Coupled Fixed Points of $\alpha$-Ψ-Contractive Type Multi Functions

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Abstract

Recently, Samet, Vetro and Vetro introduced $\alpha$-$\psi$-contractive mappings and gave some results on fixed points of the mappings. In fact, their technique generalized some ordered fixed point results. Also they have proved some results on coupled fixed points of $\alpha$-$\psi$-contractive mappings. In 1974 Ciric introduced quasicontractive mappings and obtained an important generalization of Banach’s contraction principle. Recently Mohammadi, Rezapour and Shahzad have proved some fixed point results on $\alpha$-$\psi$-contractive and $\alpha$-$\psi$-quasicontractive multifunction’s. By using the main idea of, we give some new results for coupled fixed points of $\alpha$-$\psi$-contractive multifunction.

Keywords: Coupled fixed point; $\alpha$-$\psi$-Contractive; Multifunction

Introduction

Denote by $\Psi$ the family of non-decreasing functions $\psi: [0, +\infty) \to [0, +\infty)$ such that $\sum_{n = 1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$. It is well known that $\psi(t) < t$ for all $t > 0$.

Definition 1.1: Let $(X, d)$ be a metric space and $\alpha: X \times X \to [0, \infty)$ be a map. We say that $X$ satisfies condition $\big(C_{\alpha}\big)$, if for any sequence $\{x_n\}$ in $X$, that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$, then $\alpha(x_n, x) \geq 1$ for all $n$.

Definition 1.2: Let $X$ be an arbitrary space and $\alpha: X^2 \times X^2 \to [0, +\infty)$ a map. A mapping $F: X^2 \to X$ is said to be $\alpha$-admissible whenever $\alpha((x, y), (u, v)) \geq 1$ implies

$$
\alpha(F(x, y), F(u, v)) \geq 1.
$$

Definition 1.3: Let $(X, d)$ be a metric space and $\alpha: X^2 \times X^2 \to [0, +\infty)$ be a complete metric space, the Theorem 1.2 is true if $F$ is continuous or $F$ is orbitally continuous instead of $F$ being continuous. Obviously theorem 1.2 is true if $F$ is continuous. Hence theorem 1.2 is true if $F$ is continuous or $F$ is orbitally continuous instead of $F$ being continuous. Hence theorem 1.2 is true if $F$ is continuous. Hence theorem 1.2 is true if $F$ is continuous. Hence theorem 1.2 is true if $F$ is continuous.

Theorem 1.1: Let $(X, d)$ be a complete metric space, $\alpha: X^2 \times X^2 \to [0, +\infty)$ a function, $\psi \in \Psi$ and $F: X^2 \to X$ an $\alpha$-admissible mapping such that

$$
\alpha((x, y), (u, v)) \geq 1 \quad \Rightarrow \quad \alpha(F(x, y), F(u, v)) \geq 1,
$$

for all $(x, y), (u, v) \in X^2$. Assume that the following assertions hold.

(i) There exists $(x_0, y_0) \in X^2$ such that $F(x_0, y_0) = (x_0, y_0)$.

(ii) Either $F$ is continuous or $X$ satisfies condition $\big(C_{\alpha}\big)$. Then $F$ has a coupled fixed point in $X^2$.

Let $(X, d)$ be a metric space. Define the metric $\delta$ on $X^2$ by $\delta((x, y), (u, v)) = d(x, u) + d(y, v)$ for all $(x, y), (u, v) \in X^2$. Also if $F: X^2 \to X$ then put [2]

$$
m((x, y), (u, v)) = \max\{\delta((x, y), (u, v)), \delta((y, x), (v, u)), \delta((u, v), (F(x, y), F(y, x)))\},$$

for all $(x, y), (u, v) \in X^2$. It is easy to see that $m((x, y), (u, v)) = m((v, u), (y, x))$. Recently Rezapour and H. Asl have extended theorem 1.1 to quasi-contractions as follow [3].

Theorem 1.2: Let $(X, d)$ be a complete metric space, $\alpha: X^2 \times X^2 \to [0, +\infty)$ a function, $\psi \in \Psi$ and $F: X^2 \to X$ an $\alpha$-admissible mapping such that

$$
\alpha((x, y), (u, v)) \geq 1 \quad \Rightarrow \quad \alpha(F(x, y), F(u, v)) \geq 1,
$$

for all $(x, y), (u, v) \in X^2$. Assume that the following assertions hold.

