

Eggert's Conjecture for 2-Generated Nilpotent Algebras

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Abstract

Let A be a commutative nilpotent finitely-dimensional algebra over a field F of characteristic $p > 0$. A conjecture of Eggert says that $p \cdot \dim A^{(p)} \leq \dim A$, where $A^{(p)}$ is the subalgebra of A generated by elements a^p , $a \in A$. We show that the conjecture holds if $A^{(p)}$ is at most 2-generated.

Keywords: Nilpotent algebra; Eggert's conjecture; Commutative nilpotent ring; Polynomial bases

Introduction

Let F be a field of characteristic $p > 0$ and A a commutative (associative) nilpotent finite-dimensional algebra over F . Let $A^{(p)}$ be the subalgebra generated by the set $\{a^p \mid a \in A\}$. N. Eggert [1] conjectured that

$$p \cdot \dim A^{(p)} \leq \dim A.$$

This conjecture gives an answer to the problem, when a finite abelian group is isomorphic to the adjoint group of some finite commutative nilpotent F -algebra. Recall that the adjoint group of A is the set A with the operation $x \circ y = x + y + x y$ for every $x, y \in A$.

Validity of this hypothesis would also have influence on an estimation of a (Prüfer) rank of a product of two (abelian) p -groups.

N. Eggert proved his conjecture only when $\dim A^{(p)} \leq 2$. Five years later, R. Bautista [2] proved it when $\dim A^{(p)} = 3$. C. Stack confirmed this results in Stack et al. [3,4], but provided shorter proofs. Finally, Amberg and Kazarin [5] proved the conjecture for the case $\dim A^{(p)} \leq 4$.

Another type of results presented by McLean [6,7]. He showed that this conjecture is true if the algebra A is either radical of a group algebra of a finite abelian group or A is graded and at least one of the following conditions is fulfilled:

- (i) $p = 2$ and $(A^{(p)})^4 = 0$.
- (ii) $A^{(p)}$ is 2-generated.
- (iii) $(A^{(p)})^3 = 0$.
- (iv) $n < 3p$ and $3 \leq s - 1 \leq p$, where n is the number of generators of $A^{(p)}$ and s is the index of nilpotence of $A^{(p)}$.

We also should mention the result of Gorlov [8]. He proved the conjecture for nilpotent algebras A with a metacyclic adjoint group.

One paper concerning Eggert's conjecture appeared in 2002 and the author L. Hammoudi [9] claimed he proved it. But, as Amberg [10] and McLean [7] have shown, his proof was incorrect.

In this short note we sketch out the main steps of the proof that Eggert's conjecture is true if the subalgebra $A^{(p)}$ has at most two generators. For the details, the reader is referred to Korbelaar [11].

Since we will deal with nilpotency and commutativity only, we point out that the word 'algebra' will mean a commutative one and not necessary possessing a unit.

For an algebra A and a subset $X \subseteq A$ we denote $\langle X \rangle$ ($[X]$, resp.) the algebra (vector space, resp.) generated by X .

An algebra A is called nilpotent if $A^m = 0$ for some $m \in \mathbb{N}$.

Through this paper let always F be a field of characteristic $p > 0$ and $R = F[x, y]$ be the ring of polynomials over the variables x, y and the field F .

We start with the remark, that the number of any minimal generating set of a finite generated nilpotent F -algebra A is equal to $\dim A/A^2$. This implies the following:

Lemma 1.1. *Suppose that Eggert's conjecture holds for every nilpotent 2-generated F -algebra. Then it also holds for every nilpotent F -algebra A such that $A^{(p)}$ is a 2-generated F -algebra.*

In the rest we deal with 2-generated nilpotent algebras.

Bases of Nilpotent Algebras

We will use the well-known concept of monomial ordering and standard bases.

