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Directed*-topology and Scott*-topology on Transitive Binary Relational Sets

Mohammed Khalaf M[1,2*]

[1]*High Institute of Engineering and Technology King Marioutt P.O. Box 3135, Egypt*

[2]*Mathematics Department, Faculty of Science in Zulfi, Majmaah University, Zulfi 11932, P.O. Box 1712, Saudi Arabia*

Abstract

In this work we naturally put forth an open question whether one may construct a scott-topology on transitive binary relational sets (so called *TRS*). We prove that a *TRS* gives rise to several natural topologies defined in terms of the given *TRS* structure. Mainly, we consider directed topologies and scott topologies on *TRS* and their interactions with the continuity property of *TRS*. Most of our results are generalizations of corresponding results in references as we will illustrate. Sometimes we need pre-ordered sets instead of *TRS*.

Keywords: Poset; Transitive binary relational sets; Directed topologies; Scott topologies; Pre-ordered sets

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Introduction

In domain and poset [1-3], Scott-topologies were defined. Abramsky and Jung [4] introduced the concepts of continuous directed complete posets (continuous domain) and algebraic domains. Heckmam [1] studied these conceptes by more details and explained a interactions between Scott-topology and these notions. Also, add the concepts of bounded complete posets, bounded complete domains, finitely complete posets, finitely complete domains, finitarily complete posets [5-8]. Hoffmann and Lawson [8-10] gives the concepts of continuous posets. And in more general fashion by Markowsky [11] and Eme [12]. Nino-Salcedo [3], add by deep studies the concept of algebric posets. We note that the concept of continuous posets (resp., algebraic posets) in the sense of Nino-Salcedo and continuous domain (resp., algebraic domain) of *R* are the same. Zhang [13] studied a type of continuous poset which a generalizations of the continuous poset in the sense of Nino-Salcedo. and add some interactions between bounded complete domains, Scott topology and Lawson topology. This work is devoted to introduce and study the continuity and algebraicness properties of *TRS*. Our results extended the results in posets and in domains [1-3,13]. The concepts of upper bound (for short *ub*), lower bound (for short *lb*), least upper bound (for short), gretest lower bound (for short) in any poset are clear also, some concepts in mathematical logics my building some times needs these facts [14].

To solve the problem we first introduce the following concepts.

Definition 1.1

Let $'\leq'$ be a binary relation set on $X\neq\phi$. Then;

(1) $'\leq'$is called reflexive iff $\forall xX$, $x\leq x$ [14];

(2) $'\leq'$is called antisymetric iff $\forall x,y\in X$, $x\leq y$ and $y\leq xx=y$ [14];

(3) $'\leq'$is called transitive iff $\forall x,y,z\in X$, $x\leq y$ and $y\leq zx=z$ [14];

(4) $'\leq'$is called symetric iff $\forall x,y\in X$, $x\leq y \Rightarrow y\leq x$ [14];

(5) $'\leq'$is called interpolative iff $\forall x,z\in X$, with xz, $\exists$ yX *s.t.* $xy\leq z$ [1,15].

(6) if $'\leq'$satisfies the conditions (1), (2) and (3), then $(X, \leq)$ is called Partialy order set (Poset) [14];

(7) if $'\leq'$satisfies the conditions (1), and (3), then $(X, \leq)$ is called pre-orderd set (Quasi set) [14];

(8) if $'\leq'$satisfies the conditions (1), (2), (3) and (4), then $(X, \leq)$ is called an equvalence set,

(9) if $'\leq'$satisfies the conditions (3) and (5), then $(X, \leq)$ is a Continuous information system [15,16].

(10) if $'\leq'$satisfies the conditions (3), and $\forall x\in X$, and for every finite subset$A$ of X the following axiom holds: if $\forall yA$, $y\leq x$ then $\exists$ zX *s.t.* $\forall yA$, $y\leq z$ and zx, then $(X, \leq)$ is abstract basis [17].

Definition 1.2

Let AX. Then:

(1) A is called directed subset of X iff $A\neq\phi$ and $\forall x,y\in A$, $\exists$ zA *s.t.* xz and yz [1];

(2) The lower (resp. upper) closure in X of A is denoted by $\downarrow A$ (resp. $\uparrow A$) and defined as follows: $\downarrow A=\{x\in X:\exists yAs.t.x\leq y\}$ (resp. $\uparrow A=\{x\in X:\exists yAs.t.y\leq x\}$)[1]

(3) The convex hull A is denoted by $\updownarrow A$ and defined as follows: $\updownarrow A=\downarrow A\uparrow A$[1];

(4) Let $A,B\subseteq X.B$ is called cofinal in A iff $BA\subseteq\downarrow(B)$ [1].

Definition 1.3

Let AX. Then:

(1) A subset A of the domain [1] (resp. Poset) X is called directed closed (d-closed for short) iff $\forall$ directed subset D of A,$(D)\in A$;

(2) A subset A of the Poset X is called Scott-closed iff A is d-closed lower subset of X [3];

***Corresponding author:** Mohammed Khalaf M, Mathematics Department, Faculty of Science in Zulfi, Majmaah University, Zulfi 11932, P.O. Box 1712, Saudi Arabia, Tel: 95140-0582905323, 2293641375; E-mail: Khalfmohammed2003@yahoo.com

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(3) A is called d-(resp. Scott-) open iff A^c d-(resp. Scott-) closed [1,3];

(4) Let $x,y\in X$. We say x below (resp.y is way above) y (resp. x), denoted by $x<<y$ iff $\forall D\subseteq X$, s.t.D is directed subset of X with D exists and $y\leq D$, $\exists dD$ s.t. xd. The family of all elements in X each of which way above (resp. way below) x is denoted and defined as follows: $\Uparrow x=\{y\in X:x<<y\}$(resp. $\Downarrow x=\{y\in X:y<<x\}$[3];

(5) Let $x\in X$. If $x<<x$, then x is said to be isolated. The family of all isolated points above (resp. below) $x\in X$ is denoted and defined by: $\uparrow^{\circ}x=\{y\in X:y<<y$ and $x\leq y\}$ resp. $\downarrow_{\circ} x=\{yX:y<<y$ and $yx\}$[3].

