_x \rightarrow I_x \rightarrow 0 \]

where \( I_x \) is a complete left-symmetric algebra of dimension 2 and \( \mathbb{R}_x \) is \( \mathbb{R} \) with the trivial product. At the Lie algebra level, we have a short exact sequence of Lie algebras of the form

\[ 0 \rightarrow \mathbb{R} \rightarrow G \rightarrow H \rightarrow 0 \]  \hspace{1cm} (9)

where \( H \) denotes the associated Lie algebra of \( I_x \) and \( G \) is an extension of \( H \) by \( \mathbb{R} \).

Since \( H \) is of dimension 2, then \( H \) is either isomorphic to \( \mathbb{R}^2 \) or \( \text{off}(\mathbb{R}) \).

Assume first that \( H \cong \mathbb{R}^2 \). Then, the short exact sequence (9) becomes

\[ 0 \rightarrow \mathbb{R} \rightarrow G \rightarrow \mathbb{R}^2 \rightarrow 0 \]

Let \( \{e_1, e_2\} \) be a basis for \( \mathbb{R}^2 \). On \( \mathbb{R}^2 \times \mathbb{R}^2 \), the extended Lie bracket given by (4) takes the simplified form

\[ [(a, b), (c, d)] = (0, \phi(x) b - \phi(y) a + \omega(x, y)), \]  \hspace{1cm} (10)

for all \( a, b \in \mathbb{R}, \ x, y \in \mathbb{R}^2 \).

Setting \( \hat{e}_i = (e_i, 0), i=1,2 \) and \( \hat{e}_3 = (0,1) \) we get

\[ \hat{e}_i \cdot \hat{e}_j = \omega(e_i, e_j) \hat{e}_k, \]
\[ \hat{e}_i \cdot \hat{e}_j = \phi(e_i) \hat{e}_j, \]
\[ \hat{e}_i \cdot \hat{e}_j = \phi(e_i) \hat{e}_j. \]

Since \( G \) is solvable and non-unimodular, we can, without loss of generality, assume that \( \phi(\hat{e}_i) = 0 \). That is

\[ D = \begin{pmatrix} 0 & \omega(e_i, e_j) \\ 0 & \phi(e_i) \end{pmatrix} \]

Notice that \( \phi(\hat{e}_i) \) should be non-zero, since otherwise \( G \) becomes unimodular. In other words,

\[ D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

Now, we shall determine all the complete left-symmetric structures on \( \mathbb{R}^2 \). These are described by the following lemma that we state without proof.

**Lemma 10**: Up to left-symmetric isomorphism, there are two complete left-symmetric structures on \( \mathbb{R}^2 \) given, in a basis \( \{e_1, e_2\} \) of \( \mathbb{R}^2 \), by either

(i) \( e_1 \cdot e_2 = 0 \), \( i,j = 1,2 \)

(ii) \( e_2 \cdot e_2 = e_1 \).

From now on, \( A_2 \) will denote the vector space \( \mathbb{R}^2 \) endowed with one of the complete left-symmetric structures described in Lemma 10.

The extended left-symmetric product on \( A_2 \times \mathbb{R}_x \) given by (5) turns out to take the simplified form

\[ (x, a) \cdot (y, b) = (x, y, b\lambda_1 + a\rho_1 + \omega(x, y)), \]  \hspace{1cm} (11)

for all \( x, y \in A_2 \) and \( a, b \in \mathbb{R} \). Indeed, \( \rho, \lambda \in \text{End}(\mathbb{R}) \) for all \( x \in A_2 \). So, we can identify \( \rho \) and \( \lambda \) with real numbers that we denote by \( \rho \) and \( \lambda \), respectively.
Note here that $\lambda_i = \phi(x) + \rho$, for all $x \in \mathbb{R}^3$ whereas $\phi: \mathbb{R}^3 \rightarrow \text{End}(\mathbb{R}) \cong \mathbb{R}$ in (10).

The conditions in Theorem 1 can be simplified to the following conditions

$$
\rho(x,v) = \rho_s \circ \rho_s \quad (12)
$$

$$
g(x,y,z) - g(x,z,y) + \lambda_1(g(y,z)) - \lambda_1(g(y,z)) - \rho_1(g(x,y) - g(x,y)) = 0 \quad (13)
$$

By using (10) and (11), we deduce from

$$
\omega(x,y) = g(x,y) - g(y,x) - \lambda_1(g(v,w)) - \lambda_2(g(v,w)) - \rho_1(g(x,y) - g(x,y)) = 0
$$

that

$$
\omega(x,y) = g(x,y) - g(y,x) - \lambda_1(g(v,w)) - \lambda_2(g(v,w)) - \rho_1(g(x,y) - g(x,y)) = 0
$$

Since $\omega(x,y) = 0$, then $g(e_1, e_2) = g(e_2, e_1)$. By using identity (12) to $e_1, e_2, i = 1, 2$, we deduce that $\rho = 0$. Hence $\lambda_i = 0$ and $\lambda_i \neq 0$. Therefore $\lambda_i = \alpha$, $\alpha \in \mathbb{R}^*$. In this case, the formula (6) and (7) become

$$
\delta h(x,y) = \lambda_1(h(y)) - h(x,y)
$$

And

$$
\delta g(x,y) = g(x,y) - g(y,x) + \lambda_1(g(y,z)) - \lambda_2(g(y,z)) - \rho_1(g(x,y) - g(x,y)) = 0
$$

where $h \in \mathcal{L}(A, \mathbb{R})$ and $g \in \mathcal{L}(A, \mathbb{R})$. According to Lemma 10, there are two cases to be considered.