For $\alpha = (i, j) \in \mathbb{N}_0^2$ put

$$x^\alpha = x^i y^j \in F[x, y].$$

Denote $[X]_0 = \{x^\alpha \mid \alpha \in \mathbb{N}_0^2\} \cup \{0\}$ the multiplicative monoid with the *lexicographical* ordering \leq such that

$$x^{(i,j)} \leq x^{(i',j')} \Leftrightarrow i < i' \vee (i = i' \wedge j \leq j')$$

and

$$x^{(i,j)} \leq 0$$

for every $(i, j), (i', j') \in \mathbb{N}_0^2$

For $0 \neq f = \sum_{\alpha} \lambda_{\alpha} x^{\alpha} \in F[x, y]$ put

$$m(f) = \min \{x^{\alpha} \mid \lambda_{\alpha} \neq 0\}$$

$$m(0) = 0.$$

Finally, f will be called normal iff $\lambda_{\alpha_0} = 1$, where $m(f) = x^{\alpha_0}$, and $m(f) < \pi x^{\alpha}$ implies $\lambda_{\alpha} = 0$ for every

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$$\alpha \in \mathbb{N}_0^2$$

This function $m: F[x, y] \rightarrow [X]_0$ has common properties of a valuation:

(i) $m(fg) = m(f) + m(g)$.

(ii) $m(f + g) \geq \min\{m(f), m(g)\}$. Moreover, $m(f + g) = m(f)$ if $m(f) < m(g)$.

(iii) $m(f(x^p, y^p)) = m(f)^p$.

for every $f, g \in F[x, y]$.

Finally, a set $\mathcal{X} \subseteq \{x^\alpha \mid \alpha \in \mathbb{N}_0^2\}$ will be called *upper* (*lower*, resp.) if $x^\alpha \in \mathcal{X}$ and $x^\alpha \mid x^\beta$ ($x^\beta \mid x^\alpha$, resp.) implies $x^\beta \in \mathcal{X}$ for every $x^\alpha, x^\beta \in [X]_0$.

Definition 2.1. Let A be a nilpotent F -algebra generated by $\{a_1, a_2\}$. Put

$$C_A(a_1, a_2) = \{u \in [X]_0 \mid (\exists f \in Rx + Ry) m(f) = u \wedge f(a_1, a_2) = 0\}$$

and

$$\mathcal{B}_A(a_1, a_2) = [X]_0 \setminus C_A(a_1, a_2).$$

Proposition 2.2. Let A be a nilpotent F -algebra generated by $\{a_1, a_2\}$. Then:

(i) $C_A(a_1, a_2)$ is an upper set and $0 \in C_A(a_1, a_2)$.

(ii) $\mathcal{B}_A(a_1, a_2)$ is a lower set and $1 \in \mathcal{B}_A(a_1, a_2)$.

(iii) The set $\{x^\alpha(a_1, a_2) \mid 1 \neq x^\alpha \in \mathcal{B}_A(a_1, a_2)\}$ is a basis of A . In particular, $\mathcal{B}_A(a_1, a_2)$ is finite.

(iv) $C_A(a_1, a_2) = \{u \in [X]_0 \mid (\exists f \in Rx + Ry) m(f) = u \wedge f(a_1, a_2) = 0 \wedge f$ is normal $\} \cup \{0\}$.

Definition 2.3. Let A be a nilpotent F -algebra generated by $\{a_1, a_2\}$. Denote

$$n_0 = \#\{x^\alpha \in \mathcal{B}_A(a_1, a_2) \mid \alpha \in \mathbb{N}_0 \times \{0\}\} - 1,$$

$$d_i = \#\{x^\alpha \in \mathcal{B}_A(a_1, a_2) \mid \alpha \in \{i\} \times \mathbb{N}_0\},$$

$$\overline{n}_0 = \#\{x^\alpha \in \mathcal{B}_{A^{(p)}}(a_1^p, a_2^p) \mid \alpha \in \mathbb{N}_0 \times \{0\}\} - 1,$$

$$\overline{d}_i = \#\{x^\alpha \in \mathcal{B}_{A^{(p)}}(a_1^p, a_2^p) \mid \alpha \in \{i\} \times \mathbb{N}_0\}$$

and

$$D_i = \sum_{k=pi}^{pi+p-1} d_k$$

for $i \in \mathbb{N}_0$

Lemma 2.4. Let A be a nilpotent F -algebra generated by $\{a_1, a_2\}$. Then:

(i) $\overline{d} + \overline{d}^- = (a_1^p, a_2^p) = 1 + \dim A^{(p)}$.

