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Damped Vibrations of Rectangular Plate of Variable Thickness Resting on Elastic Foundation: A Spline Technique

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Abstract

In the present paper damped vibrations of homogeneous rectangular plate of linearly varying thickness resting on elastic foundation has been studied. Following Lévy approach, the equation of motion of plate of varying thickness in one direction is solved by quintic spline method. The effect of damping, elastic foundation and taperness is discussed with permissible range of parameters. The frequency parameter Ω decreases as damping parameter D_k increases and it decreases faster in simply supported as compared to clamped-clamped boundary conditions in case of damping parameter and reverses in case of taperness.

Keywords: Taperness; Elastic foundation; Damping; Isotropic

Introduction

In the recent past, it has been observed that the research in the field of vibration is unceasingly accruing immense importance in the modern science, due to the significant role in every field of applied sciences. As fundamental structural elements, plates of various geometries are widely used in various engineering fields such as, aerospace technology, missile technology, naval ship design and telephone industry etc. Due to the appropriate variation of plate thickness, these plates provide the advantage of reduction in weight and size, and also have significantly greater efficiency for vibrations as compared to the plate of uniform thickness. Thus the vibration characteristics of plates having variable thickness have attracted the interest of researchers. An extensive survey of literature up to 1985 on linear vibration of isotropic/anisotropic plates of various geometries has been given by Leissa in his monograph and in a series of review articles [1]. Later on the studies of rectangular plate with uniform/non-uniform thickness has been carried out by many researchers. As space technology is growing rapidly, the importance of study of vibration is increasing; Gupta and Lal [2] have studied the transverse vibrations of a rectangular plate of exponentially varying thickness resting on an elastic foundation by using quintic spline technique. In reality all the vibrations are damped vibration, as free vibrations are ideal and can't be practically possible, so no vibration can be thought of being in existence without damping. In a series of papers, recently DJO'Boy [3] have analyzed the damping of flexural vibration and Alisjahbana and Wangsadinata [4] discussed the realistic vibrational problem incorporating dynamic analysis of rigid roadway pavement under moving traffic loads. In the demand of modern science, a study dealing with damped vibrations of homogeneous isotropic rectangular plate of linearly varying thickness along one direction and resting on elastic foundation is presented employing classical plate theory. Various numerical techniques such as Frobenious method [5], finite difference method [6], simple polynomial approximation [7], Galerkin's method [8,9], Rayleigh-Ritz method [10-12], finite element method [13] and Chebyshev collocation method [14] etc, have been employed to analyzed the modes of vibration of plates with different geometries. The numerical methods require small interval size to obtain the results up to the desired accuracy due to round off and truncation errors at each step. Frobenious method results in the form of series and for the purpose of computation, series is truncated which leads incorporates some errors and the characteristics orthogonal polynomials requires an appreciable number of terms for plates of variable thickness. However Quintic splines interpolation technique has the capability of producing highly accurate results with minimum computational efforts for initial and boundary value problems. Therefore in the present paper, quintic spline method is used to obtain modes of vibration because a chain of lower-order approximations yields a better accuracy than a global higher-order approximations and natural boundary conditions can be incorporated easily [15]. The frequencies and deflection corresponding to the first three modes of vibrations are computed for various values of plate parameters such as taper constants, damping parameter and elastic foundations.

Mathematical Formulation

The plate under consideration is a rectangular isotropic plate of length 'a', breath 'b', thickness 'h' and density 'ρ', with resting on a winkler- type elastic foundation 'k_f'. The plate is referred to rectangular cartesian co-ordinate (x,y,z).The middle surface being z=0 and the origin is at one of the corners of the plate. The differential equation which governs the transverse vibration of such plates is given by

$$\nabla^2(D\nabla^2 w)-(1-\upsilon)\left[\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]+\rho h\frac{\partial^2 w}{\partial t^2}+K\frac{\partial w}{\partial t}+K_f w=0 \quad (1)$$

