Dissipative Nonlinear Schrödinger Equations with Singular Data

Hayashi N*, Li C† and Naumkin PI

1Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka 560-0043, Japan
2Department of Mathematics, College of Science, Yanbian University, Goryu Road, Yanji City, Jilin Province, 133002, China
3Centro de Ciencias Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico

Abstract

We consider the long time asymptotics and dissipative nonlinear Schrödinger equations of order 1<\rho<3 with Dirac delta function in one space dimension. 2000 Mathematics Subject Classification: 35Q55, 35B40.

Keywords: Dissipative NLS equations; Large initial data; Long time asymptotics

Introduction

We consider the initial value problem for the nonlinear Schrödinger equations in one space dimension

\[ i\partial_t u + \frac{1}{2} \partial_x^2 u = \lambda |u|^{\rho-1} u, u(0,x) = \phi(x), \]  

(1)

where \( x \in \mathbb{R}, \, \lambda \in \mathbb{R}, \, \lambda = \lambda_1 + i \lambda_2, \, \lambda \in \mathbb{R}, \, j=1,2, \, \lambda_2 < 0, |\lambda_1| > \frac{2}{\sqrt{\rho}} |\lambda_2|, \, 1<\rho<3, \)

\( u(t,x) \) is an unknown complex-valued function. There are some works concerning the physical applications of (1) [1,2]. We note that \( \lambda_2<0 \) implies the dissipation of \( |u(t,x)| \) by nonlinear Ohm’s law [1].

We are interested in the initial data involving the Dirac delta function. Therefore the data are not necessarily in \( L^2 \). Related work can be seen in [3] in which homogeneous weighted \( L^2 \) space was considered. Let \( L^2_\mu \) denote the usual Lebesgue space with the norm \( \| u \|_{L^2_\mu} = \text{ess.sup}_{x} |\phi(x)| \). For \( m, t \in \mathbb{R}, \, \text{weighted homogeneous Sobolev space} \ (s,t) \). \n
Let us introduce some notations. We define the dilation operator by

\[ (D_j \phi)(x) = (\phi(x))^2 \]  

for \( t \geq 0 \) and define \( \mathcal{U} = \mathcal{U}^* \). Evolution operator \( U(t) \) is written as \( U(t) = \mathcal{M} D FM, \) where \( \mathcal{F} \) denotes the Fourier transform. We also have \( U(t) = M^{*} D FM^{*} \), which is \( \mathcal{F} \) is the inverse Fourier transform. We denote by the same letter \( C \) various positive constants. The standard generator of Galilei transformations is given by \( R(t) = U(t)JU^*(t)x + i\partial_t \). We also have commutation relations with \( J \) and \( L = \partial_x + i \partial_t \), such that \( [L,J] = 0 \).

To prove our main result, we introduce the function space:

\[ X_{\mu} = \{ \psi \in \mathcal{S}(\mathbb{R}) \mid \exists \psi_{0} \in L^2_\mu, \psi_{1} \in \mathcal{S}(\mathbb{R}), \psi_{2} \in L^{\infty}_{\mu} \mathcal{S}(\mathbb{R}) \mid \psi(t) = \psi_{0} + \psi_{1} + \psi_{2} \} \]  

with \( \mu = \mu \in (0, \infty) \) and \( m = 1,2 \).

Theorem 1: We assume that \( F \notin \mathcal{S}_{\mu} \). Then the Cauchy problem (1) with \( 0<\epsilon<3 \) has a unique global solution \( u \in X_{\mu} \) satisfying the time decay estimate

\[ \| u(t) \|_{L^2_\mu} \leq C \left( \frac{t}{\epsilon} \right)^{\frac{1}{\rho-1}} \]  

for any \( t>0 \).

Since the solution of the linear problem with the Dirac delta \( \delta \) is given by \( u(t) = -(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} |\cdot|^2} \), we look for the solution of (1) with the Dirac delta \( \delta \) as the initial function in the form

\[ u(t,x) = -(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} |\cdot|^2} \lambda e^{\frac{t}{\rho-1}} C(t), C(0) = e \]  

(2)

We have the ordinary differential equation \( \partial_{t} C(t) = \lambda C(t) e^{\frac{t}{\rho-1}} \) with \( C(0) = e \).

We change \( C(t) = w(t) e^{\frac{t}{\rho-1}} \), then

\[ \frac{d}{dt} w(t) = -\lambda C(t)^{-\frac{1}{\rho-1}} C(t) \]  

which can be solved explicitly as \( w(t) = e^{\frac{t}{\rho-1} \lambda^{-\frac{1}{\rho-1}}} \).

Thus the solution has the form (2) with \( C(t) = w(t) e^{\frac{t}{\rho-1}} \). It is expected that the solution of (1) with data involving the Dirac delta function behaves like (2). Our result says that the upper bound for solutions is the same as given in (2).

Local Existence

In this section we prove the local existence of solutions in \( X_{\mu} \).

We denote the remainder terms \( R_{\theta,\phi} = \mathcal{F} M^{\theta,\phi} \), where \( \mathcal{F} \) is the Fourier transformation.

We consider the long time asymptotics for dissipative nonlinear Schrödinger equations of order 1<\rho<3 with \( \lambda \) such that \( \lambda \in \mathbb{R}, \lambda_2 < 0 \).