(ii) $D_0 + D^- = (a_1, a_2) = 1 + \dim A$.

(iii) The set $\{x^\alpha(a_1^p, a_2^p) \mid 1 \neq x^\alpha \in (a_1^p, a_2^p)\}$ is a basis of $A^{(p)}$.

Eggert's Conjecture for 2-generated Algebras

Let $I \subseteq Rx + Ry$ be an ideal in R such that $A = Rx + Ry/I$ is a non-zero nilpotent F -algebra.

We have $A = \langle x + I, y + I \rangle$ and $A^{(p)} = \langle x^p + I, y^p + I \rangle$.

By definition of $C_A(x + I; y + I)$ there are $f_i \in Rx + Ry$, $0 \leq i \leq n_0 + 1$, such that $m(f_i) = x^{(i, d_i)}$, $f_i \in I$ and f_i are normal.

The main idea of the proof lies in the fact that taking a normal polynomial from I , dividing it by x and then multiplying by some suitable y^k , we get again a member of I (3.3). Then, using binomial formula in a suitable way, we obtain a polynomial that will estimate the number \overline{d}_i (see 3.4 and the definition of $\mathcal{B}_{A^{(p)}}(a_1^p, a_2^p)$).

Lemma 3.1. (i) $f_0 = y^{d_0} - xh_0$, where $h_0 \in R$, and $f_{n_0+1} = x^{n_0}$.

(ii) $xf_i \in Rf_{i+1} + \dots + Rf_{n_0+1}$ for $i = 0, \dots, n_0$.

Definition 3.2. Denote

$$w_A = \max \mathcal{B}_A(x + I, y + I).$$

For $0 \leq i \leq \overline{n}_0$ denote

$$m_i \in \mathbb{N}_0$$

the least integer such that $pi \leq m_i \leq pi + p - 1$ and $d_{pi} \geq \dots \geq d_{mi} = d_{mi+1} = \dots = d_{pi+p-1}$. Put

$$l_i = \left(\sum_{k=pi}^{m_i-1} (d_k - 1) \right) - (p - 1)d_{m_i}.$$

Following lemma is obtained using induction.

Lemma 3.3. Let $1 \leq i \leq n_0 + 1$ and $0 \neq f \in I$ be such that $m(f) = x^i$. Then $y^{d_{i-1}}(f/x) + I \in [w_A + I]$.

The proof of the next proposition uses only the binomial formula. It finds the particular polynomial the we need to make an estimation of the numbers D_i and thus of the dimension of $A^{(p)}$.

Proposition 3.4.

(i) If $0 \leq i < \overline{n}_0$ and $l_i \geq 0$, then $y^{l_i} x^{pi} (f_{m_i} / x^{m_i})^p + I \in [w_A + I]$

(ii) If $0 \leq i < \overline{n}_0$ and $l_i < 0$, then $x^{pi} (f_{m_i} / x^{m_i})^p \in I$

(iii) If $i = \overline{n}_0$, then $y^{D_i-1} x^{pi} + I \in [w_A + I]$.

Now, only exploring carefully the previous cases for i and l_i we get the following interesting claim. It says that the inequality " $p\overline{d}_i \leq D_i$ " holds for almost every i .

Theorem 3.5. One of the following cases takes place:

(i) $p\overline{d}_{\overline{n}_0} \leq D_{\overline{n}_0} + p - 2$ and $p\overline{d}_i \leq D_i + 1$ for every $0 \leq i < \overline{n}_0$. Moreover,

$p\overline{d}_{i_0} = D_{i_0} + 1$ for at most one $0 \leq i_0 < \overline{n}_0$

(ii) $p\overline{d}_{\overline{n}_0} \leq D_{\overline{n}_0} + p - 1$ and $p\overline{d}_i \leq D_i$ for every $0 \leq i < \overline{n}_0$

And our main result is just an easy corollary of this and 1.1.

Theorem 3.6. Let A be a nilpotent F -algebra, $\text{char } F = p > 0$, such that $A^{(p)}$ is 2-generated. Then $p \cdot \dim A^{(p)} = \dim A$.

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