(i) There exists $(x_0, y_0) \in X^2$ such that $a((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1,$

$$
a((F(x_0, y_0), F(y_0, x_0)), (F(y_0, x_0), F(x_0, y_0))) \geq 1.
$$

(ii) Either $F$ is continuous or $\psi$ is right upper semi continuous and $X$ satisfies condition $\big(C_{\alpha}\big)$.

Then $F$ has a coupled fixed point in $X^2$.

Definition 1.4: Let $(X, d)$ be a metric space and $\alpha: X^2 \times X^2 \to [0, +\infty)$ be a metric space. Define the metric $\delta$ on $X^2$ by $\delta((x, y), (u, v)) = d(x, u) + d(y, v)$ for all $(x, y), (u, v) \in X^2$. Also if $F: X^2 \to X$ then put [2].

$$
m((x, y), (u, v)) = \max\{\delta((x, y), (u, v)), \delta((y, x), (v, u)), \delta((u, v), (F(x, y), F(y, x)))\},$$

for all $(x, y), (u, v) \in X^2$. It is easy to see that $m((x, y), (u, v)) = m((v, u), (y, x))$. Recently Rezapour and H. Asl have extended theorem 1.1 to quasi-contractions as follow [3].

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Example 1.6:

Let $M_1=\left\{m_{ij}: m=0, 1, 3, 9, \ldots \right\}$ and $n=3k+1 (k \geq 0)$, $M_2=\left\{m_{ij}: m=1, 3, 9, 27, \ldots \right\}$ and $n=3k+2 (k \geq 0)$. Set $M=M_1 \cup M_2$, define $F: M^2 \to M$ by

$$F(x, y) = \begin{cases} \frac{3x+13}{13} & \text{if } x < \frac{3x}{13} \\ \frac{3x}{13} & \text{otherwise} \end{cases}$$

for all $(x, y), (u, v) \in M^2$. Then we have

$$\alpha((x, y), (u, v)) = \begin{cases} 1 & \text{if } x < \frac{3x}{13} \\ 0 & \text{otherwise} \end{cases}$$

for all $(x, y), (u, v) \in M^2$. Hence

$$d(F(x, y), F(u, v)) = \begin{cases} \frac{12}{13} & \text{if } x < \frac{3x}{13} \\ \frac{3x}{13} & \text{otherwise} \end{cases} \leq \frac{12}{13} \alpha((x, y), (u, v)).$$

To show that $F$ is $\alpha$-admissible, assume $a((x, y), (u, v)) \geq 1$. Then $(x, y), (u, v) \in M_1 \cup M_2$, $x > y$, $u > v$. Hence either $x > y$ and $F(y, x) = 0$ or $x = y$ and $F(x, y) = F(y, x)$. However $F(x, y) \geq F(x, y)$. Similarly $F(u, v) \geq F(v, u)$. On the other hand $F(x, y), F(y, x), F(u, v), F(v, u) \in M_1$. Hence

$$a((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1.$$
αψ ≤ +∞) a function, ψ ∈ Ψ a strictly increasing map and F: X 2 → CB(X) an α-admissible multifunction such that [7]

\[ α((x, y), (u, v))H((F(x, y), F(u, v)) \leq \frac{1}{2}ψ(d(x, u) + d(y, v)), \]

for all (x, y), (u, v) ∈ X 2. Assume that the following assertions hold.

(i) There exists (x, y) ∈ X 2 and (x, y) ∈ F(x, y) × F(y, x) such that

\[ α((x, y), (x, y)) = 1, \quad α((x, y), (x, y)) = 1. \]

(ii) Either F is continuous or X satisfies condition (C). Then F has a coupled fixed point in X 2.

