Journal of Generalized Lie Theory and Applications Vol. 2 (2008), No. 3, 112–116

Connection on module over a graded q-differential algebra [1]

Viktor ABRAMOV and Olga LIIVAPUU

Institute of Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia

E-mails: E-mail: viktor.abramov@ut.ee, olgai@ut.ee

Abstract

We study a concept of a q-connection on a left module, where q is a primitive Nth root of unity. This concept is based on a notion of a graded q-differential algebra whose differential d satisfies $d^N = 0$. We propose a notion of a graded q-differential algebra with involution and making use of this notion we introduce and study a concept of a q-connection consistent with a Hermitian structure of a left module. Assuming module to be a finitely generated free module we define the components of q-connection and show that these components with respect to different basises are related by gauge transformation. We also derive the relation for components of a q-connection consistent with Hermitian structure of a module.

2000 MSC: 81R60, 53C05, 81R50, 16E45, 16W50, 17B37

1 Introduction

Let q be a primitive Nth root of unity, where $N \geq 2$. A concept of a q-connection and $\mathbb{Z}_N$-graded q-connection on a left module $\mathcal{F}$ [1, 2, 3, 4] is based on a notion of a graded q-differential algebra $\mathcal{A}$ [5, 6, 7]. The differential d of a graded q-differential algebra $\mathcal{A}$ satisfies the graded q-Leibniz rule and $d^N = 0$. If $N = 2, q = -1$ then the graded q-Leibniz rule takes the form of graded Leibniz rule and $d^2 = 0$. Hence a graded q-differential algebra can be viewed as a generalization of a graded differential algebra. If $\mathcal{E}$ is a left module over the subalgebra $\mathcal{A}^0 = \mathfrak{A} \subset \mathcal{A}$ of elements of grading zero and $\mathcal{F} = \mathcal{A} \otimes_{\mathfrak{A}} \mathcal{E}$ then a q-connection on the left $\mathcal{A}$-module $\mathcal{F}$ is a linear operator D of grading one satisfying the graded q-Leibniz rule. It can be shown that the Nth power of a q-connection D is the endomorphism of the left $\mathcal{A}$-module $\mathcal{F}$ and this allows to define the curvature of q-connection as $F = D^N$. It can be proved that the curvature F of q-connection satisfies the Bianchi identity. In this paper we continue to study the concept of a q-connection started in [2, 3, 4] and propose a notion of a q-connection on the left module $\mathcal{F}$ consistent with a Hermitian structure of the module $\mathcal{F}$. A Hermitian structure on the module $\mathcal{F}$ requires an involution on a graded q-differential algebra $\mathcal{A}$, and we introduce a notion of a graded q-differential algebra with involution proving that the differential d is consistent with an involution. Assuming the left $\mathfrak{A}$-module $\mathcal{F}^0 \subset \mathcal{F}$ to be a finitely generated free left module we define the components of a q-connection with respect to a basis for the module and show that the components of a q-connection with respect to different basises are related by gauge transformation. Assuming that D is a q-connection consistent with a Hermitian structure of $\mathcal{F}$ we derive the relation for the components of D. Finally we find the expressions for components of the curvature in terms of the components of a q-connection.

[1] Presented at the 3rd Baltic-Nordic Workshop “Algebra, Geometry, and Mathematical Physics“, Göteborg, Sweden, October 11–13, 2007.

2 Modules over a graded q-differential algebra

The aim of this section is to remind a concept of a graded q-differential algebra, where q is a primitive Nth root of unity ($N \geq 2$). This algebra is a basic component in our algebraic approach to q-generalization of connection, and it may be viewed as an analog of algebra of differential forms with exterior differential satisfying $d^N = 0$. It should be noted that within the framework of this analogy the subalgebra of elements of grading zero plays a role of an algebra of functions on a base manifold. In order to have an algebraic model of differential forms with values in a vector bundle we introduce a left module over the subalgebra of elements of grading zero of a graded q-differential algebra. Assuming that this module is a finitely generated free module we describe an algebraic analog of transition from one local trivialization of a vector bundle to another.