Proposition 1.1: (Proposition 3.5.2 [1]). Let X. be a domain, we have:

(1) X and ϕ are d-closed;

(2) The intersection of a family of d-closed sets is d-closed; and

(3)The union of two of d-closed sets is d-closed. Then the family of all d-open sets in a domain X is a topology on X.

Proposition 1.2: (Proposition 3.6.2 [1]) Let X. be a poset, we have:

(1) X and ϕ are Scott-closed;

(2) Arbitrary intersection of Scott-closed sets is Scott-closed; and

(3) Finite union of two of Scott-closed sets is Scott-closed. Then the family of all Scott-open sets in a poset X is a topology on X. Furthermore it is $T_{\circ}$-topology.

Proposition 1.3: (Proposition 3.5.3(3) [1]) In a domain $(X,\leq)$. Let F be a finite set X. Then $\downarrow F$ is d-closed subset.

Definition 1.4

For any poset X consider the following topologies:

(1) $\delta_d=\{A\subseteq X : A \text{is} d-\text{open}\}$ is a topology on X (see proposition 3.5.2 [1]) in the case of X is a domain) and is called the directed topology) (d-topology for short);

(2) $\delta_{A|x}=\{A\subseteq X : A \text{isanuppersubset}\}$ is a topology on X ([1]) in the case of X is a domain) and is called the Alexandroff topology) (Alex-topology for short);

(3) $\delta_S=\{A\subseteq X : A \text{isScott opensubset}\}$ is a topology on X [1-3] and is called the Scott- topology);

(4) The upper topology on X is denoted by δ_U and is the topology generated by subbasis $\{X-\downarrow x : x\in X\}$ [2];

(5) TheLower topology on X is denoted by δ_L and is the topology generated by subbasis $\{X-\uparrow x : x\in X\}$ [2];

(6) The interval topology δ_I on X is the supremum of δ_U and δ_L i.e., $\delta_I=\delta_{U\ L}$ [2];

(7) The Lowson topology δ_{LS} on X is the supremum of δ_S and δ_L i.e., $_{LS}=\delta_L\ \delta_S$ [2].

(8) Let (X,δ) be a topological space, and let $A\subseteq X$, then the closure of λ denoted by $cl_\delta()$ defined as follows $cl_\delta(\lambda)=\cap\{F\subseteq X : F\text{is}\delta-\text{closed and } \lambda\subseteq F\}$ [14].

Theorem 1.1: (Remark 1.4 (ii) [7]). Let X be a complete lattice. Then, $\forall\ x\in X, \downarrow x.\in cl_{\delta_S}(\{x\})$.

Proposition 1.4: (Theorem 6.1.2 [1]). A point x is an isolated iff the upper cone $\uparrow x$ is Scott-open.

Proposition 1.5: (Proposition 6.7.7 [1]). Let $(X,\leq)$ be a continuous domin. Then $\forall x\in X$, the set $\Uparrow x$ is Scott-open.

Proposition 1.6: (Proposition 6.7.8 [1]). In a continuous domain X, $x<<y$ holds iff there is a Scott-open set $\bigcirc$ s.t. $y\in\bigcirc\subseteq\uparrow x$.

Theorem 1.2: (Theorem 6.7.9 [1]). A domain $(X, \leq)$ is continuous iff it has the local upper cone property: forevery $x\in X$, and for every $\bigcirc\in\delta_S$, s.t. $x\in\bigcirc, \exists\bigcirc'$ and $x'\in X$ s.t., $x\in\bigcirc'\subseteq\uparrow x'\subseteq\bigcirc$.

Definition 1.5

A space X is called a $T_{\circ}$-space iff for two distinct points $x,y\in X$, $\exists\lambda\subseteq X$ s.t. either $x\lambda,y\in X-\lambda$ or $y\lambda,x\in X-\lambda$ [14].

Definition 1.6

Let $(X,\leq_1)$ and $(X,\leq_2)$ be TRS and let $f:(X,\leq_1)\to(Y,_2)$ be a function [2]. Then:

(1) f is monotone iff $f(\lambda)\leq_2 f(\mu)$, whenever $\lambda\leq_1$,

(2) f is called Scott-continuous iff $\forall G\in(\delta_S)_Y, f^{-1}(G)\in(\delta_S)_X$, where $(\delta_S)_Y$ (resp. $(\delta_S)_X$) is the Scott-topology on Y induced by $\leq_2$(resp. on X induced by $\leq_1$).

Definition 1.7

Let $(X,\leq_1)$ and $(X,\leq_2)$ be TRS and let $f:(X,(\delta_{A|x})_X)\to(Y,(\delta_{A|x})_Y)$ is called Alexandroff-continuous iff f is monotone, $(\delta_{A|x})_X$ (resp., $(\delta_{A|x})_Y$) is the Alexandroff-topology on X (resp., on Y) induced by $\leq_1$(resp. $\leq_2$).

Directed*-Open, Scott*-Open Sets and Transitive Binary Relational Sets

Definition 2.1

Atransitive binary relational set (TRS for short) is a pair $(X,\leq)$ cwhere $X\phi$ and '$\leq$'is a transitive binary relation set.