10.1. $A_i = \{e_i, e_j : e_i e_j = 0, i, j = 1, 2\}$

In this case, using the first formula above for $\delta h$, we get

$$
\delta h = \begin{pmatrix}
  h_1 & h_2 \\
  0 & 0
\end{pmatrix}
$$

where $h_1 = \alpha h_1$ and $h_2 = \alpha h_2$. Similarly, using the second formula above for $\delta g$, we verify easily that if $g$ is a cocycle (i.e. $\delta g = 0$) and $g = g(e_i, e_j)$, then

$$
g = \begin{pmatrix}
  g_{11} & 0 \\
  0 & 0
\end{pmatrix}
$$

that is $g_{11} = g_{21} = g_{22} = 0$. In this case, the class $[\xi] \in H^1(A, \mathbb{R})$ of a cocycle $g$ may be represented, in the basis above, by a matrix of the simplified form

$$
g = \begin{pmatrix}
  s & 0 \\
  0 & 0
\end{pmatrix}
$$

We can now determine the extended complete left-symmetric structures on $A_i$. By setting $e_i = (e_i, 0), i = 1, 2$ and $e_j = (0, 1)$ and using formula (11) we obtain the non-zero relations in $A_i$ are

$$
\tilde{e}_1 \cdot \tilde{e}_2 = \alpha \tilde{e}_1,
$$

$$
\tilde{e}_1 \cdot \tilde{e}_3 = \alpha \tilde{e}_3
$$

with $\alpha = \lambda_1 \neq 0$

By setting $e_1 = \frac{1}{\alpha} \tilde{e}_1$, $e_3 = \tilde{e}_3$ and $\tilde{e}_2 = e_2$, and $t = \frac{s}{\alpha}$ we see that the new basis $\{e_1, e_2, e_3\}$ of $A_i$ satisfies

$$
e_1, e_2 = e_3
$$

and all other products are zero. We can easily see that this product is isomorphic to

$$
e_1, e_2 = e_3
$$

We set $N_{1,i} = \{e_i, e_j : e_i e_j = e_i, e_j = e_i\}$. We can now determine the extended complete left-symmetric structures on $A_i$.

10.2. $A_i = \{e_i, e_j : e_i e_j = e_i\}$

We obtain, as above, that $A_i$ is isomorphic to one of the following complete left-symmetric algebras

(i) $N_{1,1,2} = \{e_i, e_j, e_k : e_i e_j = e_i, e_k = e_i\}$

(ii) $N_{1,3,3} = \{e_i, e_j, e_k : e_i e_j = e_i, e_k = e_i\}$

Assume now that $H^1(\mathbb{R}) \cong \text{aff}(\mathbb{R})$. Then the extended Lie bracket on $\text{aff}(\mathbb{R}) \times \mathbb{R}$ given by (4) takes the form

$$
[(x,a)], (y,b)] = (x,y) + [a + \omega(x,y)],
$$

for all $a, b \in \mathbb{R}$ and $x, y \in \text{aff}(\mathbb{R})$. Let $\{e_i, e_j\}$ be a basis of $\text{aff}(\mathbb{R})$. We can easily see that $\omega(e_i, e_j) = 0$. By setting $e_i = (e_i, 0), i = 1, 2$ and $e_j = (0, 1)$ we get

$$
[e_i, e_j] = \omega(e_i, e_j)
$$

$$
[e_i, e_j] = \phi(e_i)
$$

Since $\hat{g}$ is solvable and non-unimodular, then as above we can assume that $\phi(e_i) = 0$. That is,

$$
D = \begin{pmatrix}
  0 & \omega(e_i, e_j) \\
  0 & \phi(e_i)
\end{pmatrix}
$$

Notice that $\phi(e_i) + 1 \neq 0$, since otherwise $g$ becomes unimodular.

Now, we have the following cases.

1. If $\det D = 0$, then $D = \begin{pmatrix} 1 & 0 \\
  0 & 1 \end{pmatrix}$ that is, $\phi(e_i) = 0$ and $\omega(e_i, e_j) = 0$. This means that $\hat{g}$ is identical zero, i.e. $\hat{g}$ is a central extension of $\text{aff}(\mathbb{R})$ by $\mathbb{R}$.

2. If $\det D = 0, D = \begin{pmatrix} 1 & 0 \\
  0 & 1 \end{pmatrix}$ with $\phi(e_i) = 1$. It is not hard to prove the following

**Lemma 11:** Up to left-symmetric isomorphisms, there is a unique complete left-symmetric structure on $\text{aff}(\mathbb{R})$ which is given, relative to a basis $e_i, e_j$ of $\text{aff}(\mathbb{R})$, by $e_i e_j = e_i$, $e_i e_j = e_j$.

