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Existence and Smoothness of the Navier-Stokes Equation in Two and Three-Dimensional Euclidean Space

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Abstract

A solution to this problem has been unknown for years and the fact that it hasn't been solved yet leaves a lot of unanswered questions regarding Engineering and Pure Mathematics. Turbulence is a specific topic in fluid mechanics which is a vital part of the course when it comes to real life situations. In two and three dimensional systems of equations and some initial conditions, if the smooth solutions exist, they have bounded kinetic energy. In three space dimensions and time, given an initial velocity vector, there exists a velocity field and scalar pressure field which are both smooth and globally defined that solve the Navier-Stokes equations. There are difficulties in two-dimensions and three dimensions in a possible solution and which have been unsolved for a long time and our goal is to propose a solution in three-dimensions. Lets see if we can relate a couple of courses of pure mathematics to come up with an implication.

Keywords: Navier-Stokes equation; Three-dimensional Euclidean space

Introduction

The set of real numbers R^n can also be identified as the n-dimensional Euclidean Space if we wish to emphasize its Euclidean nature. It is mentioned that the Euler and Navier-Stokes equations describe the motion of a fluid in the Euclidean Space R^n, where n could equal 2 or 3 and that these equations are to be solved for an unknown velocity vector $\vec{u}(x,t)=(\vec{u}_i(x,t))_{1\le i\le n}\in\mathbb{R}^n$ and a pressure $p(x,t)$defined for position $x\in R^n$ and time $t \geq 0$. It is also mentioned that we restrict attention here to incompressible fluids filling all of R^n. The Navier-Stokes equations are given by

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{n}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}=\nu\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}+\vec{f}_i(x,t),(x\in\mathbb{R}^n,t\ge0), \tag{1}$$

and the divergence of the velocity field $\vec{u}$ yields

$$div\,\vec{u}=\sum_{i=1}^{n}\frac{\partial\vec{u}_i}{\partial x_i}=0,(x\in\mathbb{R}^n,t\ge0) \tag{2}$$

with initial condition yielding,

$$\vec{u}(x,0)=\vec{u}^o(x)(x\in\mathbb{R}^n). \tag{3}$$

Our given $\vec{u}^o(x)$ is said to be a C^∞ divergence-free *vector field* on R^n and $\vec{f}_i(x,t)$ is the components of our given constant applied force. For example, *gravity* is a continuous force. A constant ν is a positive coefficient for *viscosity*, and

$$\Delta_x=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$$

is the Laplacian in any given space. The *Euler equations* are the previous three with ν set equal to zero. Equation 1 is Newton's Second Law of Motion $f=m\vec{a}$ for a fluid element subjected to the external force $\vec{f}=(\vec{f}_i(x,t))_{1\le i\le n}$ and to the forces being created from pressure and friction.

We can rearrange equation 1 to look like Newton's Second Law of a fluid element such that,

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{n}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}=\nu\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}+\vec{f}_i(x,t)$$

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{n}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-\nu\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=\vec{f}_i(x,t)$$

$$\vec{f}_i(x,t)=\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{n}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-\nu\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}.$$

The unit vector $\vec{u}_j$ can be brought onto the other side of the summation. Since $\Delta=\nabla*\nabla=\nabla^2$ our equation now yields,

$$\vec{f}_i(x,t)=\frac{\partial\vec{u}_i}{\partial t}+\vec{u}_j\sum_{j=1}^{n}\frac{\partial\vec{u}_i}{\partial x_j}-\nu\nabla^2\vec{u}_i+\frac{\partial p}{\partial x_i}.$$

Setting the divergence of the velocity field equal zero will specify that it is the *incompressible continuity equation*. Since we have initial conditions on the velocity field $\vec{u}$, we could possibly yield initial conditions on the force and scalar fields.

Body

The Navier-Stokes equation for an ideal fluid with zero viscosity states that the acceleration is proportional to the *derivative* of *internal pressure*. As a result, the solutions of the Navier-Stokes equation for a given physical problem must be found with the help of *calculus*. One possible way to solving the N.S. equation is to use the *conservation of mass* with boundary conditions in a system of linear or non-linear equations to produce a solution. In wave mechanics, or wave theory, "waves in one-dimension is said to be called *plane waves*. In two-dimensions, the waves are said to be called *cylindrical waves*. In three-dimensions, the waves are said to be called *spherical waves*" [1]. There is an initial velocity vector $\vec{u}^o(x)$ and a divergence-free vector field $\vec{u}_i$ on R^n. The force field $\vec{f}_i(x,t)$ is the component of a given external applied force i.e. *gravity*. The scalar ν is the viscosity and

$$\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$$

is the Laplacian with respect to x in the space variables. The Euler equations are numbers 3, 4, and 5 with the viscosity ν set equal to zero.

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The goal here is to prove letter A in the paper that states solutions of the Navier-Stokes equation exist on R^3.