Example 2.1: Partialy order set, pre-orderd set, continuous information system, equivalence set and abstract basis is TRS.

Remark 2.1: Abstract basis $\Rightarrow$ Continuous information system, the converse is not true, so give the following example

Example 2.2: Let$X=\{x,y,z\}$ and let $\leq=\{(x,x),(y,y),(x,z),(y,z)\}$, because

$x\leq x\Rightarrow\exists\ x\in X$ s.t., $x\leq x\leq x$;

$y\leq y\Rightarrow\exists\ y\in X$ s.t., $y\leq y\leq y$,

$x\leq z\Rightarrow\exists\ x\in X$ s.t., $x\leq x\leq z$, and

$y\leq z\Rightarrow\exists\ y\in X$ s.t.,$y\leq y\leq z$,

then $\leq$ transitive and interpolative. Now, Let $\lambda=\{x,y\}$. Then $x\leq z$ and $y\leq z$, but $\forall\ a\in X, a\not\leq z$. Hence $(X,\leq)$i s not abstract basis.

Definition 2.2

Let $(X,\leq)$ be a TRS. Then:

(1) D is called directed subset of X iff $D\neq\phi$ and for every distinct points $(x\neq y)\in D, \exists\ z\in D\cap ub(\{x,y\})$;

(2) D is called S-directed subset of X iff D is a directed subset and $\forall\ x\in D, (\{x\})\cap D\neq\phi$.

Remark 2.2: S-directed subset$\Rightarrow$ directed subset, the converse is not true, so give the following example.

Example 2.3: Let $\leq$ be *TRS* on $X=\{x,y,z,l,m\}$ define by $x \leq y$. Then $\{x\}$ is directed subset but not *S*-directed subset.

Definition 2.3

Let $(X,\leq)$ be a *TRS*. Then:

(1) A subset λ of X is called d^*-closed iff for every *S*-directed subset D of λ,

$(D) \subseteq \lambda$ and $\forall x \lambda$, $(\{x\}) \subseteq \lambda$,

(2) A subset λ of X is called Scott*-closed iff λ is d-closed lower subset,

(3) A subset λ of X is called d-open (resp. Scott-open) iff λ^c is d-closed (resp. Scott-closed).

Theorem 2.1: Let $(X,\leq)$ be a *TRS*. Then:

(1) A subset μ of X is called d-open iff for every *S*-directed subset D of $X, s.t. (D) \cap \mu \neq \phi$,

and $\forall\, x \in X, (\{x\}) \cap \mu \neq \phi, x \notin \mu$, and

(2) A subset μ of X is called Scott*-open iff μ is d^*-open upper subset.

Proof: (1) The proof is obtained logically as follows:

Let E_S be the set of all *S*-directed subset of *X*.And Let $P \equiv' D$ is *S*-directed subset $'\equiv' D \in E_S$;

$$Q \equiv' D \subseteq \lambda';$$
$$R \equiv' (D) \subseteq \lambda';$$
$$L \equiv' x \in X';$$
$$S \equiv' x \in \lambda';$$
$$H \equiv' (\{x\}) \subseteq \lambda'.$$

Now, λ is d^*-closed

$$\equiv P \wedge (Q \Rightarrow R) \wedge L \wedge (S \Rightarrow H) \equiv P \wedge (\neg R \Rightarrow \neg Q) \wedge L \wedge (\neg H \Rightarrow \neg S) \equiv \lambda^c$$

is d^*-open. Then λ^c is d^*-open

$$\equiv (D \in E_S) \wedge \big(((D) \cap \lambda^c) \neq \phi \Rightarrow (D \cap \lambda^c) \neq \phi\big) \wedge (x \in X) \wedge ((\{x\}) \cap \lambda^c) \neq \phi \Rightarrow x \notin \lambda^c.$$

Then μ is d-open iff $\forall\, D \in E_S\ s.t. (D) \cap \mu \neq \phi$; and $\forall\, x \in X, (\{x\}) \cap \mu \neq \phi, x \notin \mu$.

(2) By the same fashion

Theorem 2.2: Let $(X,\leq)$ be a *TRS*. If λ is d-closed (resp. d-open) subset of X then λ is d^*-closed (resp. d^*-open).

Proof: Only prove the stament. Let D be an *S*-directed subset of λ. So since λ is d-closed, then $(D) \subseteq \lambda$. Also, $\forall x \in \lambda \{x\}$, directed subset of λ so that $(\{x\} \subseteq \lambda$. Hence λ is d-closed.

Corollary 2.1: From Theorem 2.1. Let $(X,\leq)$ be a *TRS*. If λ is Scott-closed (resp. Scott-open) subset of *X*, then λ is Scott*-closed (resp. Scott*-open).

Open problem if λ is d^*-open subset of a *TRS*, then λ is d-open.

Theorem 2.3: Let $(X,\leq)$ be a pre-orderd set and $\lambda \subseteq X$. Then:

(1) λ of X is directed subset iff λ is an S-directed subset of *X;*

(2) λ is d-open (resp. d-closed, Scott-closed, Scott-open) subset of X iff λ is d^*-open (rep. d^*-closed, Scott*-closed, Scott*-open).

Proposition 2.1: Let $\lambda, \mu \in X$. If μ directed subset and cofinal in λ, then λ is directed subset and $(\lambda)=()$.