We will denote by $N_i$, the vector space $\text{aff}(\mathbb{R})$ endowed with the complete left-symmetric product given in Lemma 11. On the other hand, the extended left-symmetric product on $N_i \times \mathbb{R}$ is given by

$$
(x, a) \cdot (y, b) = \mu(x, y) + ap(x) + g(x, y),
$$

for all $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}$. 

The conditions in Theorem 1 can be simplified to the following conditions

\[ \lambda_{(x,y)} = 0 \quad (16) \]

\[ \rho_{(x,y)} = \rho_{(y,x)} \quad (17) \]

\[ g(x,y-z) \cdot g(y,z) + \lambda_{(y,z)} - \lambda_{(x,y)} = g(x,y) - g([x,y],z) \]

\[ -\rho_{z}(g(x,y) - g(y,x)) = 0 \]

By using (10) and (11), we deduce from

\[ [x,a] = (x,a) \cdot (y,b) - (y,b) \cdot (x,a) \]

that

\[ \omega(x,y) = g(x,y) - g(y,x) \]

From condition (16), we get \( \lambda_{x} = 0 \). Applying the identity (17) above to \( e_{i} \cdot e_{j} \), \( i=1,2 \), we deduce that \( \rho = 0 \) and hence \( \lambda_{x} = \phi(e_{i}) \).

Assume first that \( D \equiv \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \), that is, \( \omega(e_{i}, e_{j}) = 0 \) and \( \phi(e_{i}) = 0 \), then \( \lambda = \rho = 0 \). Thus, the extension is central.

We know that the classification of the exact central extension of \( N \) by \( R \) is, up to left-symmetric isomorphism, the orbit space of \( H^{2}(N,R) \) under the natural action of \( G = Aut(R) \times Aut(N) \) (Proposition 3). So, we must compute \( H^{2}(N,R) \). Since \( R \) is a trivial \( N \)-bimodule, then

\[ \delta_{2}h(x,y) = -h(x,y), \]

\[ \delta_{2}g(x,y,z) = g(x,y,z) - g(y,x,z) - \left( g(x,y),z \right) \]

where \( h \in C(N,R) \) and \( g \in C(N,R) \). This implies that, with respect to the basis \( e_{i}, e_{j} \) of \( N \), \( \delta_{2}h \) is of the form

\[ \delta_{2}h(x,y) = \begin{pmatrix} 0 & h_{12} \\ 0 & 0 \end{pmatrix} \]

where \( h_{12} = -h(e_{i}) \).

Observe that if \( g \) is a 2-cocycle (i.e. \( \delta_{2}g = 0 \)), then

\[ g = \begin{pmatrix} g_{11} & 0 \\ 0 & 0 \end{pmatrix} \]

where \( g_{y} = g(e_{i}, e_{j}) \). Hence, \( \left[ g \right] \in H^{2}(N,R) \) can be represented as a matrix with respect to \( \{ e_{i}, e_{j} \} \) by

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in R \]

We determine, in this case, the extended left-symmetric structure on \( A_{x} \). By setting \( \tilde{e}_{i} = (e_{i}, 0), i=1,2 \), and \( \tilde{e}_{3} = (0,1) \), and using formula (15), we find

\[ \tilde{e}_{i} \cdot \tilde{e}_{j} = t \tilde{e}_{j}, \quad \tilde{e}_{i} \cdot \tilde{e}_{j} = \tilde{e}_{2}. \]

and all other products are zero, \( t \in R \). We denote \( G \) endowed with this structure by \( N_{1x} \).

Recall that the extension

\[ 0 \rightarrow R \rightarrow A_{x} \rightarrow N \rightarrow 0 \]

is exact (i.e. \( H_{1}(R) = C(A_{x}) \)) if and only if \( I_{(x)} = \{ 0 \} \).

Let \( x = ae_{i} + be_{j} \in I_{(x)} \). Then computing all the products \( x \cdot e_{i} = e_{i} \cdot x = 0 \), we deduce that \( x = 0 \), that is the extension is exact.

Let \( N_{x}, N_{y} \) be two left-symmetric algebras as above. We know that \( N_{x} \) is isomorphic to \( N_{y} \) if and only if there exists an isomorphism \( \lambda \), \( \eta \) \( (x, \eta) \in Aut(R) \times Aut(N) \) such that for all \( x, y \in N_{x} \), we have

\[ g'(x,y) = \eta(\lambda(x,y)) \]

Now, we have to calculate \( Aut(N_{x}) \). Let \( \eta \in Aut(N_{x}) \) so that, with respect to the basis \( e_{i}, e_{j} \) of \( N_{x} \) with \( e_{i} \cdot e_{j} = e_{2} \),

\[ \eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

Since \( \eta(e_{i}) = \eta(e_{i} \cdot e_{j}) = \eta(e_{i}) \cdot \eta(e_{j}) \), then \( b = 0 \) and \( d = ad \). Also \( 0 = \eta(e_{i} \cdot e_{j}) = \eta(e_{i}) \cdot \eta(e_{j}) \) which implies that \( a = 0 \) or \( e_{i} = 0 \). Since \( \delta \eta \neq 0 \), then \( d 
eq 0 \) and hence \( a = 1 \) and \( c = 0 \). This means that

\[ \eta = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \]

with \( d \neq 0 \). We shall now apply formula (18). For this we recall first that in the basis \( e_{i}, e_{j} \), the classes \( g \) and \( g' \) corresponding to \( N_{x} \) and \( N_{y} \), have, respectively, the forms

\[ g = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad g' = \begin{pmatrix} t' & 0 \\ 0 & 0 \end{pmatrix} \]

From \( g(e_{i}, e_{j}) = g'(e_{i}, e_{j}) \), we get

\[ t' = at \]

Hence \( N_{x} \) and \( N_{y} \) are isomorphic if and only if \( t' = at \), for some \( a \in R^{*} \).

