

# Ergodic Theory and the Structure of Non-commutative Space-Time

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## Abstract

We develop further our fibre bundle construct of non-commutative space-time on a Minkowski base space. We assume space-time is non-commutative due to the existence of additional non-commutative algebraic structure at each point  $x$  of space-time, forming a quantum operator 'fibre algebra'  $A(x)$ . This structure then corresponds to the single fibre of a fibre bundle. A gauge group acts on each fibre algebra locally, while a 'section' through this bundle is then a quantum field of the form  $\{A(x); x \in M\}$  with  $M$  the underlying space-time manifold. In addition, we assume a local algebra  $O(D)$  corresponding to the algebra of sections of such a principal fibre bundle with base space a finite and bounded subset of space-time,  $D$ . The algebraic operations of addition and multiplication are assumed defined fibrewise for this algebra of sections.

**Keywords:** Quantum; Algebra; Entropy; Space

## Introduction

We characterise 'ergodic' extremal quantum states of the fibre algebra invariant under the subgroup  $T$  of local translations of space-time of the Poincare group  $P$  in terms of a non-commutative extension of entropy applied to the subgroup  $T$ . We also characterise the existence of  $T$ -invariant states by generalizing to the non-commutative case Kakutani's work on wandering projections. This leads on to a classification of the structure of the local algebra  $O(D)$  by using a 'T-Twisted' equivalence relation, including a full analysis of the  $T$ -type III case. In particular we show that  $O(D)$  is  $T$ -type III if and only if the crossed product algebra  $O(D) \times T$  is type III in the sense of Murray-von Neumann.

## Ergodic Theory in the Classical (Commutative) Case

In the commutative case, a general von Neumann algebra  $R$  is isomorphic to the set  $L^\infty(Z, \nu)$  of essentially bounded, measurable, complex-valued functions on the locally compact set  $Z$  with  $\nu$  a positive regular borel measure. If  $G$  is a group of automorphism of  $R$  then  $G$  is by definition isomorphic to a group of automorphisms of  $L^\infty(Z, \nu)$  which we also denote as  $G$ . Projections  $P$  in the algebra  $R$  become, under the isomorphism, characteristic functions of borel subsets of  $Z$ . If  $g: P \rightarrow Q$  and  $P$  is isomorphic to  $\chi_E$  for the borel set  $E$  then  $Q$  is also a projection thus is isomorphic to  $\chi_F$  for some borel subset  $F$ . Thus the group  $G$  induces a group of transformations of the  $\sigma$ -ring  $B$  of borel subsets of  $Z$ .

Let  $T$  be such a transformation which is measure preserving, then for any borel set  $E$  in  $B$ ,  $\nu(T^{-1}(E)) = \nu(E)$  i.e., the measure  $\nu$  is  $T$ -invariant.  $T$  is defined to be ergodic if it mixes the space. i.e.,  $T^{-1}(E) = E$  modulo a null set implies either  $\nu(E) = 0$  or  $\nu(Z/E) = 0$ . If  $T$  is ergodic in this sense and the measure  $\nu$  is  $T$ -invariant, then  $\nu$  is defined to be an ergodic measure [1]. It then follows that if  $Z$  is compact,  $\nu$  is a probability measure on  $Z$  which is ergodic, and  $T$  and its inverse are continuous mappings, then this is equivalent to  $\nu$  being an extreme point of the invariant measures on  $Z$ . For if  $\mathcal{E}$  denotes the set of such invariant probability measures, and  $\nu$  is an extreme point of  $\mathcal{E}$ , then any measurable set  $E$  with  $0 < \lambda = \nu(E) < 1$  allows the construction of measures  $\mu_1(*) = \frac{1}{\lambda} \nu(*) \cap E$  and  $\mu_2(*) = \frac{1}{1-\lambda} \nu(*) \cap (Z \setminus E)$  such that  $\nu = \lambda \mu_1 + (1-\lambda) \mu_2$ ; a contradiction. Conversely if  $\nu$  is a probability measure on  $Z$  which is ergodic, and  $0 < \mu < \nu$ , then we have, for any Borel set  $A$ ,  $\mu(A) = \int f(x) d\nu(x)$  for some  $f \in L^1(Z, \nu)$  and  $f$  is a  $T$ -invariant, positive function by the properties of probability measures. If  $f$  is not constant

we can define Borel sets  $S_1 = \{x \in Z; f(x) < t\}$  and  $S_2 = \{x \in Z; f(x) > t\}$  for some positive real  $t$ . Both sets are invariant and non-trivial, thus they must both have measure 1; a contradiction. Hence  $f$  is constant and the measure is an extreme point since for any convex combination of measures,  $\nu$  dominates these measures and this leads to the tautology  $\nu = \lambda \nu + (1-\lambda) \nu$  for some  $\lambda: 0 < \lambda < 1$ .