Proof: First, we prove that λ is direct subset. Since $\subseteq \lambda$, then $\lambda \neq \phi$. Let $l, m \in \lambda$ *s.t.* $l \neq m$. Then $\exists b_1, b_2 \in \mu\ s.t.\ l \leq b_1, m \leq b_2$ and $\exists\, b \in \lambda \cap ub(\{l,m\})$. Hence λ is directed subset. Scound, one can deduce that $ub(\mu) \subseteq ub(\lambda)$. Conversely, $y \notin ub(\lambda) \Rightarrow \exists\, a \in \lambda\ s.t.\ a \not\leq y \Rightarrow \exists\, a \in \downarrow(\mu) \Rightarrow \exists\, b \in \mu\ s.t.\ a \leq b$ and $a \not\leq y \Rightarrow \exists b \in \mu\ s.t.\ b \not\leq y \Rightarrow y \notin ub(\mu)$. Hence $ub(\mu) \subseteq ub(\lambda)$. Thus $(\lambda)=()$.

The following theorem is a generalization of the corresponding result in Proposition 1.1 (Proposition 3.5.2 [1]).

Theorem 2.4: Let $(X,\leq)$ be a *TRS*. If $\forall\, x \in X, (\{x\}) \neq \phi$, then $\delta_{d^*} = \{\lambda \subseteq X : \lambda \text{ is } d^* - \text{open}\}$ is a topology on X (called directed*-topology on *X*).

Proof: (1) Clearly X and ϕ are d^*-closed sets. So X and ϕ are d^*-open sets;

(2) Let λ and μ be d^*-open sets. Then λ^c and μ^c be X be d^*-closed sets. Let D be an *S*-directed subset of $\lambda^c \cup \mu^c$. Thus $D = (D \cap \lambda^c)$ *or* $(D\mu^c)$. We need to prove that either $(D \cap \lambda^c)$ *or* $(D\mu^c)$ is cofinal in *D*. Suppose $D\lambda^c$ is not cofinal in *D,* then $\exists\, d_\circ \in D\ s.t. \forall\, a \in D \cap \lambda^c,\ d_\circ \not\leq a$. If $d_\circ \in \lambda^c$, then $(\{d_\circ\}) \subseteq \lambda^c$. Thus $\exists\, m \in D \cap \lambda^c, d_\circ \not\leq a\ s.t. d_\circ \leq m$. Acontraduction. Hence $d_\circ \in \mu^c$. Let $d \in D\ s.t.d \neq d_\circ$. Then $\exists\, d' \in D \cap ub(\{d_\circ, d\})$. If $d\lambda^c$, then $d_\circ \leq d'$ which leads to a contaduction. So, $d' \in \mu^c$, *i.e.*, $D \subseteq \downarrow D \cap \mu^c$. Hence $D\mu^c$ is cofinal in *D*. Now we prove that $D\mu^c$ is directed. Let $b_1, b_2 \in D \cap \mu^c\ s.t. b_1 \neq b_2$. Thus $\exists\, d \in D \cap ub(\{b_1, b_2\})$. Thus $\exists\, k \in D \cap \mu^c\ s.t.\, d \leq k$. So, $k \in (D \cap \mu^c) \cap ub(\{b_1, b_2\})$. Hence $D \cap \mu^c$ is directed. From Proposition 2.1. $(D)=(D \cap \mu^c)$. Now $D \cap \mu^c$ is *S*-directed (Indeed, let $l \in D \cap \mu^c$. Since D is *S*-directed, $\exists\, m \in (\{l\})\ s.t.\, m \in D$. Also, since μ^c is d-closed, then $(\{l\}) \subseteq \mu^c$. Thus $m \in D \cap \mu^c$, *i.e.*, $(D \cap \mu^c) \cap (\{l\}) \neq \phi$. Hence $D\mu^c$ is directed). Then $(D) = (D \cap \mu^c) \subseteq \mu^c \subseteq \mu^c \cap \lambda^c$. Also, if $x \in \mu^c \lambda^c$, then $(\{x\}) \subseteq \mu^c \lambda^c$. So $\mu^c \cup \lambda^c$ is d^*-closed so that $\lambda \cap \mu$ is d-open, and (3) Let$\{\lambda_i : i \in I\}$ be a family of d^*-open subsets of X. Then $\{\lambda_i^c : i \in I\}$ is a family of d-closed subsets of *X*. Let D be an *S*-directed subset of ${}_{i \in I}\lambda_i^c$. So, $D \subseteq \lambda_i^c\ \ \forall i \in I$. Then $(D) = \lambda_i^c\ \ \forall i \in I$. So that $(D) =_{i \in I} \lambda_i^c$. Let $l \in_{i \in I} \lambda_i^c$, then $(\{l\}) = \lambda_i^c\ \forall i \in I$. Thus $(\{l\}) = \lambda_i^c\ \ \forall i \in I$. So that $(\{l\}) =_{i \in I} \lambda_i^c$. Then ${}_{i \in I}\lambda_i^c$ is d-closed. Hence ${}_{i \in I}\lambda_i$ is d-open.

Proposition 2.2: (1) X andϕ areLower (resp. upper) subsets; and

(2) If $\{\lambda_i : i \in I\}$ be a family of Lower (resp. upper) subsets of *X*, then ${}_{i \in I}\lambda_i$ and ${}_{i \in I}\lambda_i$ are Lower (resp. upper) subsets.

Theorem 2.5: Let $(X,\leq)$ be a *TRS*. If $\forall\, x \in X, (\{x\}) \neq \phi$, then $\delta_{S^*} = \{\lambda \subseteq X : \lambda \text{ is Scott}^* - \text{open}\}$ is a topology on X (called Scott*-topology on *X*).

Proof: Follow from Proposition 2.2. and Theorem 2.4.

Remark 2.2: Each pre-orderd set is a *TRS* satisfies $\forall x X, (\{x\}) \neq \phi$. the converse is not true, so give the following example.