Lets take $v>0$ and $n=3$. Let $\vec{u}^o(x)$ be *any* smooth, divergence-free vector field satisfying equation 4 stated in the proposal of C. Fefferman. We will also take $\vec{f}_i(x,t)$ to be equal to zero. Then there exists smooth functions $p(x,t)$ and $\vec{u}_i(x,t)$ on R^3 x[0,∞) that satisfy equations 1, 2, 3, 6, and 7. Going back to our equation of motion of a fluid element,

$$\vec{f}_i(x,t)=\frac{\partial\vec{u}_i}{\partial t}+\vec{u}_j\sum_{j=1}^{n}\frac{\partial\vec{u}_i}{\partial x_j}-v\nabla^2\vec{u}_i+\frac{\partial p}{\partial x_i}$$

we let $\vec{f}_i(x,t)=0$ such that,

$$\vec{f}_i(x,t)=\frac{\partial\vec{u}_i}{\partial t}+\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-v\nabla^2\vec{u}_i+\frac{\partial p}{\partial x_i}=0.$$

Since we let $n=3$ we have our unit vector interval $1\le j\le 3$and our $1\le i\le 3$ for the force, velocity vectors, and number of positions in space variables. Thus we have a set of force and velocity vectors. Newton's Second Law states that *The acceleration* $\vec{a}$ *of a body is parallel and directly proportional to a net force* $\vec{f}$ *and inversely proportional to a mass m such that* $\vec{f}=m\vec{a}$. In this case, the acceleration could be defined as the partial derivative of the velocity with respect to time from *i* to *infinity*. Since the acceleration is the derivative of the velocity and in this case we use partial derivatives with respect to time to describe the nature of the fluid such that,

$$\vec{a}=\frac{\partial\vec{u}_i}{\partial t}$$

where $1\le i\le 3$. The net force is inversely proportional to a corresponding mass and velocity field with respect to time $t\ge 0$ which yields,

$$\{\vec{f}_i(x_i,t)\}=\frac{\partial\{\vec{u}_i\}}{\partial t}+\{\vec{u}_j\}\sum_{j=1}^{3}\frac{\partial\{\vec{u}_i\}}{\partial\{x_j\}}-v\nabla^2\{\vec{u}_i\}+\frac{\partial p}{\partial\{x_i\}}=0.$$

Hence, the sequence yields,

$$\vec{f}_1(x_1,t)=\frac{\partial\vec{u}_1}{\partial t}+\vec{u}_1\sum_{j=1}^{3}\frac{\partial\vec{u}_1}{\partial x_1}-v\nabla^2\vec{u}_1+\frac{\partial p}{\partial x_1}=0$$

$$\vec{f}_2(x_2,t)=\frac{\partial\vec{u}_2}{\partial t}+\vec{u}_2\sum_{j=1}^{3}\frac{\partial\vec{u}_2}{\partial x_2}-v\nabla^2\vec{u}_2+\frac{\partial p}{\partial x_2}=0$$

$$\vec{f}_3(x_3,t)=\frac{\partial\vec{u}_3}{\partial t}+\vec{u}_3\sum_{j=1}^{3}\frac{\partial\vec{u}_3}{\partial x_3}-v\nabla^2\vec{u}_3+\frac{\partial p}{\partial x_3}=0.$$

Different types of partial differential equations often need to be matched with different types of boundary conditions in order for their solutions to *exist* and be *unique*.

Suppose that the force field $\vec{f}$ is not equal to zero. Let's set $\vec{f}$ to be arbitrary. Then, we go back to the initial equation derived to be,

$$f_i(x_i,t)=\frac{\partial\vec{u}_i}{\partial t}+\sum_{=1}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial}{\partial x_i} \tag{4}$$

where n equals three. Recall back to Newton's Second Law of Motion, $F=ma$. Now, let's translate this equation into partial derivative terms. We have the acceleration as $\frac{\partial\vec{u}_i}{\partial t}$ with out mass m to equal the force field as follows,

$$\vec{f}_i(x_i,t)=m\frac{\partial\vec{u}_i}{\partial t}.$$

Using the equation of force above and solving for the acceleration, we get

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=m[v\Delta\vec{u}_i-\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_i}-\frac{\partial p}{\partial x_i}+\vec{f}_i(x_i,t)].$$

After plugging in the given functions, we distribute mass inside to obtain

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=mv\Delta\vec{u}_i-m\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-m\frac{\partial p}{\partial x_i}+m\vec{f}_i(x_i,t). \tag{5}$$

We substitute the force field back in using equation 4, we get

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=mv\Delta\vec{u}_i-m\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-m\frac{\partial p}{\partial x_i}+m[\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}].$$

After another distribution of the mass into force, it becomes

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=mv\Delta\vec{u}_i-m\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-m\frac{\partial p}{\partial x_i}+m\frac{\partial\vec{u}_i}{\partial t}+m\sum_{j=1}^{3}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}-mv\Delta\vec{u}_i+m\frac{\partial p}{\partial x_i}.$$

Next, algebraic manipulation is necessary to cancel out all like terms to represent,

$$\frac{\partial\vec{u}_i}{\partial t}+\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=m\frac{\partial\vec{u}_i}{\partial t}$$

$$\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=m\frac{\partial\vec{u}_i}{\partial t}-\frac{\partial\vec{u}_i}{\partial t}$$

$$\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=\frac{\partial\vec{u}_i}{\partial t}(m-1). \tag{6}$$

I will put this into a scenario for the application of waves. Recall throwing a rock in a pond and recognize the ripple effect. I see those as mini waves ripping across the pond. Look at the rock as force into the water with created waves as a result. There could be a pressure developed as a result of a hand toss.