Let q be a primitive Nth root of unity and $\mathcal{A} = \oplus_i \mathcal{A}^i$ be an associative unital graded algebra over the complex numbers. Let us denote the identity element of this algebra by e and the grading of a homogeneous element ω of $\mathcal{A}$ by $|\omega|$. An algebra $\mathcal{A}$ is said to be a graded q-differential algebra if it is endowed with a linear mapping d of degree one, i.e $d : \mathcal{A}^i \to \mathcal{A}^{i+1}$, satisfying the graded q-Leibniz rule

$$d(\omega\,\omega') = d(\omega)\,\omega' + q^{|\omega|}\,\omega\,d(\omega')$$

where $\omega, \omega' \in \mathcal{A}$, and the N-nilpotency condition $d^N = 0$. It is easy to see that the subspace $\mathcal{A}^0$ of elements of grading zero is the subalgebra of an algebra $\mathcal{A}$. We will denote this subalgebra by $\mathfrak{A}$, i.e. $\mathfrak{A} = \mathcal{A}^0$. Obviously $\mathfrak{A}$ is the associative unital algebra over $\mathbb{C}$ with the identity element e. Given an associative unital algebra $\mathfrak{A}$ we call a graded q-differential algebra $\mathcal{A}$ an N-differential calculus over an algebra $\mathfrak{A}$, if $\mathcal{A}^0 = \mathfrak{A}$. Let us mention that taking $N = 2, q = -1$ in the definition of a graded q-differential algebra we get a graded differential algebra (with differential d satisfying $d^2 = 0$). Thus a graded q-differential algebra can be considered as a generalization of a graded differential algebra for any integer $N > 2$. It follows from the graded structure of an algebra $\mathcal{A}$ that each subspace $\mathcal{A}^i \subset \mathcal{A}$ of homogeneous elements of grading i can be considered as the bimodule over the algebra $\mathfrak{A}$. Thus we have the following sequence of bimodules

$$\ldots \to^d \mathcal{A}^{i-1} \to^d \mathcal{A}^i \to^d \mathcal{A}^{i+1} \to^d \ldots$$

The part $d : \mathfrak{A} = \mathcal{A}^0 \to \mathcal{A}^1$ of this sequence is the first order differential calculus over the algebra $\mathfrak{A}$.

We define a graded q-differential algebra with involution as a graded q-differential algebra $\mathcal{A}$ which is equipped with a mapping $* : \mathcal{A}^i \to \mathcal{A}^i$ of grading zero satisfying

$$(\alpha\,\omega + \omega')^* = \bar{\alpha}\,\omega^* + \omega'^*, \quad (\omega\,\omega')^* = \omega'^*\,\omega^*, \quad (d\omega)^* = d(\omega^*)$$

where $\alpha \in \mathbb{C}, \omega, \omega' \in \mathcal{A}$. It is easy to show that the involution is consistent with the graded q-Leibniz rule. We have

$$(d(\omega\,\omega'))^* = (d\omega\,\omega')^* + \bar{q}^{|\omega|}(\omega\,d\omega')^* = q^{(|\omega|+1)|\omega'|}\omega'^* d\omega^* + \bar{q}^{|\omega|} q^{|\omega|(1+|\omega'|)} d\omega'^*\,\omega^*$$

On the other hand,

$$d(\omega\,\omega')^* = d(q^{|\omega||\omega'|}\omega'^*\omega^*) = q^{|\omega||\omega'|} d\omega'^*\omega^* + q^{(|\omega|+1)|\omega'|}\omega'^* d\omega^*$$