Where, $D=D(x,y)=\dfrac{Eh^3(x,y)}{12(1-\upsilon^3)}$,

Where K the damping is constant, K_f is the elastic foundation constant, $w(x,y,t)$ is the transverse deflection and D is the flexural rigidity at any point in the middle plane of the plate. Let the two opposite edges y=0 and y=b of the plate be simply supported and thickness $h=h(x,y)$ varies linearly along the length i.e. in the direction of x-axis. Thus, 'h' is independent of y i.e. $h=h(x)$. For a harmonic solution, the deflection function w, satisfying the condition at y=0 and y=b, is assumed

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$$w(x,y,t) = W(x)\sin\frac{m\pi y}{b}e^{-\gamma t}\cos pt \tag{2}$$

Where ‘p’ is the circular frequency of vibration and ‘m’ is a positive integer. Thus Eq.(1) becomes

$$\begin{aligned}&\left[h^3\frac{\partial^4 W}{\partial x^4}+6h^2\frac{\partial h}{\partial x}\frac{\partial^3 W}{\partial x^3}+\left\{6h\left(\frac{\partial h}{\partial x}\right)^2+3h^2\frac{\partial^2 h}{\partial x^2}-2h^3\frac{m^2\pi^2}{b^2}\right\}\frac{\partial^2 W}{\partial x^2}-6h^2\frac{m^2\pi^2}{b^2}\frac{\partial h}{\partial x}\frac{\partial W}{\partial x}\right]\cos pt\\&+\left[\frac{m^4\pi^4}{b^4}h^3+3\upsilon\frac{m^2\pi^2}{b^2}\left\{2h\left(\frac{\partial h}{\partial x}\right)^2+h^2\frac{\partial^2 h}{\partial x^2}\right\}\right]W\cos pt+K_f\frac{12(1-\upsilon^2)}{E}W\cos pt+\\&\left[\frac{12(1-\upsilon^2)\rho h}{E}\left\{(\gamma^2-p^2)\cos pt+2\gamma\sin pt\right\}+\frac{12(1-\upsilon^2)}{E}k\left\{-p\sin pt-\gamma\cos pt\right\}\right]W=0\end{aligned} \tag{3}$$

Introducing the non-dimensional variables $H=\frac{h}{a}, X=\frac{x}{a}, \overline{W}=\frac{W}{a}$, $\beta^2 = m^2\pi^2\left(\frac{a}{b}\right)^2$

Equation (3) reduces to

$$\begin{aligned}&\left[H^3\frac{\partial^4 \overline{W}}{\partial X^4}+6H^2\frac{\partial H}{\partial X}\frac{\partial^3 \overline{W}}{\partial X^3}+\left\{6H\left(\frac{\partial H}{\partial X}\right)^2+3H^2\frac{\partial^2 H}{\partial X^2}-2H^3\beta^2\right\}\frac{\partial^2 \overline{W}}{\partial X^2}-6H^2\beta^2\frac{\partial H}{\partial X}\frac{\partial \overline{W}}{\partial X}\right]\cos pt\\&+\left[\beta^4H^3+3\upsilon\beta^2\left\{2H\left(\frac{\partial H}{\partial X}\right)^2+H^2\frac{\partial^2 H}{\partial X^2}\right\}\right]\overline{W}\cos pt+K_f\frac{12(1-\upsilon^2)}{E}\overline{W}\cos pt+\\&\left[\frac{12(1-\upsilon^2)\rho H}{E}\left\{(\gamma^2-p^2)\cos pt+2\gamma\sin pt\right\}+\frac{12(1-\upsilon^2)}{E}k\left\{-p\sin pt-\gamma\cos pt\right\}\right]\overline{W}=0\end{aligned} \tag{4}$$

Substituting $H = H_0(1-\alpha X)$, where $H_0 = (H)_{X=0}$ and ‘α’ is the taper constant due to linearly varying thickness of plate, and equating the coefficient of *sin(pt)* and *cos(pt)* independently to zero, following equation is formed

$$A_0\frac{\partial^4\overline{W}}{\partial X^4}+A_1\frac{\partial^3\overline{W}}{\partial X^3}+A_2\frac{\partial^2\overline{W}}{\partial X^2}+A_3\frac{\partial\overline{W}}{\partial X}+A_4\overline{W}=0 \tag{5}$$