Let's take a look at the the equation would look like if we set $\vec{f}_i(x_i,t)$ equal to zero as given. Refer back to equation five. If we set the force field equal to zero, we would get

$$\frac{\partial\vec{u}_i}{\partial t}+\vec{u}_j\sum_{\partial\vec{u}_i}^{\partial x_j}-v\Delta\vec{u}_i+\frac{\partial p}{\partial x_i}=mv\Delta\vec{u}_i-m\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-m\frac{\partial p}{\partial x_i}$$

$$f_i(x_i,t)=mv\Delta\vec{u}_i-m\vec{u}_j\sum_{=1}\frac{\partial}{\partial x_j}-m\frac{\partial}{\partial x_i}$$

$$0=m[v\Delta\vec{u}_i-\vec{u}_j\sum_{j=1}^{3}\frac{\partial\vec{u}_i}{\partial x_j}-\frac{\partial p}{\partial x_i}].$$

The reason why I decided to go through these derivations is because of the application mentioned above. Let us continue with the scenario of waves. Now, recall the equation of a tangent line from algebra. Let a two-dimensional wave be shaped like a bell curve similar to the normal distribution curve. Since we have a curve there exists a tangent line on all sides of the curve. Imagine the tangents keeping the shape of the wave as the force from the rock penetrates the water.

Now, we can define equations of tangent planes over a three-dimensional wave on a graph. You can use a mathematical program to visualize what I mean and verify. Recall from Calculus, the equation of tangent plane in two and three space variables such as

$$\vec{f}_i(x_i,y_i)=\frac{\partial\vec{f}_i}{\partial x_i}(x-x_0)+\frac{\partial\vec{f}_i}{\partial y_i}(y-y_0)$$

at some initial point (x_0,y_0). Then, the three-dimensional tangent plane yields

$$\vec{f}_i(x_i,y_i,z_i)=\frac{\partial\vec{f}_i}{\partial x_i}(x-x_0)+\frac{\partial\vec{f}_i}{\partial y_i}(y-y_0)+\frac{\partial\vec{f}_i}{\partial z_i}(z-z_0). \tag{7}$$

at some initial point (x_0,y_0,z_0). What if the partial derivative of any function $\vec{f}_i$ is applied to see if there are infinitely many derivatives implying an infinite force vector field? The force field may not be infinite when we think of the wave application used earlier. Figuratively, we have a derived equation of a tangent plane to look something like

$$\frac{\partial \vec{f}_i}{\partial x_i} = \frac{\partial^2 \vec{f}_i}{\partial x_i^2}(x-x_0) + \frac{\partial^2 \vec{f}_i}{\partial y_i^2}(y-y_0) + \frac{\partial^2 \vec{f}_i}{\partial z_i^2}(z-z_0).$$

What would happen if we referred back to equation four and implemented derivatives with respect to space variables and time. Mathematically, it may look like,

$$\vec{f}_i(x_i,t) = \frac{\partial \vec{u}_i}{\partial t} + \sum_{j=1}^{3} \vec{u}_j \frac{\partial \vec{u}_i}{\partial x_j} - v\Delta \vec{u}_i + \frac{\partial p}{\partial x_i}$$

$$\frac{\partial \vec{f}_i}{\partial x_i} = \vec{u}_j \sum_{j=1}^{3} \frac{\partial^2 \vec{u}_j}{\partial x_j^2} - v\frac{\partial^3 \vec{u}_i}{\partial x_i^3} + \frac{\partial^2 p}{\partial^2 x_i^2}$$

$$= \vec{u}_j \Delta \vec{u}_j - v\frac{\partial^3 \vec{u}_i}{\partial x_i^3} + \frac{\partial^2 p}{\partial^2 x_i^2}. \tag{8}$$

Deriving this equation once more and so forth may imply infinite derivatives towards the divergence free vector field.