Example 2.4: Let $\leq$ be a binary relation on $X=\{x,y,z\}$ define by $x \leq z$, $y \leq z$ and $z \leq z$. Then $(X,\leq)$ is *TRS* and $\forall x \in X, (\{x\})=\{z\} \neq \phi$. But $(X,\leq)$ is not pre-orderd set because $\leq$ is not reflexive.

Theorem 2.6: Let $(X,\leq)$ be a *TRS* and let $x \in X$. Then $\downarrow x$ is Scott-closed.

Proof: (1) Let $y \downarrow (\downarrow x)$. then $\exists z \downarrow x\ s.t.\ y \leq z$ so that $y \in \downarrow x$. Hence $\downarrow x$ is a lower subset of *X*.

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(2) Let D be a directed subsets of $\downarrow x$ and let $m\in(D)$. Now $x\in ub(D)$ and thus $m\leq x$ i.e., $m\in\downarrow x$. Then $(D)\subseteq\downarrow x$. Hence $\downarrow x$ is d-closed. From (1) and (2) we have $\downarrow x$ is Scott-closed.

Corollary 2.2: Let $(X,\leq)$ be a *TRS* $s.t. \forall\, x\in X, (\{x\})\neq\phi$. Then $\forall x\in X,\ X-\downarrow x\in\ \delta_S$.

Theorem 2.7: Let $(X,\leq)$ be pre-orderd set. Then $\downarrow x = cl_{\delta_S}(\{x\})$.

Proof: From Corollary 2.2. $X-\downarrow x\in\ \delta_S$. Since $x\leq x$, $x\leq x$,

then $cl_{\delta_S}(\{x\}) = \{F\subseteq X : F \text{ is } \delta_S-\text{closed and } \{x\}\subseteq F\}\subseteq\downarrow x\subseteq\{\downarrow F\subseteq X : F \text{ is } \delta_S-\text{closed and} \{x\}\subseteq F\}$

$=\{F\subseteq X : F \text{ is} \delta_S - \text{closed and} \{x\}\subseteq F\} = cl_{\delta_S}(\{x\})$.

The following theorem is a generalization of the corresponding result in Proposition 1.3 (Proposition 3.5.3(3) [1]).

Theorem 2.8: Let F be a finite set in pre-orderd set $(X,\leq)$. Then $X-\downarrow F\in\delta_S$.

Proof: Since $z\in\downarrow F \Leftrightarrow \exists\, x\in F\ s.t.\ z\leq x \Leftrightarrow z\in\{\downarrow x : x\in F\}$. Then On can deduce $\downarrow F=\{\downarrow x : x\in F\}$. From Theorem 2.7. $\forall\, x\in X, \downarrow x$ is Scott-closed. Since the union of finitaly number of *d*-closed is Scott-closed subset, then $\downarrow F$ is Scott-closed subset so that $X-\downarrow F\in\ \delta_S$.

Definition 2.4

Let $(X,\leq)$be a *TRS*. and let (X,δ) be a topological space,Then $(X,\leq,\delta)$ is called a topological *TRS*.

Definition 2.5

Let $(X,\leq,\delta)$ is called a topological *TRS*. The topological space (X,δ) is called transitive- $T_\circ$(*TRS*–*T* for short) iff $\forall x, y\in X,\ s.t.$ either $x\nleq y$ or $y\nleq x, \exists\, u\in\delta\ s.t.\ x\in u,\ y\notin u$ or $y\in u, x\notin u$.

Lemma 2.1: Let $(X,\leq)$ be a *TRS*. and let δ be a topology on X, Then (X,δ) is *TRS* $-T_\circ$ iff $\forall\, x, y\in X,\ s.t.\ \ x\nleq y$ or $y\nleq x$, either $x\in cl_\delta(\{y\})$ or $y\in cl_\delta(\{x\})$.

Proposition 2.3: Let $(X,\leq)$ be a *TRS*. and let δ be a topology on X, Then:

(1) If $\leq$ is antisymmetric and (X,δ) is *TRS* $-T_\circ$ then (X,δ) is T

(2) If $\leq$ is reflexive and $(X,)$ is $T_\circ$ then (X,δ) is *TRS* $-T$.

The following theorem is a generalization of the corresponding result in Theorem 1.1 (Remark 1.4 (ii) [7]).

Theorem 2.9: Let $(X,\leq)$ be pre-orderd set, then (X,δ_S) is *TRS* $-T_\circ$.

Proof: $\forall\, x, y\in X,\ s.t.$ either $x\nleq y$ or $y\nleq x$, Then $x\downarrow y$ or $y\downarrow x$. From Theorem 2.7, $x\in cl_{\delta_S}(\{y\})$ or $y\in cl_{\delta_S}(\{x\})$. Hence from Lemma 2.1, (X,δ_S) is *TRS* $-T$.

Theorem 2.10: Let $(X,\leq)$ be pre-orderd set. Then a specialization pre-ordered relation $\lesssim_{\delta_S}$ induced by $\delta_S\leq$.

Proof: $x\lesssim_{\delta_S} y \Leftrightarrow x\in cl_{\delta_S}(\{y\}) = \downarrow y \Leftrightarrow x\leq y$.

Interactions between a Continuous *TRS* and Its Scott*-Topology

The following theorem is a generalization of the corresponding result in Proposition 1.4 (Theorem 6.1.2 [1]).

Theorem 3.1: (1) Let $(X,)$be a *TRS* and xX is an isolated point, then $\uparrow x$ is Scott-open,

(2) Let $(X,\leq)$be a *TRS* and for some $x\in X$, $\uparrow x$ is d-open subset of X, then x is an isolated point.