Notice that if \( t = 0 \), we obtain the complete left-symmetric algebra \( N_{0} \) described above. If \( t \neq 0 \), we obtain, by setting \( e_{i} = e_{i}, i=1,2 \), and \( e_{3} = t \tilde{e}_{3} \), the complete left-symmetric algebra

\[ N_{3} = \{ e_{i}, e_{j}, e_{3} : e_{i} \cdot e_{j} = e_{i} \cdot e_{3} = e_{3} \cdot e_{2} = e_{3} \} \]

Assume now that \( D \equiv \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \), that is, \( \omega(e_{i}, e_{j}) = 0 \) and \( \phi(e_{i}) = 1 \).

Then \( \lambda(e_{i}) = \phi(e_{i}) = 1 \). We deduce, in this case, that, in the basis \( e_{i}, e_{j} \) of \( N_{x} \), the \( g \in H_{1}(N_{x}, R) \) of a cocycle \( g \) may be represented by a matrix of the simplified form

\[ g = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \]

We determine, in this case, the extended complete left-symmetric structure on \( A_{x} \). By setting \( \tilde{e}_{i} = (e_{i}, 0), i=1,2 \) and \( \tilde{e}_{3} = (0,1) \) and using formula (15), we obtain

\[ \tilde{e}_{i} \cdot \tilde{e}_{j} = \tilde{e}_{2} + t \tilde{e}_{3}, \quad \tilde{e}_{2} \cdot \tilde{e}_{3} = t \tilde{e}_{3}, \quad \tilde{e}_{3} \cdot \tilde{e}_{3} = \tilde{e}_{3} \]

We denote this left-symmetric algebra by \( B_{x} \). Notice that if \( t = 0 \), we obtain the complete left-symmetric algebra \( B_{3} \) with the non-zero relations...
\[ e_1 \cdot e_2 = e_2, \]
\[ e_1 \cdot e_3 = e_2, \]
If \( t \neq 0 \), we obtain, by setting \( e_i = \tilde{e}_i \), \( i = 1, 2 \) and \( e_3 = t \tilde{e}_3 \), the complete left-symmetric algebra \( B_{t_i} \), with the non-zero relations
\[ e_1 \cdot e_2 = e_2 + e_3, \]
\[ e_2 \cdot e_3 = e_1, \]
\[ e_3 \cdot e_1 = e_1. \]
Assume now that \( D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) that is, \( \omega(e_i, e_i) = 1 \) and \( \phi(e_i) = 1 \).

Hence \( \lambda(e_i) = \phi(e_i) = 1 \). Using the same method as above, it follows that the class \([g] = H^0_{\lambda, \rho}(N_1, \mathbb{R})\) of a co-cycle \( g \) takes the reduced form
\[ g = \begin{pmatrix} 0 & t \\ t & 1 \end{pmatrix}. \]

We determine, in this case, the extended complete left-symmetric structures on \( A_3 \). By setting \( \tilde{e}_i = (e_i, 0) \), \( i = 1, 2 \) and \( \tilde{e}_3 = (0, 1) \) and using formula (15), we obtain
\[ \tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2 + t \tilde{e}_3, \]
\[ \tilde{e}_2 \cdot \tilde{e}_3 = (t - 1) \tilde{e}_3, \]
\[ \tilde{e}_3 \cdot \tilde{e}_1 = \tilde{e}_3. \]

We denote such a left-symmetric algebra by \( C_{t_3} \). Notice that if \( t = 1 \), we obtain the complete left-symmetric algebra \( C_{t_3} \) with the non-zero relations
\[ e_1 \cdot e_2 = e_2 + e_3, \]
\[ e_1 \cdot e_3 = e_2, \]
and if \( t \neq 1 \), we obtain the complete left-symmetric algebra \( C_{t_3} \), with the non-zero relations
\[ e_1 \cdot e_2 = e_2 + t e_3, \]
\[ e_2 \cdot e_3 = (t - 1) e_3, \]
\[ e_3 \cdot e_1 = e_1, \]
where different values of \( t \) give non-isomorphic complete left-symmetric algebras.

Assume finally that \( D = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \), with \( 0 < |\mu| < 1 \), that is, \( \omega(e_i, e_i) = 0 \) and \( \phi(e_i) = \mu \). Hence \( \lambda(e_i) = \phi(e_i) = \mu \). It follows that the class \([g] = H^0_{\lambda, \rho}(N_1, \mathbb{R})\) of a co-cycle \( g \) is identically zero.