Given a partition  $p$  of  $Z$  into measurable subsets  $\{A_j, j=1,2,3,\dots\}$ , If  $Z$  is compact and has total measure equal to 1 then we can interpret the value  $\mu(A_j)$  as the probability of the set  $A_j$ . The expression  $-\log \mu(A_j)$  is then a measure of the description length or Kolmogorov Complexity of the partition subset  $A_j$ . A measure preserving transformation such as  $T$  transforms the partition  $p$  into the partition  $Tp$  which is  $\{TA_j; j=1,2,3\}$ . The entropy or, equivalently, the expected value of the Kolmogorov Complexity of the partition  $p$  is defined as  $-\sum_j \mu(A_j) \log \mu(A_j)$ . Since  $T$  is measure preserving, the partitions  $p$  and  $Tp$  have the same entropy.

Hopf [2] was interested in the question of when a measure representable as a Lebesgue integral is invariant under a measurable transformation  $T$ . Hopf considers a partition of the measurable set into subsets and the effect of multiples  $T^k$  of  $T$  acting on those subsets. In operator theoretic language we can express this as follows Stormer [3]. We replace measurable sets by projections and the group  $\{T^k; k \in \mathbb{Z}\}$  by a general discrete group  $G$  of automorphisms of a commutative von Neumann algebra  $R$  acting on a Hilbert space  $H$  which is implemented by the unitary representation  $g \rightarrow U_g$  from  $G$  to the set of unitaries acting on  $H$ . Two projections  $H$  and  $K$  in  $R$  are Hopf equivalent if there is an orthogonal family of projections  $E_j$  in  $R$  and group elements  $g_j \in G$  with  $H = \sum E_j$  and  $K = \sum U_{g_j}^* E_j U_{g_j}$ . This equivalence leads to a partition based criterion, 'H-finiteness', for the existence of a finite invariant measure. For this kind of orthogonal partition we can define the entropy as being derived from the partition weightings (all equal in this case).

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With this definition it is then clear that if the two projections  $H$  and  $K$  in  $R$  are Hopf equivalent then they have the same entropy relative to an invariant measure. This idea is easily extended to the noncommutative case, as we will see later.

### Noncommutative (Quantum) ergodic theory

The low energy regime beyond the standard model can be represented, we postulate, by linearised gravity with matter on a flat space-time manifold  $M$ . For this regime, invariance of quantum states to the Poincare group is a key symmetry and new results were presented [4]. Within the resulting fibre bundle construct defined there, our focus is on the local fibre algebra  $A(x)$  and the subgroup  $T$  of the Poincare group consisting of translations of space-time as a group of automorphisms of  $A(x)$ .

If  $A(x)$  is a fibre algebra and the group  $T$  of translations of space-time a subgroup of the Poincare group, we define  $\alpha: g \rightarrow \alpha_g$  to be a representation of  $T$  as automorphisms of  $A(x)$ . Let  $f$  be a faithful normal state of  $A(x)$  and define the 'induced' transformation. Since the  $\nu_g(f) = f \circ \alpha_g$  this means that  $V_g(f)$  is also a normal state and  $V_g(f)(A) = f(\alpha_g(A))$  subgroup  $T$  is abelian, and the mapping  $g \rightarrow \nu(g)$  is a group homomorphism, the set  $\{\nu(g); g \in T\}$  is a continuous group of commuting transformations of the dual space  $A(x)^*$ . If  $f$  is a state of the algebra then define  $E$  to be the weak\* closed convex hull of the set  $\{\nu(g)f; g \in T\}$ . Then  $E$  is a weak\* compact convex set and each  $\nu(g): E \rightarrow E$ . By the Markov-Kakutani fixed point theorem [5] it follows that  $E$  has an invariant element. In other words, the group  $T$  has the fixed point property and thus is amenable. In summary, because  $T$  is an abelian group and locally compact it is an amenable group, since the closed convex hull of any quantum state of the system contains a  $T$ -invariant state [6]. This leads us to define the following;

**Definition:** Let  $A(x)$  be a fibre algebra and the group  $T$  of translations of space-time a subgroup of the Poincare group. Let  $\alpha: g \rightarrow \alpha(g)$  be a representation of  $T$  as automorphisms of  $A(x)$ . The group representation  $\alpha$  acts ergodically on  $A(x)$  if given a projection  $E$  in  $A(x)$ ,  $\alpha_g(E) = E \forall g \in T$  implies that  $E=0$  or  $E=I$ .

This definition is a direct generalisation of the commutative case where  $A(x)$  is the set of essentially bounded measurable functions on a locally compact space with a regular borel measure, discussed above. We can also have the following non-commutative generalisation of an ergodic probability measure as an extreme point of the set of invariant measures; as first pointed out by Segal [7].