Existence

"Consider a quantity of fluid contained in a plane region *S bounded* by a simple closed curve ∂S. The *Law of Conservation of Mass* states that the rate at which fluid pours across ∂S into S must *balance* the rate at which the total amount of fluid in S *increases with time*" [2]. "An existence theorem for our initial-boundary value problem for the wave equation can be proved under a more restrictive hypothesis on the *smoothness* of the initial data" [3]. "Consider the PDE,

$$\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_j \frac{\partial \vec{u}_i}{\partial x_i} = K\vec{u}_i$$

where we have our given domain to be $-\infty<x_i<\infty$, $t>0$, $i=1,2\ldots m$ and C_∞, K are constants. Solving the characteristic equations and the compatibility conditions

$$\frac{dx}{dt} = \vec{u}_j, \frac{d\vec{u}}{dt} = K\vec{u}.$$

The initial manifold t=0is non-characteristic and we will select the initial state of the system described by the function $\vec{u}(x_\alpha,t)$ by the *Cauchy* data

$$\vec{u}(x_i,0) = \vec{u}(x_i), \infty < x_i < \infty\text{"}$$

Consider the equation

$$\sum_{i=1}^{3} \frac{\partial^2 \vec{u}_i}{\partial x_i^2} - \frac{1}{v^2}\frac{\partial^2 \vec{u}_i}{\partial t^2} = 0.$$

Theorem 1 *If* $\vec{u}_1(x) \in C^2$ *and* $\vec{u}_1(x) \in C^2$ *in* $-\infty<x<\infty$, $i=1,2,3$ *then the function*

$$\vec{u}(x,t) = tM(t)\vec{u}_1 + \frac{\partial}{\partial t}(tM(t)\vec{u}_0)$$

where

$$M(t)\vec{u}_i = \frac{1}{4\pi}\int_{\|v\|=1} \vec{u}_i(x+vt)d\omega, i=0,1$$

belongs to C^2 in $-\infty<(x,t)<\infty$, and is a solution of the *Cauchy* problem

$$\Delta \vec{u} - \frac{1}{v^2}\frac{\partial^2 \vec{u}}{\partial t^2} = 0$$

$$\vec{u}(x,0) = \vec{u}_0(x)$$

$$\frac{\partial \vec{u}}{\partial t}(x,0) = \vec{u}_1(x).$$

$M(t)\vec{u}_i$ denotes the mean value of $\vec{u}_i$ over the sphere with center at x and radius vt in three-dimensional space.

Proof. "Assuming that given $\vec{u}(x,t)$ holds, we first verify whether the initial conditions are satisfied:

$$\vec{u}(x,0) = M(0)\vec{u}_0$$

$$= \frac{1}{4\pi}\int_{\|v\|=1} \vec{u}_0(x)d\omega$$

$$= \vec{u}_0(x)$$

where $M_t(t)\vec{u}_i = \frac{\partial}{\partial t}M(t)\vec{u}_i$ so that

$$M_t(0)\vec{u}_0 = \frac{1}{4\pi}v\sum_{i=1}^{3}\int_{|v|=1} \vec{u}_{0\alpha i}(x)v_i d\omega$$

$$= \frac{1}{4\pi}v\sum_{i=1}^{3}\vec{u}_{0\alpha i}(x)\int_{|v|=1} v_i d\omega$$

$$= 0.(fromTheGaussIntegralTheorem)$$

It follows that both conditions satisfy the wave equation as well as the partial derivative with respect to t of $tM(t)\vec{u}_0$. Therefore, the given $\vec{u}(x,t)$ is a solution of the three-dimensional wave equation satisfying the given conditions" [4].

"In the *Goursat* problem the data specified on two interesting non-characteristic curves strictly contained in an angle between two characteristics passing through the point of intersection of the curves. Without loss of generality, we can take the point of their intersection to be the origin. Now let,

$$f(x) = F(x) + G(x)$$

$$g(\alpha) = F(\alpha(1+\beta)) + G(\alpha(1-\beta))$$

$$x = 0 and \beta x = vt, 0 < \beta < 1.$$

It is not easy to determine the functions F and G from these relations. From the given data above

$$G\{(1+\beta)x\} - G\{(1-\beta)x\} = f\{(1+\beta)x\} - g(x).$$

Let

$$x = \frac{x}{1+\beta}, 0 < \delta - \frac{1-\beta}{1+\beta} < 1, p(x) = g(\frac{x}{1+\beta}) - f(x),$$

then

$$p(x) = -G(x) + G(\delta x).$$

It follows that

$$p(\delta x) = -G(\delta x) + G(\delta^2 x)$$

and so on. Therefore

$$\sum_{i=0}^{n} p(\delta^i x) = -G(x) + G(\delta^{n+1}x).$$

Since G is continuous and 0<δ<1, letting n tend to infinity we get

$$G(x) = G(0) - \sum_{i=0}^{\infty} p(\delta^i x)$$

provided $\sum_{i=0}^{\infty} p(\delta^i x)$ exists. Using the initial data $f(x)= F(x)+G(x)$ we can now find $F(x)$. Hence, the solution is given by

$$\vec{u}(x,t) = f(x+t) + \sum_{i=0}^{\infty} p\{\delta^i(x+vt)\} - \sum_{i=0}^{\infty} p\{\delta^i(x-vt)\}.$$

The functions f and g must be such that $\sum_{i=0}^{\infty} p(\delta^1 x)$ *converges* in order for the solution to be valid. It is also unique. The region of determinacy when $f(x)$ and $g(\alpha)$ are specified for the bounds$0 \le x \le a$ and $0 \le \alpha \le b$, is the region bounded by the characteristics through $(a,0), (b,\frac{b\beta}{c})$, and the given segments on $t=0$ and $\beta x=vt$. In this case, the problem is *well-posed*. The solution $\vec{u}$ is given in terms of f and g which when equal to zero, imply that $\vec{u}$ is also zero. If *f* and *g* are arbitrarily small, then $\vec{u}$ is also the same way. Hence, the solutions are unique and *stable* therefore, the problem is well-posed" [4].