We determine, in this case, the extended complete left-symmetric structures on \( A_3 \). By setting \( \tilde{e}_i = (e_i, 0) \), \( i = 1, 2 \) and \( \tilde{e}_3 = (0, 1) \) and using formula (15), we obtain
\[ \tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2, \]
\[ \tilde{e}_1 \cdot \tilde{e}_3 = \mu \tilde{e}_3, \]
where \( 0 < |\mu| < 1 \). We set
\[ D_{t_3}(\mu) = (e_i, e_i, e_i, e_i, e_i, e_i, e_i, e_i, e_i, e_i, \mu e_i) \]
where \( 0 < |\mu| < 1 \).

Case 2: \( T \equiv \text{aff} (\mathbb{R}) \).

In this case, the short exact sequence (8) becomes
\[ 0 \rightarrow N_1 \rightarrow A_1 \rightarrow \mathbb{R} ightarrow 0 \]
where \( N_1 \) is the complete left-symmetric algebra whose associated Lie algebra is \( \text{aff} (\mathbb{R}) \) and \( \mathbb{R} \) is the trivial left-symmetric algebra over \( \mathbb{R} \).

Let \( \sigma : \mathbb{R} \rightarrow A_1 \) be a section and set \( \sigma(1) = x_i \in A_i \) and define two linear maps \( \lambda, \rho : End (N_1) \rightarrow \mathbb{R} \) by \( \lambda(y) = x_i \cdot y \) and \( \rho(y) = y \cdot x_i \). By setting \( e = x_i \cdot x_j \), we see that \( e \in N_1 \). Let \( g : \mathbb{R} \times \mathbb{R} \rightarrow N_1 \) be the bilinear map defined by \( g(a, b) = \sigma(a) \sigma(b) - \sigma(a b) \). Since the complete left-symmetric structure on \( \mathbb{R} \) is trivial, then \( g(a, b) = abe \), or equivalently \( g(1,1) = e \). Also we can show that \( \delta_g = 0 \), i.e. \( g \in Z^1_{\lambda, \rho}(N_1) \).

In this case, the extended left-symmetric product on \( \mathbb{R} \oplus N_2 \) given by (3) takes the simplified form
\[ (a, x) \cdot (b, y) = (0, x \cdot y + a \lambda(y) + b \rho(x) + abe), \]
for all \( a, b \in \mathbb{R} \) and \( x, y \in N_1 \).

The conditions in Theorem 1 can be simplified to the following conditions
\[ \lambda(x \cdot y) = \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \]
\[ \rho([x, y]) = x \cdot \rho(y) - y \cdot \rho(x) \]
\[ [\lambda, \rho] + \rho^2 = R \]

Let \( \phi : \mathbb{R} \rightarrow Der (\text{aff} (\mathbb{R})) \), be a derivation of \( \text{aff} (\mathbb{R}) \). Set
\[ \phi(1) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \]
relative to a basis \( e_1, e_2 \) of \( \text{aff} (\mathbb{R}) \) satisfying \([e_1, e_2] = e_2 \). From the identity \( \phi(1)e_i = [\phi(1), e_i] + [e_i, \phi(1)e_i] \), we deduce that \( a = c = 0 \), hence
\[ \phi(1) = \begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix} \]

Let
\[ \rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \]
relative to a basis \( e_1, e_2 \) of \( \text{aff} (\mathbb{R}) \) satisfying \([e_1, e_2] = e_2 \). Applying formula (21) to \( e_i \), we get \( \beta_i = 0 \). Since \( \phi(1) = \lambda - \rho \), we deduce that, relative to the basis \( e_1, e_2 \), we have
\[ \lambda = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 + b & \beta_2 + d \end{pmatrix} \]

Applying formula (20) to all products of the form \( e_i e_j \), \( i, j = 1, 2 \), we get \( \alpha_2 + b = 0 \). Moreover, by applying formula (22) to \( e_i \) and \( e_2 \) we get \( \alpha_i = \beta_i = 0 \). Thus
\[ \rho = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \] and \( \lambda = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \)

Now, since \( e \in N_2 \), then \( e = t e_i + s e_2 \) for some \( t, s \in \mathbb{R} \). Formula (22) when applied to \( e_i \) gives
\[ -b e_2 = s e_2 \]
for which we get that \( e = x \cdot x = t e_i - b e_2 \), \( t \in \mathbb{R} \). Hence we get a left-symmetric product on \( A_3 \). Now, let us write down the structure of \( A_3 \) using a basis. From above we have
\[ e_1 \cdot e_2 = e_2, \quad e_1 \cdot x = -b e_2. \]
\[ x = \alpha e_1 + \beta e_2 + \gamma e_3 \]

where \( x \) is the element of \( A \), and \( \alpha, \beta, \gamma \in \mathbb{R} \). Note that, in this case, \( x \) is isomorphic to one of the left-symmetric algebra \( N_{1,2} \) or \( N_{2,3} \) given above.

**Case 3: \( t \neq 0 \)**

In this case, the short exact sequence (8) becomes

\[ 0 \to A_1 \to A_2 \to \mathbb{R}_t \to 0 \]

where \( A_1 \) is a complete left-symmetric algebra whose lie algebra is \( \mathbb{R} \) and \( \mathbb{R}_t \) is the trivial left-symmetric algebra over \( \mathbb{R} \).