Proof: (1) Let D be a directed subset of X and assume that $(D)\cap\uparrow x\neq\phi$. Thus $\exists y\in X\ \ s.t.\ \ y\in\uparrow x$ and $y(D)$. Then $x\in\downarrow(D)$. Since $x<<x$, then $\exists\, d\in D\ s.t.\ d\in\uparrow x$. Hence $D\cap\uparrow x\neq\phi$. Then $\uparrow x$ is d-open. Since $\uparrow x$ is upper subset, then $\uparrow x$ is Scott-open.

(2) Let D be a directed subset of X and $x\downarrow(D)$. Then $\exists\, m\in(D)\ \ s.t.\ x\leq m$ so that $m\in(D)\cap\uparrow x$. Since $\uparrow x$ is d-open, then $D\cap\uparrow x\neq\phi$, *i.e.*, $\exists\, d\in D\ s.t.\ x\leq d$.

Hence $x<<x$.

Corollary 3.1: (1) Let $(X,)$ be a *TRS* $s.t. \forall\, x\in X, (\{x\})\neq\phi$. if $x\in X$ is an isolated point, then $\uparrow x\in\delta_S$; and

(2) If$(X,\leq)$is a pre-orderd set and for some $x\in X$, $\uparrow x\delta_S$ (moreover $_d$) then is an isolated point.

Proposition 3.1: Let $(X,\leq)$ be a *TRS*. Let $x,y,z\in X$. Then:

(1) If $x\leq y$ and $y<<z$, then $x<<z$;

(2) If $x<<y$ and $y\leq z$, then $x<<z$;

(3) If $(\{y\})\phi$ and $x<<y$, then xy;

(4) If$(\{y\})\phi$ or$(\{z\})\neq\phi$, $x<<y$ and $y<<z$ then $x<<z$.

Proof: (1) Let D be a directed subset of X s.t. $z\downarrow(D)$. Then $\exists dD$ *s.t.yd*. Then xd and hence $x<<z$.

(2) Let D be a directed subset of X s.t. $z\downarrow(D)$. Then $\exists\, k\in(D)\ \ s.t.\ z\leq k$. Thus yk and so $y\in\downarrow(D)$. Therefore $\exists l\in D\ \ s.t.\ x\leq l$. hence $x<<z$.

(3) Let D=$\{y\}$ and assume that $x<<y$. Then $\exists\, d\in D\ s.t.\ x\leq d$ but $y=d$ Thus $x\leq y$.

(4) The proof follow directly from (1) and (3) above.

The following theorem is a generalization of the corresponding result in Proposition 1.5 (Proposition 6.7.7 [1]).

Theorem 3.2: Let $(X,\leq)$ be a *TRS*. If '<< ' is interpolative, then $\forall\, x\in X, \Uparrow x$ is Scott-open.

Proof: First let $z\uparrow(\Uparrow x)$. Then $\exists yx$ s.t. yz. From Proposition 3.1.(2) $z\in\Uparrow x$. Hence $\Uparrow x$ is upper subset of X. Second Let, D be a directed subset of X with $\Uparrow x\cap\downarrow(D)\neq\phi$. Then $\exists\, z\leq X\ \ s.t. z\in\downarrow(D)$ (because for each element $z\in(D), z\leq z$) and $z\in\Uparrow x$. So, $\exists\, y\in X\ s.t.\ \ x\ll y\ll z$ and so that $\exists\, d\in D\ s.t.\ y\leq d$. From Proposition 3.1.(2) $d\Uparrow x$. Hence $D\cap\Uparrow x\neq\phi$.

Definition 3.1

$(X,\leq)$is a continuous *TRS* iff $\forall xX$, the following conditions are satisfied:

(1) $(\{x\})\neq\phi$

(2) $\Downarrow x$ be a directed subset of X, and

(3) $x\in\downarrow((\{(\Downarrow a) : a\in\Downarrow x\}))$.

Corollary 3.2: If $(X,\leq)$ is a continuous *TRS*, then $\forall x\in X$, $\Uparrow x\delta_S$.

Theorem 3.3: Let $(X,\leq)$ be a pre-orderd set and let x,yX. If $\exists$O (open set) in δ_S *s.t.* $y\in$O$\subseteq\uparrow x$, then $x<<y$.

Proof: Let D be a directed subset of X *s.t* $y\in\downarrow(D)$. Then $\exists m(D)$

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s.t. $y \leq m$. Since $y \in \uparrow y \uparrow \bigcirc \subseteq \bigcirc$, then $m \in (D) \cap \bigcirc$ *i.e.* $(D) \cap \bigcirc \neq \phi$. Then $\exists\, d \in D$ *s.t.* $d \in \bigcirc \subseteq \uparrow x$, *i.e.* $x \leq d$. Hence $x \ll y$.

The following theorem is a generalization of the corresponding result in Proposition 1.6 (Proposition 6.7.8 [1]).

Theorem 3.4: Let $(X,\leq)$ be a *TRS*. *s.t.* $\forall zX$, $(\{z\}) \neq \phi$ and assume that '<<' is interpolative. If $\forall\, x, y \in X$, *s.t.* $x \ll y$, then $\exists$ a Scott-open subset $\bigcirc$ of X *s.t.* $y \in \bigcirc \subseteq \uparrow x$.

Proof: From Theorem 3.2. $\Uparrow x$ is Scott-open. Let $y \in \Uparrow x$ i.e. $x \ll y$. From Proposition 3.1.(3), $y \in \Uparrow x$. Hence $y \in \Uparrow x \uparrow x$.

Theorem 3.5: Let $(X,\leq)$ be a continuous pre-orderd set. Then $\forall x \in X$, and $\forall \bigcirc \in \delta_S$ with $z \in \bigcirc$ for some $z \in (\Downarrow x)$, $z \in \bigcirc' \in \delta_S$ and $x' \in X$ *s.t.* $y \in \bigcirc' \subseteq \uparrow x' \subseteq \bigcirc$.