**Definition:** Let  $A(x)$  be a fibre algebra and  $\alpha: g \rightarrow \alpha(g)$  a representation of the translation subgroup  $T$  of the Poincare group  $P$  as automorphisms of  $A(x)$ . A quantum state  $f$  of  $A(x)$  is  $\alpha$ -invariant if  $f(\alpha_g(A)) = f(A) \forall A$  in  $A(x)$ . If  $f$  is a normal (i.e., density matrix) state and an extreme point of the set of invariant states, then  $f$  is defined to be a  $T$ -ergodic state.

Note that the set of invariant states is a compact convex subset of the quantum state space of  $A(x)$  and is thus generated by its extreme points [8]. There is a non-trivial invariant state for the amenable group  $T$ , as discussed earlier, thus there is an extremal  $T$ -invariant state of  $A(x)$ . Since, by definition, the Hilbert space representation on which  $A(x)$  acts is separable, the algebra contains a faithful normal state and hence a  $T$ -invariant normal state. The norm limit of a set of normal states is again normal, and thus for a separable Hilbert space, the fibre algebra  $A(x)$  with the assumptions above always contains a  $T$ -ergodic state [9].

**Definition:** The fibre algebra  $A(x)$  is a quantum operator algebra and thus always contains an identity operator  $I$ . If  $f$  is a quantum state of  $A(x)$  then by definition,  $f(I)=1$ . The support of  $f$  is the unique smallest projection  $E$  in  $A(x)$  such that  $f(E)=1$  and is denoted  $E_f$ .

We also require the following definition;

**Definition:**  $E$  is an  $\alpha$ -invariant projection in  $A(x)$  if  $\alpha_g(E)=E \forall g \in T$

The following result is well known in the finite dimensional (matrix algebra) case. For completeness we give a general proof.

**Theorem 2:** Let  $A(x)$  be a fibre algebra and  $g \rightarrow \alpha(g)$  a representation of the translation subgroup  $T$  of the Poincare group  $P$  as (gauge) automorphisms of  $A(x)$ . Assume there exists at least one state  $f$  of  $A(x)$  which is  $\alpha$ -invariant. Then the support of  $f$ ,  $E_f$ , is an invariant projection and  $f$  is a normal and ergodic state if and only if the representation  $\alpha$  acts ergodically on the cut down algebra  $E_f A(x) E_f$ .

**Proof:** We start with the observation that assuming  $f$  is  $\alpha$ -invariant implies that  $f(\alpha_g(E_f)) = f(E_f) = 1 \forall g \in T$ ; we call such states 'symmetric'. By uniqueness of the support of  $f$  it follows that  $E_f$  is an  $\alpha$ -invariant projection. Let  $\pi$  be the Gelfand-Naimark-Segal (GNS) representation of  $A(x)$  induced by the state  $f$  on the Hilbert space  $H(f)$ . We make the simplifying assumption for now that  $f$  is a faithful state; i.e.,  $E_f=I$ , and revisit this assumption later. In this case the von Neumann algebra  $\pi(A(x))$  has a separating-generating vector  $\xi$  and the representation  $\pi$  is a \*-isomorphism. Define the unitary group  $U_g \pi(A) \xi = \pi(\alpha_g(A)) \xi$  on a dense subset of  $H(f)$ , then  $U_g$  extends to a unitary on  $H(f) = \{\pi(B)\xi; B\}$  in the fibre algebra where  $\{\cdot\}$  denotes closure of the set in the norm topology. The mapping  $U: g \rightarrow g$  is then a unitary representation of the translation group  $T$  and for  $B$  a quantum observable in the fibre algebra  $A(x)$  we have  $U_g \pi(B) U_g^* = \pi(\alpha_g(B)) \forall g \in T$  i.e. the unitary representation  $U$  implements the automorphic representation  $\alpha: g \rightarrow \alpha(g)$ .

Consider now the involution mapping on  $\pi(A(x))$  defined as  $A \rightarrow A^*$ . \* This induces an anti-linear mapping on a dense subset of the Hilbert space  $H(f)$ ;  $S: A\xi \rightarrow A^*\xi$ . Moreover, this extends to a mapping with closed graph which we also denote by  $S$ . By the theorem of Tomita-Takesaki  $S$  has a polar decomposition  $S = J\Delta^{\frac{1}{2}}$  such that  $J(A(x))J = \pi(A(x))'$ ; the commutant of the fibre algebra  $\pi(A(x))$  [9]. If  $x=B\xi$  is in the domain of  $S$ , then it follows that  $U_g x$  also lies in the domain of  $S$ , and we have the relationship;

$$U_g S B \xi = U_g B^* \xi = \alpha_g(B^*) \xi = \alpha_g(B)^* \xi = S \alpha_g(B) \xi = S U_g B \xi$$