At the lie algebra level, we have a short exact sequence of lie algebras of the form

\[ 0 \to \mathbb{R}^2 \to \mathbb{G} \to \mathbb{R} \to 0 \]

where \( \mathbb{R}^2 \) is a basis for \( \mathbb{G} \). Let \( \phi : \mathbb{R} \to \text{Der}(\mathbb{R}^2) \approx \text{End}(\mathbb{R}^2) \), be a derivation of \( \mathbb{R}^2 \). Relative to a basis \( e_1, e_2 \) of \( \mathbb{R}^2 \), set \( \phi(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

In this case, the extended lie bracket on \( \mathbb{R} \times \mathbb{R}^2 \), given by (4), takes the simplified form

\[ [(a,x),(b,y)] = (0, \phi(a)y - \phi(b)x + \alpha(a,b)) \]

for all \( a, b, x, y \in \mathbb{R} \). Setting \( \hat{e}_i = (1, 0) \) and \( \hat{e}_{i1} = (0, e_i) \), \( i = 1, 2 \) we obtain

\[ [\hat{e}_1, \hat{e}_2] = \hat{e}_3 + \beta \hat{e}_2 \]

By Lemma 5, we obtain that, relative to the basis \( e_1, e_2 \),

\[ D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

with \( a + d = 0 \). Note that, in this case, \( \omega \) may not be zero, that is, the extensions of \( \mathbb{R} \) by \( \mathbb{R}^2 \) are not necessarily semi direct products of \( \mathbb{R} \).

According to Lemma 5, there are five cases to be considered

\[ D \cong \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -\zeta \\ 1 & \zeta \end{pmatrix} \]

Where \( \zeta > 0 \) and \( 0 < |\mu| < 1 \).

Let \( \sigma : \mathbb{R}_x \to A_2 \) be a section and set \( \sigma(1) = x, \in A_1 \) and define two linear maps \( \lambda, \rho \in \text{End}(A_2) \) by putting \( \lambda(x) = x \cdot y \) and \( \rho(y) = x \cdot y \). By setting \( e = x \cdot e_0 \), we see that \( e \in A_2 \). Let \( g : \mathbb{R}_x \times \mathbb{R}_y \to A_2 \) be the bilinear map defined by \( g(a,b) = \sigma(a) \cdot \sigma(b) - \sigma(a \cdot b) \).

The extended left-symmetric product on \( \mathbb{R}_x \oplus A_1 \), given by (5) is then taken the simplified form

\[ (a,x) \cdot (b,y) = (0, x \cdot y + a\lambda(y) + b\rho(x) + ab) \]

for all \( x, y \in A_2 \) and \( a, b \in \mathbb{R} \).

The conditions in Theorem 1 can be simplified to the following conditions

\[ \lambda(x \cdot y) = \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \]

\[ x \cdot \rho(y) = x \cdot \rho(y) \]

\[ [\lambda, \rho] + \rho^2 = R \]

According to Lemma 10, we have the following cases of \( A_2 \):

1. \( A_2 = \{ e_1, e_2, e_3 | e_i = 0, i, j = 1, 2 \} \)

Assume first that \( D \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and let

\[ \rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \]

relative to the basis \( e_1, e_2 \) of \( A_2 \). Since \( \phi(1) = \lambda - \rho \), we deduce that, relative to the basis \( e_1, e_2 \), we have

\[ \lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \]

Applying formula (26) to \( e_1 \), we obtain \( \beta_3 = \beta_1 = 0 \). The same formula when applied to \( e_2 \) yields \( \alpha_2 = \alpha_1 = 0 \). It follows that \( \rho \) is identically zero and

\[ \lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

We can easily show that the condition (26) above is satisfied for all \( e = x \cdot e_1 + te_2 \), \( t \in \mathbb{R} \). Hence we get a left-symmetric product on \( A_2 \).

Now, let us write down the structure of \( A_2 \) using a basis. From
We can easily prove that,\( x \cdot x = s e_1 + t e_2 \).

Since \( \alpha, \beta \) are arbitrary, we can choose \( \alpha, \beta \) so that \( e_1 = x - s e_1 \).

Hence the left-symmetric product on \( A_2 \) is given by the basis \( \{ e_1, e_2, e_3 \} \) of \( A_2 \), by the non-zero relations

\[
e_3 \cdot e_1 = e_1, \\
e_3 \cdot e_2 = te_2.
\]

Notice that if \( t = 0 \), we find the complete left-symmetric algebra \( N_{\alpha, \beta} \).

Assume then that \( D \ni \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and let

\[
\rho = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}
\]

relative to the basis \( e_1, e_2 \) of \( A_2 \). Since \( \phi(1) = \lambda - \rho \), we deduce that, relative to the basis \( e_1, e_2, e_3 \), we have

\[
\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 \\ \alpha_2 & \beta_2 + 1 \end{pmatrix}
\]

By applying formula (26) to \( e_1, e_2 \), and let

\[\rho = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[
\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}
\]

and \( e = x \cdot x = \alpha^2 e_1 + \alpha e_2 \).