Where

$$A_0=(1-\alpha X)^4, A_1=-6\alpha(1-\alpha X)^3, A_2=6\alpha^2(1-\alpha X)^2-2\beta^2(1-\alpha X)^4, A_3=6\alpha\beta^2(1-\alpha X)^3$$
$$A_4=\beta^4(1-\alpha X)^4-6\alpha^2\beta^2\upsilon(1-\alpha X)^2-\left\{D_K{}^2 I^{*2}+\Omega^2 I^*(1-\alpha X)^2-E_f(1-\alpha X)^3 C^*\right\}$$

$$D_K=\frac{3(1-\upsilon^2)K^2}{E\rho}, I^*=\frac{1}{H_0{}^2}, C^*=\frac{1}{H_0{}^3}, E_f=\frac{12(1-\upsilon^2)a}{E_0}, \Omega^2=\frac{12(1-\upsilon^2)a^2\rho p^2}{E},$$

Ω, D_K, E_f are frequency parameter, damping parameter and elastic foundation parameter respectively.

The solution of Eq.(5) together with boundary conditions at the edge x=0 and x=1 constitutes a two-point boundary value problem. As the PDE has several plate parameters, therefore it becomes quite difficult to find it’s exact solution. Keeping this in mind, complex for the purpose of computation, the quintic spline interpolation technique, is used. Let f(x) be a function with continuous derivatives in the range [0,1] and interval [0,1] be divided into ‘n’ subintervals by means of points X_i such that $0 = X_0 < X_1 < X_2 < \ldots < X_n = 1$. Where $\Delta X = 1/n, X_i = i\Delta X (i = 0,1,2,\ldots,n)$. Let the approximating function $\overline{W}(X)$ for the f(x) be a quintic spline with the following properties:

(i) $\overline{W}(X)$ is a quintic polynomial in each interval (X_k, X_{k+1}).

(ii) $\overline{W}(X) = F(X_k), k = 0,1,2,\ldots,n.$

(iii) $\frac{\partial\overline{W}}{\partial X}, \frac{\partial^2\overline{W}}{\partial X^2}, \frac{\partial^3\overline{W}}{\partial X^3}$ *and* $\frac{\partial^4\overline{W}}{\partial X^4}$ are continuous.

In view of above axioms, the quintic spline takes the form

$$\overline{W}(X) = a_0 + \sum_{i=0}^{4} a_i (X - X_0)^i + \sum_{j=0}^{n-1} b_j (X - X_j)_*^5$$

Where $(X - X_j)_* = \begin{cases} 0, & if\ X \le X_j \\ (X - X_j), & if\ X > X_j \end{cases}$

and $a_0, a_1, a_2, a_3, a_4, b_{n-1}$ are (n+5) unknown constants. Thus for the satisfaction at the n[th] knot, Eq.(6) reduced to

$$\begin{aligned}&A_4a_0+[A_4(X_m-X_0)+A_3]a_1+[A_4(X_m-X_0)^2+2A_3(X_m-X_0)+2A_2]a_2\\&+[A_4(X_m-X_0)^3+3A_3(X_m-X_0)^2+6A_2(X_r-X_0)+6A_1]a_3\\&+[A_4(X_m-X_0)^4+4A_3(X_m-X_0)^3+12A_2(X_m-X_0)^2+24A_1(X_m-X_0)+24A_0]a_4\\&+\sum_{j=0}^{n-1}b_j[A_4(X_m-X_j)_*{}^5+5A_3(X_m-X_j)_*{}^4+20A_2(X_m-X_j)_*{}^3+60A_1(X_m-X_j)_*{}^2+120A_0(X_m-X_j)_*]=0\end{aligned} \tag{7}$$

For m=0(1)n, above system contains (n+1) homogeneous equation with (n+5) unknowns, ai, i=0(1)4, bj, j=0,1,2,…,(n-1), and can be represented in matrix form as [A]{B}={0} (8)

Where [A] is a matrix of order (n+1)×(n+5) while {B} and {0} are column matrices of order (n×5).