Copyright © 2016 Hayashi N. et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

*Corresponding author: Hayashi N, Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka 560-0043, Japan, Tel: +81668505328; E-mail: nhayashi@math.sci.osaka-u.ac.jp

Received April 27, 2016; Accepted April 28, 2016; Published May 02, 2016


Copyright: © 2016 Hayashi N, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
In the same way as in the proof of (6), we have a dissipation for the difference of two solutions, where \( u_0(x) = \phi(x) \), \( u(0,x) = \psi(x) \), \( j=1,2 \). By (10) and (11), we find that there exists a time \( T \) such that the transformation \( u = Sv \) is a contraction mapping from \( X_{tr} \) into itself. This implies the local existence of solutions in \( X_{tr} \).

**Proof of Theorem 1**

Under the assumptions \( \lambda \in [0,1] \), \( \frac{1}{2p+1} \). From the local existence theorem, it is enough to prove the a-priori estimate of \( F(U(-t)) \). Using the representation of (8) in the form \( \psi = e^{r\theta}, r = |\psi| \), \( w = \arg \psi \), then we find

\[
\partial_t F - p\partial_r F - r \rho \leq C \rho^{1-\frac{1}{2p}},
\]

if \( \rho \leq 0 \), since we have \( \| F \| \leq C(1+ |\theta|) \rho^\frac{1}{2p} \) for \( j=1,2 \) by Lemma 1. Define

\[
F(t) = \frac{r(t)}{1 - \frac{2\lambda(p-1)}{3-p} r(t)^{p-1}}.
\]

By a direct calculation we get \( F(t)^{p-1} = \frac{2\lambda(p-1)}{3-p} r(t)^{p-1} - F(t)^{p-1} - F(t) \). Multiplying both sides of the above inequality by \( F(t)^{p-1} \), we obtain

\[
\frac{d}{dt} (F(t)^{p-1}) + \lambda(p-1)(F(t)^{p-1}) \leq C \rho F(t)^{p-1} + C \rho F(t)^{p-1} |\theta|^{-2}.
\]

By the Young inequality \( \rho F(t)^{p-1} \leq \lambda(p-1)(F(t)^{p-1}) + C \rho F(t)^{p-1} + C \rho F(t)^{p-1} |\theta|^{-2} \). Integrating in time, we get

\[
\int r(t)F(t) + C \rho F(t)^{p-1} + C \rho F(t)^{p-1} |\theta|^{-2} dt = F(t) + C \rho F(t) + C \rho F(t)^{p-1} + C \rho F(t)^{p-1} |\theta|^{-2} dt.
\]

since by the definition \( (p(t)^{p-1})^r = r(0) + 2\lambda(p-1)(p-1) \rho \). Let us consider the second term of the right-hand side of (12). We find

\[
(F(t)^{p-1})^{r} = r(0) + 2\lambda(p-1)(p-1) \rho \frac{1}{3-p} (F(t)^{p-1})^\frac{p-1}{p} dt.
\]
for \( p \leq \frac{3}{2} \) \((p-1)>0\) and \( p \geq 2 \) \( \frac{5}{4} \) \( \frac{1}{p-1} \), which is satisfied if \( p > \frac{3}{4} \). Hence \[
\|u(t)\|_{L^{2}} \leq C(t^{\frac{1}{2}}) \left( t \right)^{-\frac{1}{4}}. 
\]

Since \( u(t) = MD \psi + MD \mathcal{F}(M-1) \mathcal{F}^{-1} \psi \), we have \[
\|u(t)\|_{L^{2}} \leq C(t^{\frac{1}{2}}) \left( \|\psi\|_{L^{2}} + C(t^{\frac{1}{2}}) \|\mathcal{F}(M-1) \mathcal{F}^{-1} \psi\|_{L^{2}} \right) \leq C(t^{\frac{1}{2}}) \left( \|\psi\|_{L^{2}} + \|\mathcal{F}(M-1) \mathcal{F}^{-1} \psi\|_{L^{2}} \right). 
\]

This completes the proof of the theorem.

Finally we make a remark. In the same way as in the proof of Theorem 1.2 from [5], we obtain the following result.

**Theorem 2:** Assume that \( \mathcal{F} \phi \in Y_1 \). Then the Cauchy problem (1) with \( \frac{3}{2} < p \leq \frac{5}{4} \) has a unique global solution \( u \in X_2,\infty \) satisfying the time decay estimate

\[
\|u(t)\|_{L^{2}} \leq C(t^{\frac{1}{2}}) \left( t \right)^{-\frac{1}{4}}
\]

for all \( t > 0 \).

**Acknowledgments**

The work of N.H. is partially supported by JSPS KAKENHI Grant Numbers 25220702, 15H03630. The work of C.L. is partially supported by the Education Department of Jilin Province ([2015] No. 34) and NNSFC Grant No.11461074. The work of P.I.N. is partially supported by CONACYT and PAPIIT project IN100616.

**References**


**OMICS International: Publication Benefits & Features**

**Unique features:**
- Increased global visibility of articles through worldwide distribution and indexing
- Showcasing recent research output in a timely and updated manner
- Special issues on the current trends of scientific research

**Special features:**
- 700+ Open Access Journals
- 50,000+ editorial team
- Rapid review process
- Quality and quick editorial, review and publication processing
- Indexing at major indexing services
- Sharing Option: Social Networking Enabled
- Authors, Reviewers and Editors rewarded with online Scientific Credits
- Better discount for your subsequent articles

Submit your manuscript at: www.omicsonline.org/submission