Proof Define β: X 2 × X 2 → [0, +∞) by

\[ β((ξ, ξ), (η, η)) = \min{α((ξ, ξ), (η, η)), α((η, η), (ξ, ξ))} \]

(3)

for all (ξ, ξ), (η, η) ∈ X 2. Also suppose T: X 2 → C B(X 2) is defined by T(x, y) = F(x, y) × F(y, x). Obviously the metric space (X 2, δ) is complete. Now since F is α-admissible, it is easy to see that T is β-admissible. Also by assumption

\[ α((x, y), (u, v))H((F(x, y), F(u, v))) \leq \frac{1}{2}ψ(d(x, u) + d(y, v)), \]

(4)

\[ α((u, v), (x, y))H((F(u, v), F(x, y))) \leq \frac{1}{2}ψ(d(y, v) + d(u, x)). \]

for all (x, y), (u, v) ∈ X 2. By adding the above two relations we obtain

\[ β((x, y), (u, v))[H((F(x, y), F(u, v)) + H(F(u, v), F(y, x))] \leq ψ(δ((x, y), (u, v))). \]

(5)

Assume that H 2 is the Hausdorff metric on (X 2, δ). We should show that

\[ H_2(T(x, y), T(u, v)) ≤ H(F(x, y), F(u, v)) + H(F(u, v), F(y, x)). \]

(6)

For this we have

\[ H_2(T(x, y), T(u, v)) = \sup_{(ξ, ξ), (η, η) ∈ (X 2, δ)} δ((ξ, ξ), F(x, y)), \sup_{(ξ, ξ), (η, η) ∈ (X 2, δ)} δ((η, η), F(u, v)) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η).

(7)

Let (ξ, ξ), (η, η) ∈ (X 2, δ) × (X 2, δ). Then

\[ δ((ξ, ξ), F(x, y)) = \sup_{(ξ, ξ) ∈ (X 2, δ)} δ((ξ, ξ), F(x, y)) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η).

(8)

Similarly for (η, η) F: X 2 → X 2 we have

\[ δ((η, η), F(x, y) × F(y, x)) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η) \leq H(F(x, y), F(u, v)) + H(F(y, x), F(v, u)). \]

Hence (5) holds. By (4) and (5) we have

\[ β((x, y), (u, v))H_2(T(x, y), T(u, v)) ≤ ψ(δ((x, y), (u, v))). \]

Hence for any ξ = (ξ, ξ) ∈ (X 2, δ) we have

\[ β((ξ, ξ), (η, η)) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η) = \inf_{ξ ∈ T(x, y)} d(ξ, η) + \inf_{η ∈ T(u, v)} d(ξ, η). \]

(9)

So T: X 2 → C B(X 2) is a β-ψ-contractive multifunction. Put P x = (x, y), P y = (y, x). Then P 1 ∈ T P x P y ≥ 1. Now let F be continuous. We show that T is continuous. To see this let (x, y) be a sequence in X 2 such that δ((x, y), (x, y)) → 0. Then d(x, y) + d(y, y) → 0 and hence d(x, y) + d(y, y) → 0. Now since F is continuous, hence

\[ H(F(x, y), F(y, x)) → 0, F(y, x) → 0. \]

Therefore

\[ H_2(T(x, y), T(y, x)) ≤ H(F(x, y), (x, y)) + H(F(y, x), (y, x)) → 0. \]

This shows that T: X 2 → C B(X 2) is continuous.

Also if X has (C 2) condition, it is easy to see that X 2 has condition C 2. Now all of the conditions of Lemma 1.3 hold. Hence by the lemma there exists (x, y) ∈ X 2 such that (x, y) ∈ T((x, y)) = F(x, y) × F(y, x), hence x, y ∈ F(x, y), y ∈ F(y, x). That is (x, y) is a coupled fixed point of F in X 2. Now by using the following simple definitions we want to prove another version of theorem 2.1.

Definition 2.3: Let (X, d) be a metric space and α: X 2 → [0, +∞) is a map. A multifunction F: X 2 → C B(X) is said to be modified α-admissible whenever if (x, y) ∈ X 2, (u, v) ∈ F(x, y) × F(y, x) and α((x, y), (u, v)) ≥ 1, then α((u, v), (w, z)) ≥ 1, for all (w, z) ∈ F(u, v) × F(u, v).

Definition 2.4: Let (X, d) be a metric space and α: X 2 × X 2 → [0, +∞) we say that X satisfies condition (B*). If for any two sequences {x n} and {y n} in X, that x n → x, y n → y and α((x n, y n), (x n+1, y n+1)) ≥ 1 for all n, then we have α((x n, y n), (x, y)) ≥ 1 for all n.