"There are equations of a viscous incompressible fluid that are called *stationary* that yield,

$$\vec{u}_1\frac{\partial\vec{u}_1}{\partial x}+\vec{u}_2\frac{\partial\vec{u}_1}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\Delta\vec{u}_1,$$

$$\vec{u}_1\frac{\partial\vec{u}_2}{\partial x}+\vec{u}_2\frac{\partial\vec{u}_2}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\Delta\vec{u}_2,$$

$$\frac{\partial\vec{u}_1}{\partial x}+\frac{\partial\vec{u}_2}{\partial y}=0$$

can be reduced to an equation in question by defining a *stream function w* such that $\vec{u}_1=\frac{\partial w}{\partial y}$ and $\vec{u}_2=-\frac{\partial w}{\partial x}$ followed by the elimination of the pressure *p* from the first two equations" [5,6]. "Going back to the system of stationary *hydrodynamic* equations we add *F*(*y*) to the first equation such that,

$$\vec{u}_1\frac{\partial\vec{u}_1}{\partial x}+\vec{u}_2\frac{\partial\vec{u}_1}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\Delta\vec{u}_1+F(y).$$

The above equation along with the other two equations could describe the *plane flow* of a viscous incompressible fluid under the *action of a transverse force*. Then,

$$f(y)=\frac{\partial F}{\partial y}.$$

Letting *F*(*y*)=*asin*(*λy*) corresponds to *A.N. Kolmogorov's* model, which is *used for describing sub-critical and transitional* (laminar-to-turbulent) flow" [7]. Again, we yield the equations of a viscous incompressible fluid. Only this time we make the equations "*non-stationary* such that,

$$\frac{\partial\vec{u}_1}{\partial t}+\vec{u}_1\frac{\partial\vec{u}_1}{\partial x}+\vec{u}_2\frac{\partial\vec{u}_1}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\Delta\vec{u}_1,$$

$$\frac{\partial\vec{u}_2}{\partial t}+\vec{u}_1\frac{\partial\vec{u}_1}{\partial x}+\vec{u}_2\frac{\partial\vec{u}_1}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\Delta\vec{u}_1,$$

$$\frac{\partial\vec{u}_1}{\partial x}+\frac{\partial\vec{u}_2}{\partial y}=2a,$$

Describing the motion of a viscous incompressible fluid by two parallel disks moving towards each other is reduced to the given equation. In this case, *a* is the relative velocity of the disks which can also be denoted as $v_r e$ *l*while $\vec{u}_1$ and $\vec{u}_2$ are the horizontal velocity components and $\vec{u}_3=-2az$ is the vertical velocity component. Then a new stream function is defined such that $\vec{u}_1=ax+\frac{\partial w}{\partial y}$ and $\vec{u}_2=ay-\frac{\partial w}{\partial x}$ followed by the elimination of the pressure *p* leading to the equation we are looking for" [5]. "We shall derive a formula expressing the *Law of Conservation of Momentum* in the *x*-direction (one-dimension),(*x*,*y*)-directions (two-dimensions), and the (*x*,*y*,*z*)directions (three-dimensions) "when not only viscosity but also external fields of force, such as gravity are neglected" [2]. The horizontal component of force exerted on a small section *S* of our fluid by the *scalar pressure p a*long it boundary ∂*S* is given by the integral,

$$-\int_{\partial S}pdy=-\iint_S\frac{\partial p}{\partial x}dxdy.$$

Since the velocity field is $dx/dt=\vec{u}$ in one-dimension, the horizontal acceleration d^2x/dt^2 of this particle is equal to

$$\frac{d\vec{u}}{dt}=\frac{\partial\vec{u}}{\partial t}+\vec{u}\frac{\partial\vec{u}}{\partial x}+\vec{v}\frac{\partial\vec{u}}{\partial y}$$

in two-dimensions respectively such that

$$\frac{\partial\vec{u}_i}{\partial t}+\sum_{j=1}^{n}\vec{u}_j\frac{\partial\vec{u}_i}{\partial x_j}=v\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}+\vec{f}_i(x_i,t)$$

$$\frac{d\vec{u}}{dt}=v\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}+\vec{f}_i(x_i,t)$$

and since we let $f_i(x,t)=0$ in the beginning we have now,

$$\frac{d\vec{u}}{dt}=v\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}.$$