Proof: Let $\forall xX$ and let $\bigcirc \in \delta_S$ with $z \in \bigcirc$ for some $z(\Downarrow x)$. Then $(\Downarrow x) \cap \bigcirc \neq \phi$. Thus $\exists\, x' \in \Downarrow x \cap \bigcirc$ so that $\uparrow x' \subseteq \uparrow \bigcirc \subseteq \bigcirc$. From Theorem 3.4. and since $x' \subseteq x$, $\exists\, \bigcirc' \in \delta_S$ *s.t.* $x \in \bigcirc' \subseteq \uparrow x' \subseteq \uparrow \bigcirc \subseteq \bigcirc$.

Theorem 3.6: Let $(X,\leq)$ be domain pre-orderd set. If $\forall x \in X$, and $\forall \bigcirc \in \delta_S$, $\exists\, \bigcirc' \in \delta_S$ and $\exists\, x' \in X$ *s.t.* $x \in \bigcirc' \subseteq \uparrow x' \subseteq \bigcirc$, then $\exists$ a directed subsets D of $\Downarrow x$ *s.t.* $x \in \downarrow (D)$.

Proof: Let $D = \{u \in X : \exists\, \bigcirc_u \in \delta_S$ with $x \in \bigcirc_u \subseteq \uparrow u\}$. From Theorem 3.3, $D \subseteq \Downarrow x$. Since X iteslf is Scott-open and xX, then there are $y \in X$ and $\bigcirc_y \in \delta_S$ *s.t.* $x \in \bigcirc_y \subseteq \uparrow y$. Then $D\phi$. Let $u,v \in D$ *s.t.uv*. Then, there $\exists\, \bigcirc_u, \bigcirc_v \in \delta_S$ *s.t.* $x \in \bigcirc_u \subseteq \uparrow u$ and $x \in \bigcirc_v \subseteq \uparrow v$. Since $x \in \bigcirc_u \cap \bigcirc_v$, there $\exists wX$ and $\bigcirc_w \delta_S$ *s.t.* $x \in \bigcirc_w \subseteq \uparrow w \subseteq \bigcirc_u \cap \bigcirc_v$. Then wD and $w \uparrow u \cap \uparrow v$. So $u \leq w$ and $v \leq w$. Then D is directed subsets of $\Downarrow x$. Now there are $y \in X$ and $\bigcirc_y \in \delta_S$ *s.t.* $x \in \bigcirc_y \subseteq \uparrow y \subseteq \bigcirc$. Thus $yD \cap \bigcirc$. Let $m \in (D)$, then ym. Since $\bigcirc$ is upper subset, then $m\bigcirc$. Now $x \lesssim_{\delta_S} m$. From Theorem 2.11, xm. Hence $x \downarrow (D)$.

The following theorem is a generalization of the corresponding result in Theorem 1.2 (Theorem 6.7.9 [1]).

Theorem 3.7: Let $(X,\leq)$ be domain pre-orderd set. Assume that If $\forall xX$, and $\forall \bigcirc \in \delta_S$, $\exists\, z \in (\Downarrow x) \cap \bigcirc$. Then the following statments are equivalent:

(1) $(X,\leq)$ be a continuous TRS, and

(2) (a) '<<' is interpolative,

(b) $\forall xX$, $\forall \bigcirc \in \delta_S$ with $x \in \bigcirc$ there are $\bigcirc' \in \delta_S$ and $x \in X$ *s.t.* $x \in \bigcirc' \subseteq \uparrow x' \subseteq \bigcirc$.

Proof: Applying Theorem 3.4, Theorem 3.5, Theorem 3.6. the results holds.

An Application on Topological Spaces

From Theorem 2.6, one can assign for any topology δ, a new topology $S(\delta)$ where $S(\delta)$ is Scott-topology induced by the specialization pre-ordered relation $\lesssim_\delta$ induced by δ.

Theorem 4.1: Let (X,δ) be a topological space Then:

(1) $(X,S(\delta))$ be a topological space is *TRS–T.w.r.t*, the pre-ordered relation $\lesssim_\delta$; and

(2) If (X,δ) is $T_\circ$-space, then $(X,S(\delta))$ so is.

Proof: (1) The result is a corollary from Theorem 2.9.

(2) From Proposition 4.3.3 [1], if (X,δ) is $T_\circ$-space, then $\leq\delta$ is a partially ordered relation. So, from Theorem 2.9. and proposition 2.3. one can have the $(X,S(\delta))$ is $T_\circ$-space.

Theorem 4.2: Let $(X,\leq)$be a pre-orderd set. Then $S(_S)=_S$.

Proof: The result follows from Theorem 2.10.

More Topologies on *TRS* and More on Functions between *TRS*

Proposition 5.1: (page 10 [2]). In $(X,)$, let $d \in X$. Then:

(1) $\downarrow d$ is a principle ideal; and

(2) $\uparrow d$ is a principle filter.

Definition 5.1

Let $(X,\leq)$be a *TRS*. $\forall \in X$, $(\{x\}) \neq \phi$. Define topologies:

(1) $\delta_{\lambda|x} = \{\lambda \subseteq X : \lambda \text{ is anuppersubset}\}$ (see Proposition 5.1.) is a topology on X called the Alexandroff topology;

(2) The upper topology on X is induced by δ_α and is the topology generated by subbasis $\{X - \downarrow x : x \in X\}$;

(3) The lower topology on X is denoted by δ_β and is the topology generated by subbasis $\{X - \uparrow x : x \in X\}$;

(4) The interval topology δ_χ on X is the supremum of δ_α and δ_β, i.e $\delta_\chi = \delta_{\alpha\beta}$;

(5) The Lowson topology $\delta_{\beta S}$ on X is the supremum of δ_β and δ_S, i.e $\delta_{\beta S} =_\beta \delta_S$.