From the above formulae and

$$\bar{q}^{|\omega|} q^{|\omega|(1+|\omega'|)} = q^{-|\omega|+|\omega|+|\omega||\omega'|} = q^{|\omega||\omega'|}$$

it follows that the involution $*$ is consistent with the graded q-Leibniz rule. Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. Considering a graded q-differential algebra $\mathcal{A}$ as the $(\mathfrak{A}, \mathfrak{A})$-bimodule we take the tensor product $\mathcal{A} \otimes_{\mathfrak{A}} \mathcal{E}$ of modules which clearly has the structure of left $\mathfrak{A}$-module. Let us denote this left $\mathfrak{A}$-module by $\mathcal{F}$, i.e. $\mathcal{F} = \mathcal{A} \otimes_{\mathfrak{A}} \mathcal{E}$. Obviously $\mathcal{F}$ inherits the graded structure of $\mathcal{A}$. Indeed for every i we have the left $\mathfrak{A}$-submodule $\mathcal{F}^i = \mathcal{A}^i \otimes_{\mathfrak{A}} \mathcal{E}$ of the left $\mathfrak{A}$-module $\mathcal{F}$. It is easy to see that the left $\mathfrak{A}$-module $\mathcal{F}$ is the direct sum of its submodules $\mathcal{F}^i$, i. e. $\mathcal{F} = \oplus_i \mathcal{F}^i$. It is worth noting that the left $\mathfrak{A}$-submodule $\mathcal{F}^0$ of elements of grading zero is isomorphic to a left $\mathfrak{A}$-module $\mathcal{E}$, i. e. $\mathcal{F}^0 \cong \mathcal{E}$, where the isomorphism $\varphi : \mathcal{E} \to \mathcal{F}^0$ can be defined by $\varphi(\xi) = e \otimes \xi$. The left $\mathfrak{A}$-module $\mathcal{F}$ can be also considered as the left $\mathcal{A}$-module and in the next section we will use this structure to describe a concept of q-connection. Let us mention that multiplication by elements of $\mathcal{A}^i$, where $i \neq 0$, does not preserve the graded structure of the module $\mathcal{F}$.

Since $\mathcal{A}$ is a graded algebra the tensor product $\mathcal{F} = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{E}$ of vector spaces is the graded vector space $\mathcal{F} = \oplus_i \mathcal{F}^i$ over $\mathbb{C}$, where $\mathcal{F}^i = \mathcal{A}^i \otimes_{\mathbb{C}} \mathcal{E}$. Hence we have the graded algebra of linear operators of the graded vector space $\mathcal{F}$, which we denote by $\mathfrak{L}(\mathcal{F}) = \oplus_i \mathfrak{L}^i(\mathcal{F})$, where $\mathfrak{L}^i(\mathcal{F})$ is the subspace of homogeneous linear operators of grading i. If $A : \mathcal{F} \to \mathcal{F}$ is a homogeneous linear operator then we can extend it to the linear operator $L_A : \mathfrak{L}(\mathcal{F}) \to \mathfrak{L}(\mathcal{F})$ on the whole graded algebra of linear operators $\mathfrak{L}(\mathcal{F})$ by means of the graded q-commutator as follows

$$L_A(B) = [A, B]_q = A \cdot B - q^{|A||B|} B \cdot A$$

where B is a homogeneous linear operator and $A \cdot B$ is the product of two linear operators.

In order to have an algebraic analog of the local structure of a vector bundle in this approach we assume $\mathcal{E}$ to be a finitely generated free left $\mathfrak{A}$-module. Let $\mathfrak{e} = \{\mathfrak{e}_\mu\}_{\mu=1}^r$ be a basis for a left module $\mathcal{E}$. This basis induces the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, where $\mathfrak{f}_\mu = e \otimes \mathfrak{e}_\mu$, for the left $\mathfrak{A}$-module $\mathcal{F}^0$. Taking into account that $\mathcal{F}^0 \subset \mathcal{F}$ and $\mathcal{F}$ is the left $\mathcal{A}$-module we can multiply the elements of the basis $\mathfrak{f}$ by elements of $\mathcal{A}$. It is easy to see that if $\omega \in \mathcal{A}^i$ then for any μ we have $\omega\, \mathfrak{f}_\mu \in \mathcal{F}^i$. Consequently we can express any element of $\mathcal{F}^i$ as a linear combination of $\mathfrak{f}_\mu$ with coefficients from $\mathcal{A}^i$. Indeed let $\omega \otimes \xi$ be an element of $\mathcal{F}^i = \mathcal{A}^i \otimes_{\mathfrak{A}} \mathcal{E}$. Then