Boundary Conditions and Frequency Equation

The following two cases of boundary conditions have been considered:

(i) (c-ss-c-ss): clamped at both the edge X=0 and X=1.

(ii) (c-ss-ss-ss): clamped at X=0 and simply supported at X=1.

The relations that should be satisfied at clamped and simply supported are

$W = \frac{\partial W}{\partial X} = 0$; $W = \frac{\partial^2 W}{\partial X^2} = 0$; respectively. (9)

Applying the boundary conditions c-ss-c-ss to the displacement function by Eq. (9) one obtains a set of four homogeneous equations in terms of (n+5) unknown constants which can be written as

[B[cc]]{B}={0} (10)

Where B^{cc} is a matrix of order 4×(n+5). Therefore the Eq. (7) together with the Eq. (10) gives a complete set of (n+5) homogeneous equations having (n+5) unknowns which can be written as

$$\left[\frac{A}{B^{cc}}\right]\{B\} = \{0\} \tag{11}$$

For a non-trival solution of Eq. (11), the characteristic determinant must vanish, i.e.

$$\left|\frac{A}{B^{cc}}\right| = 0 \tag{12}$$

Similarly for (c-ss-ss-ss) plate the frequency determinant can be written as

$\left|\frac{A}{B^{ss}}\right| = 0$, Where Bss is a matrix of order 4×(n+5) (13)

Numerical Results and Discussion

In the present paper, first three frequency modes of vibration have been computed for the above mentioned two boundary conditions for different values of foundation parameter E_f=0.0(0.005)0.02, damping parameter D_k=0.0(0.01)0.04 and taper parameter α=0.0(0.1)0.4 for Poisson ratio’s ν=0.3, thickness of plate h=0.03 and aspect ratio a/b=0.25. The numerical method provides approximate values therefore in order to minimize the error, there is an urgent need to determine the optimum size of interval length ΔX. In the present problem, a computer

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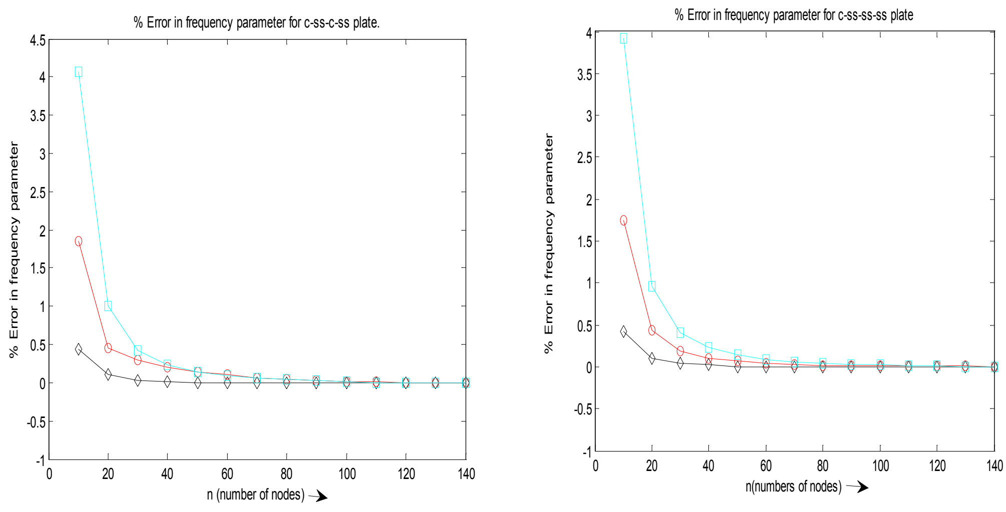

(a) (b)