Definition 5.2

Let $(X,\leq_1)$and $(X,\leq_2)$ be *TRS* and let $\lambda,\mu \in X$ and let $f:(X,\leq_1) \rightarrow (Y,_2)$ be a function. Then f is monotone iff $f(\lambda) \leq_2 f(\mu)$, whenever $\lambda \leq_1$, and $\mu \leq_1$

Lemma 5.1: If $(X,\leq_1)$ and $(X,\leq_2)$ be are pre-order sets then the function $f:(X,\leq_1) \rightarrow (Y,_2)$ is *TRS* monotone iff it is monotone.

Proof: Obvious.

Theorem 5.1: Let $(X,\leq)$ be pre-orderd set. Then the following statments are hold:

(1) $\delta_\alpha \leq \delta_S$ (2) $\delta_S \leq \delta_{\lambda|x}$ (3) $\delta_S \leq \delta_d$

(4) $\delta_S \leq \delta_{\beta S}$ (5) $\delta_\beta \leq \delta_{\beta S}$ (6) $\delta_\alpha \leq \delta_\chi$

(7) $\delta_\beta \leq \delta_\chi$ (8) $\delta_S = \delta_d \cap \delta_{\lambda|x}$ (9) $\delta_\alpha \leq \delta_{\lambda|x}$

(10) $\delta_\alpha \leq \delta_d$ (11) $\delta_\alpha \leq \delta_{\beta S}$.

In the following Theorem we give a characterization of Alexandroff-continuous between *TRS*.

Theorem 5.2: Let $(X,\leq_1)$ and $(X,\leq_2)$ be *TRS* and let $f:(X,\leq_1) \rightarrow (X,_2)$ be a function. Then f is Alexandroff-continuous iff f is monotone.

Proof: ($\Rightarrow$) Let $\lambda,\mu \in X$ *s.t.* $\lambda \leq_1$. Since $\downarrow f(\mu)$ is a lower set in Y, then $f^1(\downarrow f(\mu))$ is a lower set in X. From the fact $\mu \in f^1(\downarrow f(\mu))$ one can have that $\lambda \in f^1(\downarrow f(\mu))$. Then $f(\lambda) \leq_2 f(\mu)$.

($\Leftarrow$) Let F be a lower set in Y, *i.e.*, an Alexandroff-closed in Y. We need to prove that $f^1(F)$ is a lower set in X. Let $\lambda \in \downarrow f^{-1}(F)$. Then $\exists \mu \in f^1(F)$ s.t., $\lambda \leq_1$. So, $f() \leq_2 f(\mu)$. Since F be a lower set, then $f(\lambda) \in F$, i.e., $\lambda \in f^1(F)$. Hence $f^{-1}(F)$ is Alexandroff-closed subset in X.

Lemma 5.1: Let $(X,\leq_1)$ and $(Y,\leq_2)$ be *TRS* and let $f:(X,\leq_1) \rightarrow (X,_2)$ be a function and let

(1) If f is Scott*-continuous, then f is *TRS*–monotone, and

(2) If for any direcrted subset D of X $s.t._X\sup_X(D)\neq\phi, f(_X(D))=_Y(f(D))$, then f is *TRS*–monotone.

Proof: (1) Let $\lambda,\mu\in X$ $s.t.$ $\lambda\leq_1\mu$. and $\mu\leq_1$. From Theorem 2.6. $\downarrow f(\mu)$ is Scott* -closed so that $f^{-1}(\downarrow f(\mu))$ is a lower set in X. It is dear that $\mu\in f^{-1}(\downarrow f(\mu))$. Then $\lambda\in f^1(\downarrow f(\mu))$ so that $f(\lambda)\leq_2 f(\mu)$.

(2) Let $\lambda,\mu\in X$ $s.t.\lambda\leq_1$ and $\mu\leq_1$. Then $D=\{,\mu\}$ is direcrted subset of X and $\mu\in_X(D)$. So $f(\mu)\in f(X(D))=X(f(D))$. Thus $f(\mu)\geq_2 f(\lambda)$. Hence f is *TRS*–monotone.

From Lemma 5.1 one can have the following Corollary.

Corollary 5.1: Let $(X,\leq_1)$ and $(X,\leq_2)$ be are pre-order sets and let $f:(X,\leq_1)\to(Y,\leq_2)$be the function. Then:

(1) If f is Scott*-continuous, then f is monotone, and

(2) If for any direcrted subset D of X $s.t.\ \mu\in_X \sup_X(D)\neq\phi,\ f(_X(D))=_Y(f(D))$, then f is monotone.

Lemma 5.2: Let $(X,\leq_1)$ and $(X,\leq_2)$ be *TRS* and let $f:(X,\leq_1)\to(Y,_2)$ be a monotone function. If D is directed subset of X, then $f(D)$ is a directed subset of Y

Proof: Let $\lambda_1,\lambda_2\in f(D)\ \ s.t.\ \lambda_1\neq\lambda_2$ then $\exists\ \mu_1,\mu_2\in D.\ s.t.\ \mu_1\neq\mu_2$, and $f(b_1)=a_1$ and $f(\mu_1)=\lambda_1$. Since D is directed, then $c\in D.\ s.t.\ c\geq_1\mu_1$, and $c\geq_{12}$. So, $f(c)\in f(D)$ $s.t.$ $f(c)\geq_2\ \in f(\mu_1)=\lambda_1$ and $f(c)\geq_2\ f(\mu_2)=\lambda_2$. Then $f(D)$ is a directed subset of Y.

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