Theorem 2.2: Let (X, d) be a complete metric space, α: X 2 × X 2 → [0, +∞) a function, ψ ∈ Ψ a strictly increasing map and F: X 2 → C B(X) a modified α-admissible multifunction such that

\[ α((x, y), (u, v))H((F(x, y), F(u, v))) ≤ \frac{1}{2}ψ(d(x, u) + d(y, v)). \]

For all (x, y), (u, v) ∈ X 2. Assume that the following assertions hold.

(i) There exist (x, y) ∈ X 2 and (x, y) ∈ F(x, y) × F(y, x) such that

\[ A((x, y), (x, y)) ≥ 1, \quad A((x, y), (x, y)) ≥ 1. \]

(ii) Either F is continuous or X satisfies condition (B*).

Then F has a coupled fixed point in X 2.

Proof Define φ: X 2 × π 2 → [0, +∞) by

\[ φ((ξ, ξ), (η, η)) = \min{α((ξ, ξ), (η, η)), α((η, η), (ξ, ξ))}. \]

(10)

for all (ξ, ξ), (η, η) ∈ X 2. The remain of proof is completely similar to proof of theorem 2.1.

Example 2.5: Let X be the space of real numbers with the usual metric d(x, y) = |x − y| and define F: X 2 → C B(X) by

\[ F(x, y) = \begin{cases} 0, & \text{if } x < y, \\ \infty, & \text{if } x ≥ y. \end{cases} \]

(11)

Also define α: X 2 × X 2 → [0, +∞) by
\[ \alpha((x, y), (u, v)) = \begin{cases} 1 & \text{if } d(x, y) = 0 \text{ or } x = y = u = v = 0 \\ \frac{1}{2} & \text{otherwise} \end{cases} \] (9)

For \((x, y), (u, v) \in X^2\) and \(x \geq y, u \geq v\), then
\[ H(F((x, y)), F((u, v))) = \frac{1}{3} |x - y|\]
\[ = \frac{1}{3} |u - v| \leq \frac{2}{3} (d(x, u) + d(y, v)). \]

Now put \(\psi(t) = \frac{2t}{3}\). We see that
\[ \alpha((x, y), (u, v))H(F((x, y)), F((u, v))) \leq \frac{1}{2} \psi(d(x, u) + d(y, v)). \]

Also for \((x_0, y_0) = (1, 1), (x, y) = (0, 0)\) we have
\[ F(x_0, y_0) = F(y_0, x_0) = \emptyset, (x_0, y_0) \subseteq (0, 0) \implies F(x_0, y_0) \times F(y_0, x_0), \]
\[ \alpha((x_0, y_0), (y_0, x_0)) = \alpha((0, 0), (0, 0)) = 1, \]
\[ \alpha((y_0, x_0), (y_0, x_0)) = \alpha((1, 1), (0, 0)) = 1. \]

Now let \((x, y) \in X^2\), \((u, v) \in F(x, y) \times F(y, x)\) and \(\alpha((x, y), (u, v)) \geq 1\). Then \(x \geq y, u \geq v\). Hence \(F((v, u)) = \emptyset\). Now if \((w, z) \in F(x, u) \times F(v, u)\), then \(w \geq 0\). Therefore \(\alpha((w, u), (v, z)) \geq 1\). This shows that \(F\) is modified \(\alpha\)-admissible. It is easy to see that \(F\) is continuous. Hence by theorem 2.2, \(F\) has a coupled fixed point in \(X^2\). For example \((0, 0)\) is a coupled fixed point of \(F\).

**Example 2.6:** Let \(X\) be the space of real numbers with the usual metric \(d(x, y) = |x - y|\) and define \(F : X^2 \to C(B(X))\) by \(F(x, y) = [0, 2]|x - y|\) for all \(x, y \in X\). Also define \(\alpha : X^2 \times X^2 \to [0, \infty)\) by
\[ \alpha((x, y), (u, v)) = \begin{cases} 0 & \text{if } (x, y) = (u, v) = (0, 0) \\ \frac{1}{2} & \text{otherwise} \end{cases} \]

Now put \(\psi(t) = \frac{4t}{5}\). We see that
\[ \alpha((x, y), (u, v))H(F((x, y)), F((u, v))) \leq \frac{1}{2} \psi(d(x, u) + d(y, v)). \]