$$\omega \otimes \xi = (\omega\, e) \otimes (\xi^\mu \mathfrak{e}_\mu) = (\omega\, e\, \xi^\mu) \otimes \mathfrak{e}_\mu = (\omega \xi^\mu\, e) \otimes \mathfrak{e}_\mu = \omega \xi^\mu\, (e \otimes \mathfrak{e}_\mu) = \omega^\mu \mathfrak{f}_\mu$$

where $\omega^\mu = \omega \xi^\mu \in \mathcal{A}^i$.

Denote by $\mathfrak{M}_r(\mathcal{A})$ the vector space of $r \times r$-matrices whose entries are the elements of an algebra $\mathcal{A}$. This vector space is a graded vector space with graded structure induced by the graded structure of a graded q-differential algebra $\mathcal{A}$. Hence $\mathfrak{M}_r(\mathcal{A}) = \oplus_i \mathfrak{M}_r^i(\mathcal{A})$, where $\mathfrak{M}_r^i(\mathcal{A})$ is the subspace of homogeneous matrices of grading i, i.e. if $\Omega = (\omega^\mu_\nu) \in \mathfrak{M}_r^i(\mathcal{A})$ then $\omega^\mu_\nu \in \mathcal{A}^i$. The vector space $\mathfrak{M}_r(\mathcal{A})$ of $r \times r$-matrices becomes the associative unital graded algebra if we define the product of two matrices $\Omega = (\omega^\mu_\nu), \Omega' = (\omega'^\mu_\nu)$ by $\Omega \cdot \Omega' = (\omega^\mu_\sigma\, \omega'^\sigma_\nu)$. In the next section we shall use the graded q-commutator of homogeneous matrices which is defined by

$$[\Omega, \Omega']_q = \Omega \cdot \Omega' - q^{|\Omega||\Omega'|} \Omega' \cdot \Omega$$

We extend the differential d of a graded q-differential algebra $\mathcal{A}$ to the algebra $\mathfrak{M}_r(\mathcal{A})$ as usual: $d\Omega = d(\omega^\mu_\nu) = (d\omega^\mu_\nu)$.

Let $\mathfrak{f}' = \{\mathfrak{f}'_\mu\}_{\mu=1}^r$ be another basis for the left $\mathfrak{A}$-module $\mathcal{F}^0$ with the same number of elements (this will always be the case if $\mathfrak{A}$ is a division algebra or if $\mathfrak{A}$ is commutative). Then $\mathfrak{f}'_\nu = g^\mu_\nu\, \mathfrak{f}_\mu$, where $G = (g^\mu_\nu) \in \mathfrak{M}_r^0(\mathcal{A})$, $g^\mu_\nu \in \mathfrak{A}$, is the transition matrix from the basis $\mathfrak{f}$ to the basis $\mathfrak{f}'$. It is well known [8] that in the case of finitely generated free module transition matrix is an invertible matrix, and we denote the inverse matrix of G by $G^{-1} = (\tilde{g}^\mu_\nu)$.

In order to define a Hermitian structure on the left $\mathcal{A}$-module $\mathcal{F}$ we assume $\mathcal{A}$ to be a graded q-differential algebra with involution $*$. We will call the left module $\mathcal{F}$ a Hermitian (left) module if $\mathcal{F}^0$ is endowed with a bilinear form $h : \mathcal{F}^0 \times \mathcal{F}^0 \to \mathfrak{A}$ which satisfies $h(\omega\xi, \omega'\xi') = \omega\omega'^* h(\xi, \xi')$, where $\omega, \omega' \in \mathfrak{A}$ and $\xi, \xi' \in \mathcal{F}^0$. It is easy to extend a Hermitian form h to the whole left $\mathcal{A}$-module $\mathcal{F}$ if we put