This leads to the conclusion that, on the domain of  $S$ , we have  $U_g S = S U_g$

Then we have;

$$S = U_g S U_g^* = U_g J \Delta^{\frac{1}{2}} U_g^* = U_g J U_g^* U_g \Delta^{\frac{1}{2}} U_g^*$$

By uniqueness of the polar decomposition,  $J = U_g J U_g^*$ ;  $J$  from this we deduce  $J$  and  $U_g$  commute for all  $g \in T$

$$B \in \pi(A(x)) \cap \{U_g; g \in T\}' \text{ implies that } J B J U_g = U_g J B J \text{ for all } g \in T$$

$$\text{Thus } J B J \in \{U_g; g \in T\}' \cap \pi(A(x))'$$

Conversely if  $C \in \{U_g; g \in T\}' \cap \pi(A(x))'$  then  $C = J B J$  for some  $B \in \pi(A(x))$  and

$$B J U_g = U_g J B J \text{ implies } J B U_g J = J U_g B J \text{ and thus } B U_g = U_g B \text{ so that } B \in \{U_g; g \in T\}'$$

that; We conclude that  $J\left\{\pi(A(x)) \cap \{U_g; g \in T\}\right\}' = \pi(A(x))' \cap \{U_g; g \in T\}'$

The automorphic representation  $\alpha: g \rightarrow \alpha_g$  of  $T$  acts ergodically if and only if  $\pi(A(x)) \cap \{U_g; g \in T\}'$  is trivial, containing only the projections 0 and I and thus consisting of the set of complex multiples of I. From the reasoning above it follows that the representation  $\alpha: g \rightarrow \alpha_g$  of  $T$  acts ergodically if and only if  $\pi(A(x))' \cap \{U_g; g \in T\}'$  invariant is also trivial.

If  $E$  is a projection in the set  $\pi(A(x))' \cap \{U_g; g \in T\}'$  we can define a state  $f_E(A) = \frac{\langle \xi, E\pi(A)\xi \rangle}{\langle \xi, E\xi \rangle}$  on the fibre algebra  $A(x)$ . Then  $f_E = \omega_{E\xi} \circ \pi$  is a state dominated by  $f = \omega_\xi \circ \pi$  and we have;

$$f(A) = \frac{\langle \xi, \pi(A)\xi \rangle}{\langle \xi, \xi \rangle} = \lambda \frac{\langle \xi, E\pi(A)\xi \rangle}{\langle \xi, E\xi \rangle} + (1-\lambda) \frac{\langle \xi, (I-E)\pi(A)\xi \rangle}{\langle \xi, (I-E)\xi \rangle} \text{ for } A \in A(x),$$

where  $\lambda = \frac{\langle \xi, E\xi \rangle}{\langle \xi, \xi \rangle} = \frac{\|E\xi\|^2}{\|\xi\|^2}$  and  $1-\lambda = \frac{\|E\xi\|^2 - \|E\xi\|^2}{\|\xi\|^2} = \frac{\langle \xi, (I-E)\xi \rangle}{\|\xi\|^2}$

Thus  $f$  is an extremal invariant state if and only if the projection  $E=0$  or  $I$ . The result follows for the support of  $f$  equal to 1. Finally, we need to extend the result to a general invariant state  $f$  with support  $E_f$ ,  $0 < E_f < I$ . This follows from what we have already proved, since the restriction of  $f$  to  $E_f A(x) E_f$  is a faithful state, and a state extremal among the invariant states of the cut down algebra  $E_f A(x) E_f$  is also extremal among the invariant states of the full fibre algebra  $A(x)$ . This follows from the fact that if  $f$  is a convex combination of states from the full fibre algebra, then each of them has a support less than or equal to  $E_f$ . In the next section we develop and prove a noncommutative version of a well-known result in classical ergodic theory and use it to characterize the existence of such symmetric states.

## Wandering Projections and Invariant Symmetry States

(Hajian and Kakutani, 1964) defined a wandering set as follows; [10]

**Definition:** Let  $(X, B, \mu)$  be a measure space with finite measure;  $\mu(X) < \infty$  and where  $B$  is the set of all measurable subsets of  $X$ . Let  $T$  be a bijective transformation of  $X$  such that both  $T$  and its inverse are measurable mappings. A wandering set for  $T$  is a measurable subset  $S$  of  $X$  such that the sets  $\{T^{nk}(S)\}$  are disjoint, for some infinite sequence of integers  $nk$ .

**Definition:** Two measures  $\nu$  and  $\mu$  on the measure space  $X$  are said to be equivalent if they share the same null sets. A measure  $\nu$  is  $T$ -invariant if  $\nu(T(E)) = \nu(E)$  for all measurable subsets  $E$  of  $X$ .