Also for \((x_0, y_0) = (x, y) = (0, 0)\) we have
\[ F(x_0, y_0) = F(y_0, x_0) = \emptyset, (x_0, y_0) = (0, 0) \implies F(x_0, y_0) \times F(y_0, x_0), \]
\[ \alpha((x_0, y_0), (y_0, x_0)) = \alpha((0, 0), (0, 0)) = 1, \]
\[ \alpha((y_0, x_0), (y_0, x_0)) = \alpha((1, 1), (0, 0)) = 1. \]

It is easy to see that \(F\) is modified \(\alpha\)-admissible. Obviously \(F\) is continuous. Hence by theorem 2.2, \(F\) has a coupled fixed point in \(X^2\). For example \((0, 0)\) is a coupled fixed point of \(F\). Note that coupled fixed point of \(F\) is not unique for example \((2, 1)\) is a coupled fixed point of \(F\) too. In fact for any \(x, y \geq 0\) that \(x \geq 2y\) or \(y \geq 2x\), \((x, y)\) is a coupled fixed point of \(F\).

**Corollary 2.3:** Let \((X, d)\) be a complete metric space and \(\succeq\) be an order on \(X^2\). Fix \((x', y') \in X^2\). Suppose \(\psi \in \Psi\) a strictly increasing map and \(F : X^2 \to C(B(X))\) a multifunction such that
\[ H(F((x, y)), F((u, v))) \leq \frac{1}{2} \psi(d(x, u) + d(y, v)). \]

for all comparable elements \((x, y), (u, v) \in X^2\). Assume that the following assertions hold.

(i) If \((x, y) \in X^2, (u, v) \in F(x, y) \times F(y, x)\) and \((x, y), (u, v)\) are comparable, then \((u, v), (w, z)\) are comparable, for all \((w, z) \in F(u, v) \times F(v, u)\).

(ii) There exist \((x_0, y_0) \in X^2\) and \((x', y') \in F(x_0, y_0) \times F(y_0, x_0)\) such that \((x_0, y_0), (x', y')\) are comparable and \((y_0, x_0), (y', x')\) are comparable.

(iii) Either \(F\) is continuous or for any two sequences \(x_n\) and \(y_n\) in \(X\), that \(x_n \to x'\) \(y_n \to y'\) and \((x_n, y_n), (x_{n+1}, y_{n+1})\) are comparable, then \((x_n, y_n), (x, y)\) are comparable for all \(n\).

Then \(F\) has a coupled fixed point in \(X^2\).

Proof Define \(\alpha : X^2 \times X^2 \to [0, +\infty)\) by \(\alpha((x, y), (u, v)) = 1\) if \((x, y), (u, v)\) are comparable and \(\alpha((x, y), (u, v)) = 0\) otherwise and apply theorem 2.2

**Corollary 2.4:** Let \((X, d)\) be a complete metric space and \(\succeq\) be an order on \(X^2\). Fix \((x', y') \in X^2\). Suppose \(\psi \in \Psi\) a strictly increasing map and \(F : X^2 \to C(B(X))\) a multifunction such that
\[ H(F((x, y)), F((u, v))) \leq \frac{1}{2} \psi(d(x, u) + d(y, v)). \]

for all comparable elements \((x, y), (u, v) \in X^2\). Assume that the following assertions hold.

(i) If \((x, y) \in X^2, (u, v) \in F(x, y) \times F(y, x)\) and \((x, y), (u, v)\) are comparable, then \((u, v), (w, z)\) are comparable, for all \((w, z) \in F(u, v) \times F(v, u)\).

(ii) There exist \((x_0, y_0) \in X^2\) and \((x', y') \in F(x_0, y_0) \times F(y_0, x_0)\) such that \((x_0, y_0), (x', y')\) are comparable and \((y_0, x_0), (y', x')\) are comparable.

(iii) Either \(F\) is continuous or for any two sequences \(x_n\) and \(y_n\) in \(X\), that \(x_n \to x'\) \(y_n \to y'\) and \((x_n, y_n), (x_{n+1}, y_{n+1})\) are comparable, then \((x_n, y_n), (x, y)\) are comparable for all \(n\).

Then \(F\) has a coupled fixed point in \(X^2\).

Proof Define \(\alpha : X^2 \times X^2 \to [0, +\infty)\) by \(\alpha((x, y), (u, v)) = 1\) if \((x, y), (u, v)\) are comparable and \(\alpha((x, y), (u, v)) = 0\) otherwise and apply theorem 2.2

**References**