$$h(\omega \otimes \xi, \omega' \otimes \xi') = \omega\, \omega'^*\, h(\xi, \xi')$$

where $\omega \in \mathcal{A}^i, \xi \in \mathcal{F}^0, \omega \otimes \xi \in \mathcal{F}^i$ and $\omega' \in \mathcal{A}^j, \xi' \in \mathcal{F}^0, \omega' \otimes \xi' \in \mathcal{F}^j$. Consequently it holds $h : \mathcal{F}^i \times \mathcal{F}^j \to \mathcal{A}^{i+j}$. The matrix of this Hermitian form with respect to a basis $\mathfrak{f}$ is denoted by $H = (h_{\mu\nu}) = (h(\mathfrak{f}_\mu, \mathfrak{f}_\nu)) \in \mathfrak{M}^0_r(\mathcal{A})$.

3 q-connection on module $\mathcal{F}$

In this section we describe a concept of q-connection [2, 3, 4] on the left $\mathcal{A}$-module $\mathcal{F}$, curvature of q-connection and Bianchi identity. Assuming that graded q-differential algebra $\mathcal{A}$ is an algebra with involution and $\mathcal{F}$ is the Hermitian module over this algebra we define a q-connection consistent with a Hermitian structure of $\mathcal{F}$. Then assuming the submodule $\mathcal{F}^0 \subset \mathcal{F}$ to be a finitely generated free module we introduce the matrices of q-connection and its curvature.

A q-connection on the left $\mathcal{A}$-module $\mathcal{F}$ is a linear operator $D : \mathcal{F} \to \mathcal{F}$ of degree one satisfying the condition

$$D(\omega\, \xi) = d\omega\, \xi + q^{|\omega|}\, \omega\, D\xi \tag{3.1}$$

where $\omega \in \mathcal{A}, \xi \in \mathcal{F}$, and d is the differential of a graded q-differential algebra $\mathcal{A}$. If the left $\mathcal{A}$-module $\mathcal{F}$ is the Hermitian left module with Hermitian form h a q-connection D on $\mathcal{F}$ is said to be consistent with a Hermitian structure of $\mathcal{F}$ if it satisfies

$$dh(\xi, \xi') = h(D\xi, \xi') + h(\xi, D\xi')$$

where $\xi, \xi' \in \mathcal{F}^0$.

It can be shown that the N-th power of any q-connection D is the endomorphism of degree N of the left $\mathcal{A}$-module $\mathcal{F}$. This allows us to define the curvature of a q-connection D as the endomorphism $F = D^N$ of degree N of the left $\mathcal{A}$-module $\mathcal{F}$. The curvature F of any q-connection D on $\mathcal{F}$ satisfies the Bianchi identity $L_D(F) = 0$ [3], where $L_D : \mathfrak{L}(\mathcal{F}) \to \mathfrak{L}(\mathcal{F})$ is the extention of D to the algebra of linear operators of $\mathcal{F}$.

Let $\mathcal{F}^0$ be a finitely generated free module with a basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, and $\xi = \xi^\mu \mathfrak{f}_\mu \in \mathcal{F}^0$, where $\xi^\mu \in \mathfrak{A}$. Obviously $D\xi \in \mathcal{F}^1$. The coefficients of a q-connection D with respect to a basis $\mathfrak{f}$ are defined by $D\mathfrak{f}_\nu = \theta^\mu_\nu\, \mathfrak{f}_\mu$. The matrix $\Theta = (\theta^\mu_\nu) \in \mathfrak{M}^1_r(\mathcal{A})$ is called the matrix of q-connection D with respect to $\mathfrak{f}$. Using (3.1) we obtain

$$D\xi = D(\xi^\mu \mathfrak{f}_\mu) = d\xi^\mu\, \mathfrak{f}_\mu + \xi^\mu\, D\mathfrak{f}_\mu = (d\xi^\mu + \xi^\nu \theta^\mu_\nu)\, \mathfrak{f}_\mu = (\nabla\xi)^\mu \mathfrak{f}_\mu \tag{3.2}$$