Figure 1: percentage error in frequency parameter Ω: (a) c-ss-c-ss plate (b) c-ss-ss-ss plate, for a/b=0.25, α=0.0, D_k=0.0, Percentage error=$[(\Omega_n-\Omega_{140})/\Omega_{140}]$x100; n=10(10)140.

	h=0.03	v=0.3	m=1	a/b=0.25	Dk=0.0	E_f=0.0
	c-ss-c-ss plate			c-s-ss-ss plate		
taper parameter (α)	**mode1**	**mode2**	**mode3**	**mode1**	**mode2**	**mode3**
0	0.6816	1.8643	3.6432	0.4766	1.515	3.1449
0.1	0.6471	1.7699	3.4587	0.4584	1.4442	2.9916
0.2	0.6117	1.6729	3.2689	0.4396	1.3715	2.8341
0.3	0.5752	1.5727	3.0729	0.42	1.2963	2.6741
0.4	0.5373	1.4686	2.8692	0.3995	1.2184	2.5025

Table 1a: Values of frequency parameter Ω for different values of taper constant α.

	h=0.03	v=0.3	m=1	a/b=0.25	Dk=0.0025	E_f=0.0
	c-ss-c-ss plate			c-ss-ss-ss plate		
taper parameter (α)	**mode1**	**mode2**	**mode3**	**mode1**	**mode2**	**mode3**
0	0.6765	1.8624	3.6422	0.4692	1.5128	3.1438
0.1	0.6411	1.7677	3.4576	0.4498	1.4416	2.9903
0.2	0.6046	1.6703	3.2676	0.4293	1.3683	2.8325
0.3	0.5666	1.5695	3.0713	0.4076	1.2924	2.6695
0.4	0.5266	1.4647	2.8671	0.3841	1.2135	2.5001

Table 1b: Values of frequency parameter Ω for different values of taper constant α.

program was developed and executed for n=10(10)150 and observed that no consistent improvement in results while n ≥ 140 for clarity (Figures 1a and 1b). Therefore the results are obtained for n=140 and depicted through table and graphs. It has also been observed that the frequencies for c-ss-c-ss plates are more than the frequencies of c-ss-ss-ss plates for the same set of values of other parameters. Tables 1a-1c and Figures 2a-2c show the behavior of frequency parameter Ω with the increasing value of taper constant (α) for the fixed value of damping parameter D_k=0.0, 0.0025, and foundation parameter E_f=0.0, 0.01 for first three modes of vibration of c-c and c-ss plate, it is observed that the frequency parameter Ω decreases with the increasing values of taper parameter α. Tables 2a-2b and Figures 3a and 3b, provide the inference of damping parameter D_k on frequency parameter Ω for two values of and foundation parameter E_f=0.0, and 0.01 respectively, for the fixed value of taper parameter α=0.4. It is observed that the frequency parameter decreases with the increases of damping parameter D_k

	h=0.03	v=0.3	m=1	a/b=0.25	Dk=0.0025	E_f=0.01
	c-ss-c-ss plate			c-ss-ss-ss plate		
taper parameter (α)	**mode1**	**mode2**	**mode3**	**mode1**	**mode2**	**mode3**
0	**0.8894**	1.9499	3.6877	0.744	1.6192	3.1964
0.1	**0.853**	**1.8551**	**3.503**	**0.7187**	**1.5472**	**3.0427**
0.2	0.8153	1.7576	3.3131	0.6922	1.4731	2.8848
0.3	0.776	1.6568	3.1168	0.6641	1.3964	2.7216
0.4	0.7348	1.552	2.9127	0.6341	1.3166	2.5521

Table 1c: Values of frequency parameter Ω for different values of taper constant α.

	h=0.03	v=0.3	m=1	a/b=0.25	α=0.4	E_f=0.0
	c-ss-c-ss plate			c-ss-ss-ss plate		
Damping parameter (D_k)	**mode1**	**mode2**	**mode3**	**mode1**	**mode2**	**mode3**
0	0.5373	1.4686	2.8692	0.3995	1.2184	2.5025
0.001	0.5356	1.468	2.8688	0.397	1.2176	2.5021
0.002	0.5305	1.4661	2.8679	0.3896	1.2152	2.501
0.003	0.5218	1.4629	2.8662	0.377	1.2113	2.4991
0.004	0.5094	1.4585	2.8639	0.3585	1.2058	2.4964