In one-dimensional flow, *Euler's equations of motion* reduce to,

$$\frac{\partial\vec{u}}{\partial t}+\vec{u}\frac{\partial\vec{u}}{\partial x}=-\frac{\partial p}{\partial x}.$$

Then in two and three-dimensions, the Euler's equation will yield,

$$\frac{\partial\vec{u}}{\partial t}+\hat{u}\frac{\partial\vec{u}}{\partial x}+\hat{v}\frac{\partial\vec{u}}{\partial y}=-\frac{\partial p}{\partial x\partial y}$$

and

$$\frac{\partial\vec{u}}{\partial t}+\hat{u}\frac{\partial\vec{u}}{\partial x}+\hat{v}\frac{\partial\vec{u}}{\partial y}+\hat{w}\frac{\partial\vec{u}}{\partial z}=-\frac{\partial p}{\partial x\partial y\partial z}$$

Where *u*,*v* and *w* are the unit vectors in the specified directions.

Theorem 2 *"The solution of*

$$\frac{\partial^2\vec{u}}{\partial t^2}-v^2\frac{\partial^2\vec{u}}{\partial x^2}=\vec{f}(x,t)$$

subject to the *boundary conditions*

$$\vec{u}(a,t)=\vec{u}(b,t)=0$$

$$\vec{u}(x,0)=g_1(x)$$

$$\frac{\partial\vec{u}}{\partial t}(x,0)=g_2(x)$$

is

$$\vec{u}(x,t)=\int_0^t\int_a^b G(x,t,\eta,\tau)\vec{f}(\eta,\tau)d\eta d\tau+$$

$$\int_a^b[g_2(\eta)G(x,t,\eta,0)-g_1(\eta)\frac{\partial G}{\partial\tau}(x,t,\eta,0)]dx,$$

Where(*a*,*b*) is(-∞,∞) and *G* is *Green's function* for the wave equation" [8].

The proof is located in the referenced book above. As we know already "mathematically, the partial differential equation to be solved is non-linear and is of *fourth order*, with two, three, or even four independent variables. With these numerical techniques they tend to require very large computer time, tend to lack accuracy due to the non-linearity, and tend to be unstable" [9]. It is said that "when a body moves through a viscous fluid, the Navier-Stokes equations are satisfactorily approximated by the *boundary layer equations* in a narrow region adjacent to the body" [9].

Smoothness

The solution to the three-dimensional wave equation given by $\vec{u}(x,t)$ is of class C^2 for t ≥ 0 when $\vec{u}_0\in C^3(\mathbb{R}^3)$ and $\vec{u}_1\in C^2(\mathbb{R}^3)$. Therefore the solution can be less smooth than the data. There is a possible loss of one derivative. This loss could be due to what happens for *m*>1, where $\vec{u}(x_1,x_2,...,x_m,t)$ in -space variables. For *m*=1the solution is *smooth* for all *t* as the initial data at *t=0.* The solution of $\vec{u}(x,t)$ of the three-dimensional wave equation given in Theorem 1 depends on the values of $\vec{u}_0$, $\vec{u}_1''$, and the first derivatives of $\vec{u}_0$ on the surface of the sphere of center *x* and radius *vt*.If $\vec{u}_0$ and $\vec{u}_1$ have support in a *closed bounded region* Ω of R^3, for example, if they are oth zero outside of Ω, then at t>0 $\vec{u}(x,t) \neq 0$ at those points x which lie on a sphere of radius vt and centered at a point y∈Ω and $x \in S_{y,yt}$ for some y∈Ω. $S_{y,yt}$ is the sphere with respect to point y and radius vt. We begin to learn about the development of shock waves from the initial-value problem for $\vec{u}(x,t)$

$$\frac{\partial \vec{u}}{\partial t} + p(\vec{u})\frac{\partial \vec{u}}{\partial x} = 0, -\infty < x < \infty, t > 0,$$

$$\vec{u}(x,0) = \vec{u}_0(x), -\infty < x < \infty,$$

where $p(\vec{u})$ and $\vec{u}(x)$ are $C^1(\mathbb{R})$ functions of their are arguments, that is, they are *smooth functions*. There are characteristic equations that correspond with the equation above which yield,

$$\frac{dt}{1} = \frac{dx}{p(\vec{u})} = \frac{d\vec{u}}{0}.$$

These equations imply that,

$$\frac{d\vec{u}}{dt} = 0 and \frac{dx}{dt} = p(\vec{u}).$$

The solution of $dx/dt = p(\vec{u})$ represents characteristics of the first above equation along the condition that,

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{u}}{\partial t} = p(u)\frac{\partial \vec{u}}{\partial x} + \frac{\partial \vec{u}}{\partial t} = 0.$$