where $(\nabla\xi)^\mu = d\xi^\mu + \xi^\nu \theta^\mu_\nu$. Let $\mathfrak{f}' = \{\mathfrak{f}'_\mu\}_{\mu=1}^r$ be another basis for the left $\mathfrak{A}$-module $\mathcal{F}^0$, and $\mathfrak{f}'_\mu = g^\nu_\mu \mathfrak{f}_\nu$, where $G = (g^\nu_\mu) \in \mathfrak{M}^0_r(\mathcal{A})$ is a transition matrix. If we denote by θ'^μ_ν the coefficients of D with respect to basis $\mathfrak{f}'$ and $\tilde{g}^\mu_\nu$ are the entries of the inverse matrix G^{-1} then $\theta'^\mu_\nu = dg^\sigma_\nu \tilde{g}^\mu_\sigma + g^\sigma_\nu \theta^\tau_\sigma \tilde{g}^\mu_\tau$, and this clearly shows that the components of D with respect to different basises of module $\mathcal{F}^0$ are related by the gauge transformation. Let $\mathcal{A}$ be a graded q-differential algebra with involution $* : \mathcal{A} \to \mathcal{A}$, $\mathcal{F}$ be a Hermitian module with a Hermitian form h, and D be a q-connection on $\mathcal{F}$ consistent with a Hermitian structure of $\mathcal{F}$. Then the components θ^μ_ν of D obey the relation

$$\theta^\sigma_\mu\, h_{\sigma\nu} + \theta^{*\tau}_\nu\, h_{\mu\tau} = dh_{\mu\nu}$$

Our next aim is to express the components of the curvature F of a q-connection D in terms of the coefficients of a q-connection D. We define the components of curvature F with respect to a basis $\mathfrak{f}$ by $F(\mathfrak{f}_\mu) = \psi^\nu_\mu\, \mathfrak{f}_\nu$ and denote the matrix of curvature by $\Psi = (\psi^\mu_\nu)$. Straightforward computation gives for different $k = 1, 2, \ldots, N$ the polynomial

$$D^k\xi = \sum_{l=0}^{k} C_q(k, l)\, d^{k-l}\xi^\mu\, \psi^{l,\nu}_\mu\, \mathfrak{f}_\nu$$

where $C_q(k,l)$ are q-binomial coefficients, $\psi^\nu_\mu = \psi^{N,\nu}_\mu$ and $\psi^{0,\nu}_\mu = \delta^\nu_\mu\, e$. From this polynomial we get the recursion formula for the components of curvature

$$\psi^{l,\nu}_\mu = d\psi^{l-1,\nu}_\mu + q^{l-1}\,\psi^{l-1,\sigma}_\mu\,\theta^\nu_\sigma$$

This recursion formula gives the following expressions for the first three values of k:

$$\psi^{1,\nu}_\mu = \theta^\nu_\mu,\;\; \psi^{2,\nu}_\mu = d\theta^\nu_\mu + q\,\theta^\sigma_\mu\theta^\nu_\sigma,\;\; \psi^{3,\nu}_\mu = d^2\theta^\nu_\mu + (q+q^2)\,d\theta^\sigma_\mu\theta^\nu_\sigma + q^2\,\theta^\sigma_\mu\,d\theta^\nu_\sigma + q^3\,\theta^\tau_\mu\theta^\sigma_\tau\theta^\nu_\sigma \qquad (3.3)$$

Let us consider the expressions for curvature in two cases when $N = 2$ and $N = 3$. If $N = 2, q = -1$ then a graded q-differential algebra $\mathcal{A}$ is a differential superalgebra ($\mathbb{Z}_2$-graded), and we have for the components of curvature $\psi^\nu_\mu = \psi^{2,\nu}_\mu = d\theta^\nu_\mu - \theta^\sigma_\mu\theta^\nu_\sigma$. Assuming $\mathcal{A}$ to be a super-commutative algebra we can put the expression for components of curvature into the form $\psi^\nu_\mu = d\theta^\nu_\mu + \theta^\nu_\sigma\theta^\sigma_\mu$ or by means of matrices $\Psi = d\Theta + \Theta\cdot\Theta$ in which we recognize the classical expression for the curvature.