Similarly, we find that, relative to the basis \( \{ e_1, e_2, e_3 \} \) of \( A_2 \), the left-symmetric product on \( A_2 \) is given by the non-zero relations

\[
e_1 \cdot e_1 = e_1, \\
e_1 \cdot e_2 = \alpha e_2 + e_2, \\
e_1 \cdot e_3 = \alpha e_3.
\]

Notice that if \( \alpha = 0 \), we get, by setting \( \tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \) and \( \tilde{e}_3 = te_2 \) that \( A_2 \) is isomorphic to the complete left-symmetric algebra \( N_{\alpha, \beta, 0} \).

Assume finally that \( D \ni \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} \), where \( \zeta > 0 \), and let

\[
\rho = \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}, \quad \alpha \in \mathbb{R}
\]

relative to the basis \( e_1, e_2 \) of \( A_2 \). Since \( \phi(1) = \lambda - \rho \), we deduce that, relative to the basis \( e_1, e_2, e_3 \), we have

\[
\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 + 1 \\ \alpha_2 & \beta_2 + 1 \end{pmatrix}
\]

By applying formula (26) to \( e_1, e_2 \), we get

\[
\rho = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix}, \quad a \in \mathbb{R}
\]

and \( e = x \cdot x = \alpha^2 e_1 + \alpha e_2 \).

Similarly, we find that, relative to the basis \( \{ e_1, e_2, e_3 \} \) of \( A_2 \), the left-symmetric product on \( A_2 \) is given by the non-zero relations

\[
e_1 \cdot e_1 = e_1, \\
e_1 \cdot e_2 = \mu e_2, \\
e_1 \cdot e_3 = \mu e_3.
\]

By setting \( \tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \) and \( \tilde{e}_3 = e_3 \), we get the complete left-symmetric algebra \( D_{\alpha, \beta, 0} \),\( \alpha \) and \( \beta \) are the relative to the basis \( e_1, e_2, e_3 \) of \( A_2 \), since \( \phi(1) = \lambda - \rho \), we deduce that, relative to the basis \( e_1, e_2, e_3 \) of \( A_2 \), since \( \phi(1) = \lambda - \rho \), we deduce that, relative to the basis \( e_1, e_2, e_3 \), we have

\[
\lambda = \begin{pmatrix} \alpha_1 + 1 & \beta_1 + 1 \\ \alpha_2 & \beta_2 + 1 \end{pmatrix}
\]
By applying formula (26) to $e_i$ and $e_j$, we obtain that $\rho$ is identically zero,
$$\lambda = \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix}$$
and $e = x \cdot x = 2\zeta e_i + (\zeta^2 - 1)e_j$.

Similarly, we find that, relative to a basis $\{e_i, e_j, e_k\}$ of $A_3$, with
$e_i = x - \zeta e_j + e_k$, the left-symmetric product on $A_3$ is given by the non-zero relations
$$e_i \cdot e_j = e_i + \zeta e_j$$
$$e_i \cdot e_k = -\zeta e_j + e_i$$

Set $\tilde{e}_i = e_i, \tilde{e}_j = e_j$ and $\tilde{e}_k = e_k$. Then, the non-zero relations above become
$$\tilde{e}_i \cdot \tilde{e}_j = \tilde{e}_i + \zeta \tilde{e}_j$$
$$\tilde{e}_i \cdot \tilde{e}_k = -\zeta \tilde{e}_j + \tilde{e}_i$$

We set
$$E_{ij} := \langle e_i, e_j, e_k; e_i \cdot e_j = e_k \rangle.\quad 2. \quad A_3 = \langle e_i, e_j, e_k; e_i \cdot e_j = e_k \rangle.$$

Let $$\rho = \begin{pmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{pmatrix}$$
relative to the basis $e_i, e_j$ of $A_3$. By applying formula (25) to $e_i$ and $e_j$, we get that $\alpha_i = \beta_j = 0$.

Assume first that $D \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then, as $\phi(1) = \lambda - \rho$, we deduce that, relative to the basis $e_i, e_j$,
$$\lambda = \begin{pmatrix} \alpha_i + 1 & \beta_j \\ 0 & \beta_j \end{pmatrix}$$

By applying formula (26) to $e_i$ and $e_j$, we get that $\alpha_i = \beta_j = 0$. Moreover, by applying formula (24) to all products of the form $e_i \cdot e_j, i, j = 1, 2$, we get that $1 = 0$, a contradiction. Thus $D$ cannot be of this form. Similarly, we can prove that $D$ cannot be of the forms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & -\zeta \end{pmatrix}$, $\lambda > 0$.

Assume that $D \equiv \begin{pmatrix} 0 & 0 \\ 1 & \mu \end{pmatrix}$, where $0 < |\mu| < 1$. Then, as $\phi(1) = \lambda - \rho$, we deduce that
$$\lambda = \begin{pmatrix} \alpha_i + 1 & \beta_j \\ 0 & \beta_j + \mu \end{pmatrix}$$

By applying formula (26) to $e_i$ and $e_j$, we get that $\alpha_i = \beta_j = 0$. Moreover, by applying formula (24) to all products of the form $e_i \cdot e_j, i, j = 1, 2$, we get that $\mu = \frac{1}{2}$. Thus
$$\rho = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix}, \lambda = \begin{pmatrix} 1 & \alpha \\ 0 & \frac{1}{2} \end{pmatrix}, \alpha \in \mathbb{R}$$

and $e \equiv x \cdot x = \zeta e_i + \frac{1}{2} e_j, \alpha \in \mathbb{R}$.