With these definitions, showed that there is a finite  $T$ -invariant measure  $\nu$  on  $X$ , equivalent to  $\mu$ , if and only if there are no wandering subsets of  $X$  [10].

If now we consider an abelian von Neumann algebra  $R$ , then  $R$  is isomorphic to  $C(X)$  with  $X$  a compact stonean space of finite measure, and the positive, normal, regular borel measures on  $X$  correspond to the normal states of  $R$ . By Dixmier we can characterise these normal measures as being equivalent to measures which annihilate each nowhere dense subset of  $X$  [11]. It follows that if measures  $\nu$  and  $\mu$  on the measure space  $X$  are equivalent and measure  $\nu$  is normal, then measure  $\mu$  is also normal.

If  $\theta$  is a continuous automorphism of the abelian algebra  $R$ , isomorphic to  $C(X)$ , then we can define the homeomorphism  $T$  of  $X$  by  $\theta f(x) = f(Tx)$  for  $f \in C(X)$ . By the result quoted above, if  $\mu$  is a normal measure on  $X$  with support equal to  $X$ , there is a measure equivalent to  $\mu$  which is  $T$ -invariant if and only if there are no wandering measurable subsets  $E$  of  $X$ .

If such a set  $E$  did exist, such that the sets  $\{T^{nk}(E)\}$  are disjoint, for some infinite sequence of integers  $nk$ , then by regularity of  $\mu$  we can assume that  $E$  is closed. Since  $X$  is a stonean space,  $E$  is both open and closed. Thus the characteristic function  $\chi_E$  corresponds to a projection in the algebra  $R$  and the set of projections  $\theta^{nk}(\chi_E)$  is an orthogonal set. From the algebraic perspective then we can say the following. Given an abelian von Neumann algebra  $R$ , an automorphism  $\theta$  of  $R$  and a faithful normal state acting on  $R$ . Then there is a faithful normal  $\theta$ -invariant state acting on  $R$  if and only if there are no non-trivial projections  $E$  in  $R$  such that for some infinite sequence of integers  $nk$ , the projections  $\theta^{nk}(E)$  are mutually orthogonal. It can be easily shown that for a commutative algebra this condition on the set of projections is equivalent to the requirement that there are no nonzero projections  $E$  with  $\theta^{nk}(E) = 0$  in the ultraweak topology  $nk \rightarrow \infty$  for some infinite sequence  $nk$  of integers. This new formulation now generalises easily to the non-commutative (quantum) case as follows.

**Definition:** Let  $R$  be a von Neumann algebra,  $G$  a group of automorphisms of  $R$ . Then a nontrivial projection  $E$  in  $R$  is wandering if  $E$  is such that;  $g_{nk}(E) \rightarrow 0$  for some infinite sequence  $g_{nk}$  in  $G$ . Convergence is defined in the weak operator topology.

If  $A(x)$  is a fibre algebra then it is a von Neumann algebra with trivial centre and is countably decomposable. Let  $\alpha: g \rightarrow \alpha_g$  be a group representation of the translation subgroup  $T$  of the Poincare group which is ultraweakly continuous.

**Theorem 3:** There is a faithful normal translation invariant quantum state on the fibre algebra  $A(x)$  if and only if there are no wandering projections in  $A(x)$ .

Clearly if  $E$  is a projection in  $A(x)$  such that  $g_{nk}(E) \rightarrow 0$  for some infinite sequence  $g_{nk}$  in  $G$  and  $f$  is a faithful, normal  $\alpha$ -invariant state, then  $f(E) = 0$ , thus  $E = 0$ .

The proof of the converse is based on work by Takesaki on singular states [12]. We assume that there are no wandering projections in  $A(x)$ . The fibre algebra  $A(x)$  has a faithful normal state  $f$ . By the fixed point property, applied to the set;  $E = \text{weak}^* \text{ closed convex hull of } \{v(g)f; g \in T\}$ ,  $A(x)$  has an invariant state which we denote as  $h$ . We need to show that  $h$  is both normal and faithful. By Takesaki  $h$  has a unique decomposition  $h = h_n + h_s$  with  $h_n$  a normal positive linear functional and  $h_s$  a singular positive linear functional [13]. By uniqueness of the decomposition, both of these linear functionals are also  $\alpha$ -invariant. Let  $S$  be the support of  $h_n$  so that  $0 \leq S \leq I$ . If  $S \neq I$  we can choose a projection  $F$  with  $0 < F < I - E$ ,  $h_s(F) = 0$  and  $h_n(F) = 0$  [12]