"The condition means that $\vec{u}$ is *constant* on the characteristics which propagate with *speed* $p(\vec{u})$. The dependence of p and $\vec{u}$ produces a *gradual non-linear distortion* of the wave profile as it propagates. It follows that $p(\vec{u})$ is also *constant* on the characteristics, and therefore must be straight lines in the (x,t)-plane with a constant slope of $1/p(\vec{u})$." [5] "If there are two points(ξ,0) and(η,0) withξ<η then the characteristics starting at (ξ,0) and (η,0) will intersect at a *pressure point p(x,t)* for t>0. "At the *point of intersection p(x,t)*, the solution of $\vec{u}(x,t)$ has two different values $\vec{u}(\xi)$ and $\vec{u}(\eta)$. This means that $\vec{u}$ is double valued, and hence, the solution is *not unique* at the point of intersection *of the characteristics*. Thus, the solution *must* be *discontinuous* at the point of intersection. The result is that if no two characteristic lines intersect in the *half plane* t>0, *there exists a solution of the initial-value problem* as a *differentiable function* for all t>0. This can happen only if the reciprocal of the slope $p(\vec{u})$ is an *increasing function* of the intercept. In other words, the family of characteristics spreads only for t>0 and generates a solution of the problem that is at least *as smooth as* $\vec{u}(x)$. Such as solution is called an *expansive* or a *refractive wave*" [5]. "Let the periodic boundary conditions yield,

$$\vec{u}(0,t) = \vec{u}(1,t) = 0 \, for \, t > 0$$

and

$$\frac{\partial \vec{u}}{\partial x}(0,t) = \frac{\partial \vec{u}}{\partial x}(1,t) = 0 \, for \, t > 0$$

and the initial condition be

$$\vec{u}(x,0) = \vec{u}(x) \forall x \in \mathbb{R}^n.$$

"We assume that this initial-boundary problem for an equation possesses a *smooth function* which is uniquely determined by the initial data $\vec{u}$. Two invariants of the problem which are constants of the motion are given [10].

$$I(t) = \int_{\mathbb{R}^n} |\vec{u}(x,t)|^2 \, dx.$$

We consider *any* function $\vec{u}(x,t)$, not necessarily a solution of a certain equation, which is defined for t>0 and sufficiently smooth. We will define a *strict solution* for the initial-boundary value problem to be a function $\vec{u}(x,t)$. This velocity is continuous together with its first and second-order derivatives and satisfies a specific PDE for -∞<x<∞, t>0. The initial and boundary conditions are satisfied in the sense of *equality*. We assume that $\vec{u}(x,t)$ satisfies the given boundary conditions and evolves in time, so that function *I(t)* is a constant in time. A solution $\vec{u}(x,t)$ is said to be *Lagrangian stable* if there exists a constant C independent of t, but at the same time could be dependent on initial data such that,

$$|\vec{u}(x,t)|_\infty^2 = \max_{x \in \mathbb{R}^n} |\vec{u}(x,t)|^2 \leq C, forallt \geq 0.$$

"A *strict solution* is one of showing the continuity of $\vec{u}(x,t)$ which implies that,

$$\lim_{t \to 0} \vec{u}(x,t) = \vec{u}(x,0) = \vec{u}(x)$$

uniformly for -∞<x<∞ and this becomes

$$\lim_{t \to 0} \int_{-\infty}^{\infty} \vec{u}(x,t)dx = \int_{-\infty}^{\infty} \vec{u}(x,0)dx = \int_{-\infty}^{\infty} \vec{u}(x)dx$$

where the initial conditions yield

$$\vec{u}(x,0) = \vec{u}(x)$$

$$\frac{\partial \vec{u}}{\partial t}(x,0) = 0.$$

We have our given velocity field $\vec{u}$ and let our scalar pressure field be *p(x,t)* be equal to a function $g_i(x,t)$. "Let $\vec{u}(x,t)$ be *generalized solution* of the initial conditions if there is a *sequence* of *strict solutions* $\{\vec{u}_i(x,t)\}$ with $\vec{u}_i(x)$,

$p_i(x,t) = g_i(x,t)$ such that

$$\lim_{i \to \infty} \vec{f}_i(x,t) = \vec{f}(x,t), \lim_{i \to \infty} g_i(x,t) = p(x,t)$$

uniformly for -∞<x<∞, and

$$\lim_{i \to \infty} \vec{u}_i(x,t) = \vec{u}(x,t),$$

uniformly for -∞<x<∞ and t ≥ 0" [8]. Take the solution of the *plucked string* problem to be generalized as well. The *i-th* partial sum yields,

$$\vec{u}_i(x,t) = \sum_{i=1}^{3} b_i cos(i\pi vt) sin(i\pi x)$$

where $\{\vec{u}_i(x,t)\} = \{\vec{u}_1(x,t), \vec{u}_2(x,t), \vec{u}_3(x,t)\}$ for 1 ≤ i ≤ 3 is clearly a strict solution with initial data,