Table 2a: Values of frequency parameter Ω for different values of damping parameter D_k.

	h=0.03	v=0.3	m=1	a/b=0.25	α=0.4	E_f=0.01
	c-ss-c-ss plate			c-ss-ss-ss plate		
Damping parameter (D_k)	**mode1**	**mode2**	**mode3**	**mode1**	**mode2**	**mode3**
0	0.7425	1.5557	2.1947	0.6436	1.3211	2.5544
0.001	0.7412	1.5551	2.9144	0.6421	1.3204	2.554
0.002	0.7375	1.5533	2.9135	0.6375	1.3182	2.5529
0.003	0.7313	1.5504	2.9119	0.6298	1.3146	2.551
0.004	0.7225	1.5462	2.9096	0.6189	1.3095	2.5484

Table 2b: Values of frequency parameter Ω for different values of damping parameter D_k.

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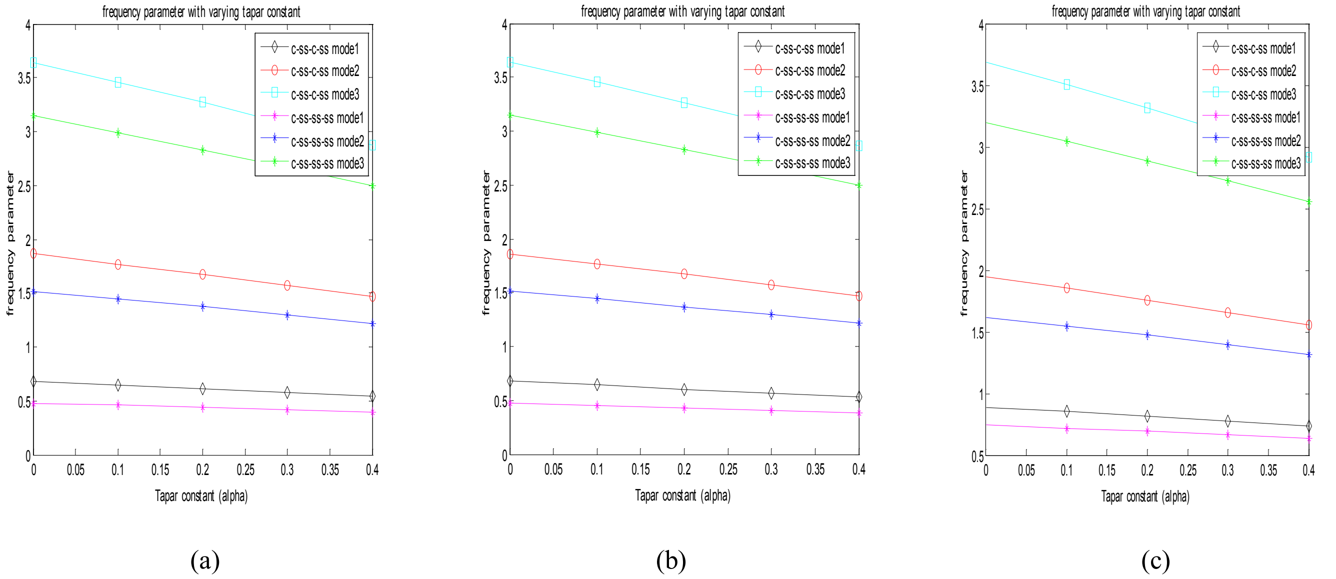

Figure 2: Frequency parameter for clamped, simply supported plates: (a) modes for D_k=0.0, E_f=0.0; (b) modes for D_k=0.0025, E_f=0.0; (c) modes for D_k=0.0025, E_f=0.01.