Let  $\lambda = \inf_{g \in T} (f \circ \alpha_g(F))$ . Since  $h = h_n + h_s$ , we have  $h(F) = 0$ . Therefore  $\lambda = 0$ . Thus there is a sequence  $g_{nk}$  with  $f \circ \alpha_{g_{nk}}(F) \rightarrow 0$ . Since  $f$  is faithful and normal this implies that  $\alpha_{g_{nk}}(F) \rightarrow 0$  in the weak operator topology; i.e.  $F$  is a wandering projection. This contradiction shows that the support of  $h_n$  equals  $I$  and  $h_n$  is the required normal, faithful invariant state

## The Structure of the Local Algebra $O(D)$

For each event point  $x$  in Minkowski space-time, we have a fibre algebra  $A(x)$  defined as a von Neumann algebra with trivial centre and a faithful representation as an algebra of operators acting on a separable Hilbert space. Thus  $O(D)$  is an associative principal fibre subbundle; *associative* in the sense that a Lie group (the translation subgroup of the Poincare group) acts on each fibre; a *subbundle* in the sense that only that subset of  $\{A(x); x \in M\}$  with  $x \in D$  is of physical interest.

The local von Neumann algebra  $O(D)$  does not necessarily have a trivial centre; its structure is more complex in some ways. We assume



that the quantum system it represents has an energy operator with discrete countable eigenstates and we thus assume also that  $O(D)$  is separable. We propose to use the ideas of noncommutative ergodic theory to gain insight into the structure of  $O(D)$ , as we now describe.

Von Neumann introduced the idea of equivalence of measurement ‘projection’ operators as a way of gaining traction on the structure of a general von Neumann algebra [9]. Much of this analysis centres around the question of whether or not the algebra possesses a finite trace, extending the idea of the trace of a finite matrix operator as the sum of its observable eigenvalues. This analysis was enhanced to take account of groups of (unitarily implemented) automorphisms of the algebra by Stormer [3]. This allows him to define a ‘G-equivalence’ of projections which generalises to the non-commutative quantum case the definition used in standard commutative ergodic theory [2]. One of us, extended this work to a characterisation of the tensor product of ‘G-type III’ algebras. As a result of this previous work we can now develop a classification of the structure of our local algebra  $O(D)$  [14]. We do this by applying these earlier results where the group concerned is now the subgroup  $T$  of local translations of space-time of the Poincare group  $P$ .

**Definition:** Let  $\alpha: g \rightarrow \alpha_g$  be a group representation of the translation subgroup  $T$  of the Poincare group as a discrete group acting on the von Neumann algebra  $O(D)$ . A representation  $\pi$  of  $O(D)$  acting on a Hilbert space  $H$  is covariant if there is a homomorphism  $g \rightarrow U_g$  from  $G$  to the group of unitary operators on  $H$  with  $\pi(\alpha_g(A)) = U_g \pi(A) U_g^* \quad \forall A \in O(D)$ .

If  $\varphi$  is a normal state of  $O(D)$  then  $\phi \circ \alpha_g$  is also a normal state since each automorphism preserves the algebraic structure and hence preserves complete additivity. If  $S$  denotes the set of all normal states of  $O(D)$  then the direct sum  $\pi = \bigoplus \{\pi_\varphi; \varphi \in S\}$  of their Gelfand-Naimark-Segal (GNS) representations is a faithful representation of  $O(D)$  as a von Neumann algebra acting on a Hilbert space  $H$  which is the direct sum of the GNS Hilbert spaces. If we define

$U_g \left( \bigoplus_{\varphi \in S} \pi_\varphi(A_\varphi) x_\varphi \right) = \bigoplus_{\varphi \in S} \pi_\varphi(\alpha_g(A_{\varphi \circ \alpha_g}) x_\varphi)$  as a mapping on each of the pre-Hilbert spaces for the GNS constructions, then  $U_g$  extends to a unitary operator on  $H$  and the representation  $\{A_g; g \in T, A_g \in O(D)\}$  is a faithful normal representation of  $O(D)$ . We can therefore assume that  $T$  acting on  $O(D)$  as a discrete group of automorphisms is unitarily implemented, if necessary.

**Definition:** Let  $\alpha: g \rightarrow \alpha_g$  be a group representation of the translation subgroup  $T$  of the Poincare group as a discrete group acting on the von Neumann algebra  $O(D)$ . If  $E$  and  $F$  are projections in  $O(D)$  we say that  $E$  and  $F$  are  $T$ -equivalent if there is a set of operators  $\{A_g; g \in T, A_g \in O(D)\}$  with  $E = \sum_g A_g^* A_g$  and  $F = \sum_g \alpha_g(A_g A_g^*)$ .

**Definition:** We write this  $T$ -equivalence as  $E \approx F$  and call it a  $T$ -twisted equivalence. In the special case that each  $A_g$  is a projection, this definition is a direct non-commutative generalisation of Hopf equivalence.