Similarly, we find that, relative to a basis $\{e_1, e_2, e_3\}$ of $A_3$ with $e_1 = x + \{a - 1\}e_2 - \zeta e_3$, the left-symmetric product on $A_3$ is given by the non-zero relations
$$e_1 \cdot e_2 = e_1$$
$$e_1 \cdot e_3 = e_1$$
$$e_2 \cdot e_3 = e_2$$

Set $\tilde{e}_1 = e_1, \tilde{e}_2 = e_2$ and $\tilde{e}_3 = e_3$. Then the non-zero relations above become
$$\tilde{e}_1 \cdot \tilde{e}_2 = \tilde{e}_2$$
$$\tilde{e}_1 \cdot \tilde{e}_3 = \tilde{e}_1$$
$$\tilde{e}_2 \cdot \tilde{e}_3 = \tilde{e}_3$$

We set
$$D_{ij} := \langle e_1, e_2, e_3; e_i \cdot e_j = e_k \rangle.$$

**Conclusion**

We can now state the main result of this paper

**Theorem 12:** Let $A_3$ be a three dimensional complete left-symmetric algebra whose associated Lie algebra $G$ is solvable and non-unimodular. Then $A_3$ is isomorphic to one of the following left-symmetric algebras (Table 1).

<table>
<thead>
<tr>
<th>Name</th>
<th>Non-zero product</th>
<th>Lie algebra</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{1,3}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{1,1}$</td>
<td>N.D.S</td>
</tr>
<tr>
<td>$N_{1,4}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{1,1}$</td>
<td>N.D.S</td>
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<tr>
<td>$N_{3,2}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,1}$</td>
<td>S</td>
</tr>
<tr>
<td>$N_{3,3}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,1}$</td>
<td>S</td>
</tr>
<tr>
<td>$B_{1,3}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,2}$</td>
<td>N.D.S</td>
</tr>
<tr>
<td>$B_{1,4}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,2}$</td>
<td>D</td>
</tr>
<tr>
<td>$C_{1,1}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,1}$</td>
<td>N.D.S</td>
</tr>
<tr>
<td>$C_{1,2}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,2}$</td>
<td>D</td>
</tr>
<tr>
<td>$D_{1,1}$ (1)</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,1}$</td>
<td>N.D.S</td>
</tr>
<tr>
<td>$D_{1,2}$</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,2}$</td>
<td>N</td>
</tr>
<tr>
<td>$E_{1,3}$ (2)</td>
<td>$e_1 \cdot e_2 = e_3$</td>
<td>$\mathfrak{g}_{3,3}$</td>
<td>N.D.S</td>
</tr>
</tbody>
</table>

Table 1: Left-symmetric algebras.

Here, the letter $N$ that the left-symmetric algebra $A_3$ is Novikov, the letter $D$ means that $A_3$ is derivation and the letter $S$ means that $A_3$ satisfying $\{x, y, z\} = 0$ for all $x, y, z \in A_3$.

**Remark 1:** We note that left-symmetric algebras satisfying the identity $(x \cdot y) \cdot z = (y \cdot x) \cdot z$ for all $x, y, z \in A$ (or equivalently, the
By using the exponential maps, Theorem 12 can now be stated, in terms of simply transitive actions of subgroups of the affine group $\text{Aff}(\mathbb{R}^3) = GL(\mathbb{R}^3) \mathbb{R}^3$, as follows.

To state it, define the continuous functions $f, g, h, k$ and $\phi$ by

$$f(x) = \begin{cases} \frac{e^{r} - 1}{x}, & x \neq 0, \\ 1 & x = 0 \end{cases}, \quad g(x) = \begin{cases} \frac{e^{r} - x - 1}{x}, & x \neq 0, \\ \frac{1}{2} & x = 0 \end{cases},$$

$$h(x) = \frac{\cos x - 1}{x^2}, \quad k(x) = \frac{\sin x - x}{x},$$

$$\phi(x) = \sum_{n=1}^{\infty} \frac{nx^n}{(n+1)!}.$$

Theorem 13: Suppose that the Lie group $G$ of the non-unimodular Lie algebra $\mathcal{G}$ of dimension 3 acts simply transitively by affine transformations on $\mathbb{R}^3$. Then, as a subgroup of $\text{Aff}(\mathbb{R}^3)$, $G$ is conjugate to one of the following subgroups:

$$G_{x+a} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad a, b, c \in \mathbb{R},$$

$$G_{x+b} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad b, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x+c} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad c, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad e, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad g, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad h, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad k, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad \phi, f(a), \quad a, b, c \in \mathbb{R},$$

$$G_{x} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad \psi, f(a), \quad a, b, c \in \mathbb{R}.$$

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