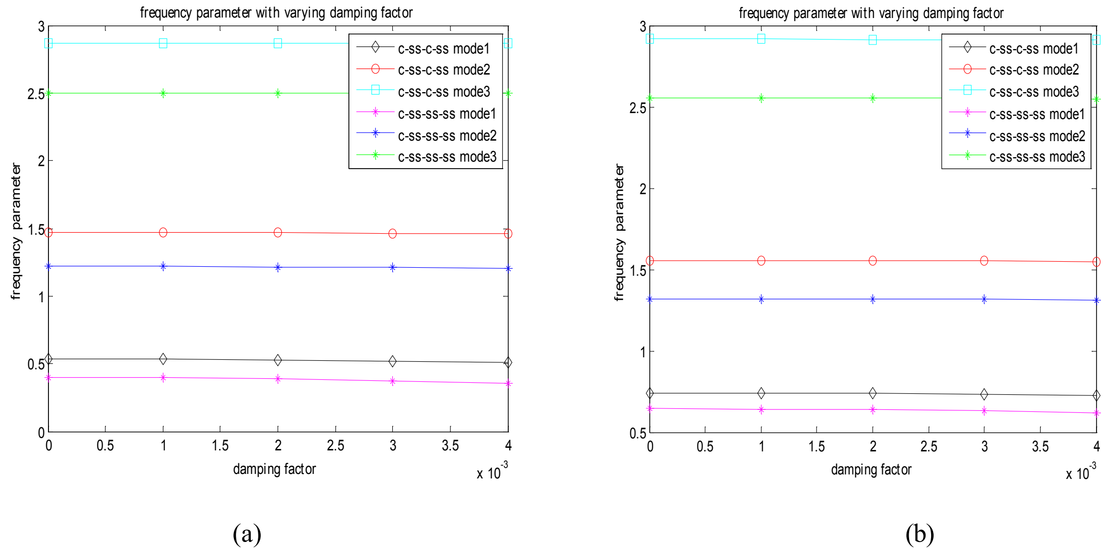

Figure 3: Frequency parameter for clamped, simply supported plates: (a) modes for α=0.0, E_f=0.0; (b) modes for α=0.4, E_f=0.01.

	h=0.03	v=0.3	m=1	a/b=0.25	α=0.4	D_k=0.0
	c-ss-c-ss plate			c-ss-ss-ss plate		
Foundation parameter (E_f)	mode1	mode2	mode3	mode1	mode2	mode3
0	**0.5373**	**1.4686**	**2.8692**	**0.3995**	**1.2184**	**2.5025**
0.005	0.6481	1.5128	2.892	0.5357	1.2707	2.5286
0.01	0.7425	1.5557	2.9147	0.6436	1.3211	2.5544
0.015	0.8261	1.5975	2.9373	0.7357	1.3696	2.5799
0.02	0.9019	1.6383	2.9596	0.8175	1.4165	2.6052

Table 3a: Values of frequency parameter Ω for different values of foundation parameter E_f.

	h=0.03	v=0.3	m=1	a/b=0.25	α=0.4	D_k=0.0025
	c-ss-c-ss plate			c-ss-ss-ss plate		
Foundation parameter (E_f)	mode1	mode2	mode3	mode1	mode2	mode3
0	0.5266	1.4647	2.8671	0.384	1.2135	2.5011
0.005	0.6392	1.5089	2.89	0.5242	1.2661	2.5262
0.01	0.7348	1.552	2.9127	0.6341	1.3166	2.5521
0.015	0.8191	1.5939	2.9353	0.7274	1.3653	2.5776
0.02	0.8955	1.6347	2.9577	0.8099	1.4123	2.603

Table 3b: Values of frequency parameter Ω for different values of foundation parameter E_f.