**Definition:** A projection  $F$  is defined to be  $T$ -finite if  $F$  contains no proper sub-projections which are  $T$ -equivalent to  $F$ . The algebra  $O(D)$  is defined to be  $T$ -finite, or  $T$ -Type II(1), if the identity of  $O(D)$  is a  $T$ -finite projection.  $O(D)$  is  $T$ -semifinite, or  $T$ -Type II( $\infty$ ) if every projection in  $O(D)$  dominates a  $T$ -finite projection.  $O(D)$  is  $T$ -purely infinite, or  $T$ -Type III, if  $O(D)$  does not contain any  $T$ -finite projections.

The  $T$ -type III case is the most difficult to analyse. In the  $T$ -type III

case there is not even the ‘shadow’ of a trace. A  $T$ -invariant trace is a bounded faithful normal linear mapping  $\tau: O(D) \rightarrow \mathbb{C}$  with;

$$\tau(AB) = \tau(\alpha_g(AB)) = \tau(BA) \quad \forall g \in T; A, B \in O(D).$$

If  $\tau$  is a trace, then by the earlier remarks we can assume that the group representation of  $T$ , as a discrete group, is unitarily implemented by the unitary representation  $U: g \rightarrow U_g$  so that  $\tau$  is automatically  $T$ -invariant. Further, if  $F$  is a  $T$ -finite projection and  $E \sim F$  in the sense of Murray and von Neumann then  $E \leq F$  and  $E \sim F$  imply that  $E \leq F$  and  $E \approx F$  (using only the identity of the group). Thus  $E = F$  and  $F$  being  $T$ -finite implies  $F$  is finite.

Stormer established that  $O(D)$  is  $T$ -semifinite if and only if there is a faithful normal semifinite  $T$ -invariant trace on  $O(D)$ .

## The Crossed Product Algebra of $O(D)$

Assume (by taking a faithful representation if necessary) that  $O(D)$  acts on a Hilbert space  $H$ . Define the Dirac function  $\varepsilon_g$  to take the value 1 at  $g$  and zero elsewhere on  $T$ . Then  $\{\varepsilon_g; g \in T\}$  is an orthonormal basis for the Hilbert space  $L^2(T)$ . Given  $L^2(T)$  and  $H$  we can form the tensor product Hilbert space  $H \otimes L^2(T)$ . Define;

$$U_h(x \otimes \varepsilon_g) = x \otimes \varepsilon_{gh^{-1}} \quad \text{for } x \in H, g, h \in T$$

$$A(x \otimes \varepsilon_g) = \alpha_g(A)x \otimes \varepsilon_g \quad \text{for } A \in O(D), g \in T$$

Then  $U_h$  extends to a unitary operator on  $H \otimes L^2(T)$  and the mapping  $h \rightarrow U_h$  is a group homomorphism from the translation group  $T$  into the group of unitaries acting on  $H \otimes L^2(T)$ .

Similarly  $\Phi(A)$  extends to a bounded linear operator on  $H \otimes L^2(T)$  for all  $A$  in  $O(D)$  and the mapping  $h \rightarrow U_h$  implements the automorphic representation  $h \rightarrow \alpha_h: O(D) \rightarrow O(D)$ .

The transformation  $\Phi$  is an ultraweakly continuous  $*$ isomorphism of  $O(D)$  and it follows that  $\Phi(O(D))$  is a von Neumann algebra. Finite sums  $\sum_j U_{g_j} \Phi(A_j)$  form a  $*$ algebra denoted  $(O(D) \times T)_0$  which contains  $\Phi(O(D))$ . The cross product algebra  $O(D) \times T$  is defined as the closure of the  $*$ algebra  $(O(D) \times T)_0$  for the ultraweak operator topology. The crossed product algebra  $O(D) \times T$  can be used to prove the following structural result.

Assume  $O(D1)$  and  $O(D2)$  are local von Neumann algebras in space-time regions  $D1$  and  $D2$  which are not space-like separated. Let  $G$  and  $H$  be discrete representations of the translation subgroup of the Poincare group as automorphisms of  $O(D1)$  and  $O(D2)$  respectively. Then if either  $O(D1)$  or  $O(D2)$  is  $G/H$ -purely infinite ( $G/H$ -Type III), the joint algebra  $O(D1) \otimes O(D2)$  is  $G \times H$ -purely infinite (equivalently  $G \times H$ -type III) under the action of the joint representation  $G \times H$  of the translation group. If both  $O(D1)$  and  $O(D2)$  are  $G/H$  finite or  $G/H$  semifinite, then the same applies to the joint algebra  $O(D1) \otimes O(D2)$ . These results follow from the fact that;  $O(D1) \times G \otimes O(D2) \times H$  is spatially  $*$ isomorphic to  $(O(D1) \otimes O(D2)) \times (G \times H)$  [14].