If $N = 3$ then $q = \exp(\frac{2\pi i}{3})$ is the cubic root of unity satisfying the relations $q^3 = 1$, $1+q+q^2 = 0$. This is the first non-classical case of a q-connection, and we have for components

$$\begin{aligned}\psi^\nu_\mu &= d^2\theta^\nu_\mu + (q+q^2)\,d\theta^\sigma_\mu\theta^\nu_\sigma + q^2\,\theta^\sigma_\mu\,d\theta^\nu_\sigma + q^3\,\theta^\tau_\mu\theta^\sigma_\tau\theta^\nu_\sigma = d^2\theta^\nu_\mu - d\theta^\sigma_\mu\theta^\nu_\sigma + q^2\,\theta^\sigma_\mu\,d\theta^\nu_\sigma + \theta^\tau_\mu\theta^\sigma_\tau\theta^\nu_\sigma\\ &= d^2\theta^\nu_\mu - (d\theta^\sigma_\mu\theta^\nu_\sigma - q^2\,\theta^\sigma_\mu\,d\theta^\nu_\sigma) + \theta^\tau_\mu\theta^\sigma_\tau\theta^\nu_\sigma\end{aligned}$$

Finally we derive the form of Bianchi identity in terms of the components of a q-connection and its curvature. The curvature F of a q-connection satisfies the Bianchi identity $L_D(F) = [D,F]_q = 0$. If $\theta^\mu_\nu, \psi^\mu_\nu$ are the components of a q-connection D and its curvature F with respect to a basis $\mathfrak{f}$ for the module $\mathcal{F}$ then the Bianchi identity takes on the form

$$d\psi^\mu_\nu = \theta^\sigma_\mu\,\psi^\nu_\sigma - \psi^\sigma_\mu\,\theta^\nu_\sigma$$

Acknowledgement

The authors gratefully acknowledge the financial support of their research by the Estonian Science Foundation under the grant ETF 7427.

References

[1] M. M. Kapranov. On the q-analog of homological algebra. Preprint `arXiv:math.QA/9611005`.

[2] V. Abramov. q-connection and its matrix. In *"Generalized Lie Theory in Mathematics, Physics and Beyond"*. V. Abramov, E. Paal, S. Silvestrov, and A. Stolin, Eds. Springer-Verlag, Berlin–New York, 2008 (to be published).

[3] V. Abramov. Generalization of superconnection in non-commutative geometry. Proc. Estonian Acad. Sci. Phys. Math. **55** (2006), 3–15.

[4] V. Abramov and O. Liivapuu. Geometric approach to BRST-symmetry and $\mathbb{Z}_N$-generalization of superconnection. J. Nonlinear Math. Phys. **13** Supplement (2006), 9–20.

[5] M. Dubois-Violette and R. Kerner. Universal q-differential calculus and q-analog of homological algebra. Acta Math. Univ. Comenian, **65** (1996), 175-188.

[6] M. Dubois-Violette. Lectures on graded differential algebras and noncommutative geometry. In *"Noncommutative differential geometry and its applications to physics. Proceedings of the workshop held in Shonan, May 31–June 4, 1999"*. Y. Maeda, H. Moriyoshi, H. Omori, D. Sternheimer, T. Tate, and S. Watamura, Eds. Kluwer Academic Publishers, Math. Phys. Stud. **23** (2001), 245-306.

[7] M. Dubois-Violette. Lectures on differentials, generalized differentials and on some examples related to theoretical physics. Preprint `arXiv:math.QA/0005256`.

[8] R. Godement. *Algebra.* Hermann, Paris; Kershow Publishing Company Ltd, London, 1969.

Received December 20, 2007
Revised March 13, 2008