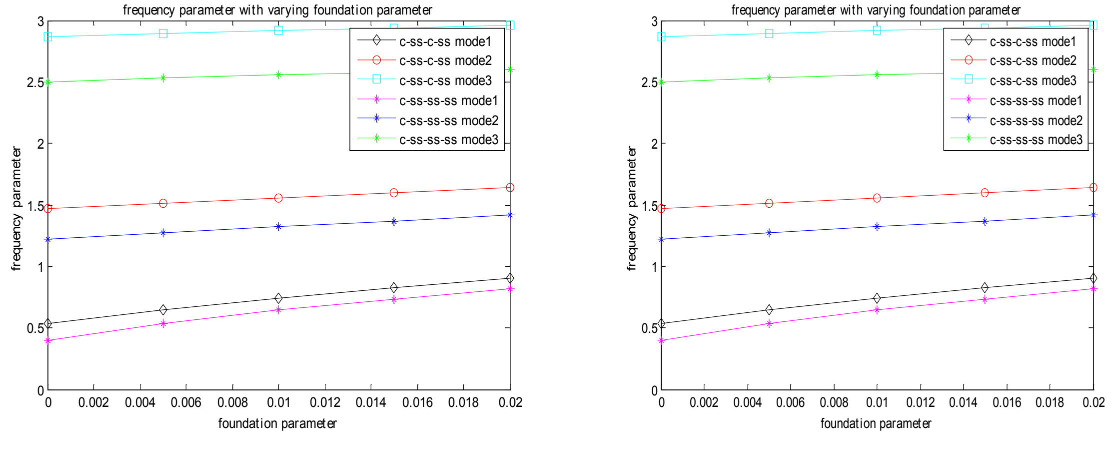

Figure 4: Frequency parameter for clamped, simply supported plates: (a) modes for α=0.04, D_k=0.0; (b) modes for α=0.4, D_k=0.0025.

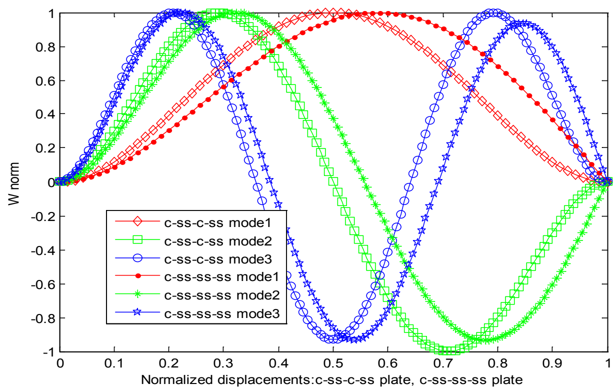

Figure 5: Normalized displacements for clamped and simply supported plates, for a/b=0.025; h=0.03,D_k=0.0, Ef=0.0.

and the rate of decreases with the increases in the value of damping parameter D_k for c-ss plates is higher than that for c-c plate keeping all other plate parameter fixed. Tables 3a and 3b and Figures 4a and 4b show the increase in frequency parameter Ω verses foundation parameter E_f for two different value of damping parameter D_k=0.0, 0.0025, for the fixed value of taper parameter α=0.4. It is noticed that the frequency parameter Ω increases continuously with the increasing value of foundation parameter E_f for c-ss-c-ss and c-ss-ss-ss plates, whatever be the value of other plate parameters. It is found that the rate of increases of frequency parameter Ω for c-ss-ss-ss plate is higher than c-ss-c-ss plate for three modes. The normalized displacements for the two boundary conditions c-ss-c-ss and c-ss-ss-ss, considered in this paper are shown in Figure 5. The plate thickness varies linearly in X-direction and the plate is considered resting on elastic foundations. Figure 5 Shows the transverse displacements for the first three modes. The nodal lines are seen to shift towards the edge, i.e. X=1 as the boundary conditions change from c-ss-c-ss to c-ss-ss-ss for all the three transverse modes. The slope of the normalized displacement curves on both edges for c-ss-c-ss condition, are nearly zero which indicates the correctness of the solution.

Conclusion

In the present study results are computed using MATLAB within the permissible range of parameters up to the desired accuracy (10^{-8}), which validates the actual phenomenon of vibrational problem. Variation in thickness, elastic foundation and damping parameter are of great interest since it provides reasonable approximation to linear vibrations. Thus the present study may be useful for design engineers especially in rigid roadway pavement under moving traffic loads.

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