## A Symmetry of Types

In this part of our analysis of the structure of  $O(D)$  we find a pleasing symmetry for purely infinite type III algebras between the  $T$ -type of  $O(D)$  and the corresponding Murray-von Neumann type of its cross product algebra  $O(D) \times T$ . First we have to prove the following key result. Recall that, by construction, the crossed product von Neumann algebra  $O(D) \times T$  contains the embedded closed sub-algebra  $\Phi(O(D))$ , isomorphic to  $O(D)$ .

**Theorem 4:** There is an ultraweakly continuous mapping, denoted  $\Gamma$ , from  $O(D) \times T$  to  $O(D)$  such that the restriction  $\Gamma|_{\Phi(O(D))} = \Phi^{-1}$ , the inverse of the embedding of the algebra  $O(D)$ ; and the composite map  $\Gamma \circ \Phi : O(D) \times T \rightarrow \Phi(O(D))$  is a continuous projection of norm one.

**Proof:** Continuing with the notation introduced earlier; the map  $x \rightarrow x \otimes \varepsilon_g : H \rightarrow H_g$ , where  $H_g$  is a closed linear subspace of  $H \otimes P(T)$ , is both isometric and linear. Since the set  $\{\varepsilon_g; g \in T\}$  is orthonormal, the Hilbert space  $K = H \otimes P(T)$  is the direct sum of the  $H_g$ 's and every element  $x$  of  $K$  can be represented as  $x = \sum_{g \in T} x_g \otimes \varepsilon_g$  with  $\|x\|^2 = \sum_{g \in T} \|x_g\|^2 < \infty$ .

If  $E_g$  is the projection from  $K$  onto  $H_g$  and  $B = \sum_g U_g \Phi(A_g)$  is an element of  $(O(D) \times T)_0$  then straightforward arguments show that  $E_s B E_t = E_s U_{s^{-1}t} \Phi(A_{s^{-1}t}) E_t$ . Taking the weak closure, we have  $B \in O(D) \times T$  with  $B = \lim \{B^\alpha; E_s B^\alpha E_t = E_s U_{s^{-1}t} \Phi(D_{s^{-1}t}^\alpha) E_t\}$ . From the Kaplansky density theorem we can choose  $B^\alpha$  with  $\|B^\alpha\| \leq \|B\|$  and the net  $D_{s^{-1}t}^\alpha$  is then a bounded net in the ultraweakly compact ball of radius  $\|B\|$  [9]. It thus has a subnet converging to an element  $D_{s^{-1}t}$  of  $O(D)$ . From this we have the following expression;

$$E_s B E_t = E_s U_{s^{-1}t} \Phi(D_{s^{-1}t}) E_t \quad (1)$$

In particular we have  $E_e B E_e = E_e \Phi(D_e) E_e$ . If we define  $\Gamma(B) = D_e$  then clearly the mapping  $\Gamma$  is linear, and  $\Gamma|_{\Phi(O(D))} = \Phi^{-1}$ . Finally if  $B^\alpha \rightarrow B$  ultraweakly then  $E_e B^\alpha E_e \rightarrow E_e B E_e$  and the mapping  $\Gamma$  is ultraweakly continuous.

If  $B$  is in the kernel of  $\Gamma$  then  $D_e = 0$  from equation (1) above this implies that  $E_s B E_s = 0 \forall s$  and thus  $B = 0$ ; the kernel of  $\Gamma$  is  $\{0\}$  and  $\Gamma$  is a faithful mapping. This shows that  $\Gamma$  has the required properties, and completes the proof.

This allows us to now prove the following key structural result.

**Theorem 5:**  $O(D)$  is  $T$ -type III if and only if the crossed product algebra  $O(D) \times T$  is type III in the sense of Murray-von Neumann.

**Proof:** If  $O(D)$  is not  $T$ -type III then it contains a non-trivial  $T$ -finite projection  $E$ . Then it follows that if  $\Phi$  is the identification of  $O(D)$  within the crossed product algebra  $O(D) \times T$  then  $\Phi(E)$  is finite in the sense of Murray-von Neumann. Thus  $O(D) \times T$  is not type III. Conversely assume the crossed product algebra  $O(D) \times T$  is not type III. From Theorem 4 we know that there is a faithful normal projection  $\Gamma$  of norm one from  $O(D) \times T$  onto  $O(D)$ . From Sakai [15] it follows that  $O(D)$  cannot be type III. Thus  $O(D)Z$  is semifinite for some projection  $Z$  in the centre of  $O(D)$ . From Stormer [16,17]  $O(D)Z$  is  $T$ -semifinite thus  $O(D)$  cannot be  $T$ -type III. This completes the proof.

## Conclusion

Through the experiment we explained fibre bundle construct of non-commutative space-time on a Minkowski base space.

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