$$\vec{f}_i(x) = \sum_{i=1}^{3} b_i sin(i\pi x), g_i(x) = p_i(x),$$

where $\{\vec{f}_i(x,t)\} = \{\vec{f}_1(x,t), \vec{f}_2(x,t), \vec{f}_3(x,t)\}$. The limit of $\vec{u}$ by the definition of $\vec{u}(x,t)$ and the limit of $\vec{f}$ hold by the *pointwise convergence theorem* for *Fourier Sine Series* and the limit of $g_i(x)$ trivially" [3]. The process of limiting is one way that can explain the continuity of each function. As we can see, $\vec{f}$,p, and $\vec{u}$ *converges uniformly* with respect to the domain -∞<x<∞ and $t \geq 0$. In this perception, there are partial derivatives of the functions $\vec{u}$ and p that are continuous with initial functions of $\vec{u}$ and p that are also continuous. The set of these functions are denoted as the infinite *differentiability class* on the set R^n with time $t \geq 0$. Higher order differentiability classes should correspond to the existence of higher order derivatives. Functions that have derivatives of all orders could be named as *smooth* functions. We must now show that p and $\vec{u}$ is *infinitely differentiable* on the set R^n x[0,∞)where as in this case we are letting n equal 3 such that R^3 x[0,∞).The notation of C^∞ means that the scalar field p_i and velocity field $\vec{u}_i$ are in a specific type of differentiability class of *smooth functions* if and only if they have

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derivatives of all orders. We must show that these functions are in this class with respect to the given space R^n x $[0,\infty)$ where we let n equal to 3. When we look at equation 3 we see that we have $\frac{\partial p_i}{\partial x_i}$ where $\{x_i\}$ is the sequence of positions in the x-direction such that

$$\frac{\partial p_1}{\partial x_1}$$
$$\frac{\partial p_2}{\partial x_2}$$
$$\frac{\partial p_3}{\partial x_3}$$
.
.
.
$$\frac{\partial p_n}{\partial x_n}$$

If this is the case, there could be a corresponding sequence of *pressure intersection points* $p_i\{(x_i, t)\}$ in a given body from i to ∞ such that

$$p_1(x_1,t)$$
$$p_2(x_2,t)$$
$$p_3(x_3,t)$$
.
.
.
$$p_n(x_n,t).$$

Analysis of the Navier-Stokes equation

$$\frac{\partial \vec{u}_i}{\partial t}+\sum_{j=1}^{n}\vec{u}_j\frac{\partial \vec{u}_i}{\partial x_j}=v\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}+\vec{f}_i(x_i,t)$$

The motion of a *non-turbulent Newtonian fluid* could be governed by the equation above where the first term is the time-derivative of any fluid's velocity or the *acceleration* in Newton's Second Law of motion for a fluid element. This element is subjected to an external force that is stated above and also to forces coming from the pressure field and friction in this equation which is known as the *material derivative* such that,

$$\frac{\partial \vec{u}_i}{\partial t}+(\vec{u}*\nabla\vec{u})$$

then

$$\frac{\partial \vec{u}_i}{\partial t}+\vec{u}_j\nabla\vec{u}_i=v\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}+\vec{f}_i(x_i,t)$$

where ($\vec{u}*\nabla\vec{u}$) is a term of convection in fluid mechanics. The equation could also be used to model turbulent flows where the fluid parameter could be interpreted as time averages. The second term could be the velocity in the change of coordinates by the Law of Coordinate Transformation also known as a *contravariant*. The fourth term could be the gradient vector of the scalar pressure field p in all space dimensions where x is in the real numbers from i to *infinity*.

The fifth term represents a sequence of external force vectors $\vec{f}_i$ from i to *infinity* that correspond to a mass m and a velocity field $\vec{u}_i$ as stated above from Newton's Second Law. As shown earlier, we seen a perception on the scalar field p_i and velocity field u_i being elements of the differentiability class C^∞ with respect to all space dimensions (x,y,z) and the time interval $t \geq 0$ on R^3 x$[0,\infty)$. This confirms equation six which states the scalar and velocity field p *and* $\vec{u}_i$ are elements of the infinite differentiability class C^∞ (R^3 x$[0,\infty)$). Equation seven is of bounded energy or *global regularity* that could be expressed as the magnitude or *modulus* of the velocity field squared with respect to all space dimensions and time $t \geq 0$. In non-relativistic wave mechanics, there could exist a wave function $\vec{u}(x,t)$ of a particle that satisfies a certain wave equation where,

$$\frac{\partial \vec{u}_i}{\partial t}=\frac{-h^2}{2m}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z})\vec{u}_i$$

so that

$$\frac{\partial \vec{u}_i}{\partial t}=\frac{-h^2}{2m}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z})\vec{u}_i$$
$$\frac{\partial \vec{u}_i}{\partial t}=\frac{-h^2}{2m}\nabla\vec{u}_i$$

since $\frac{-h}{2m}$ is constant let it be equal to v such that

$$\frac{\partial \vec{u}_i}{\partial t}=v\nabla\vec{u}_i.$$

The velocity vector field $\vec{u}$ can be looked at as the *average velocity* that was differentiated to obtain the *average acceleration*. Now we take the a second derivative of $\vec{u}_i$ such that

$$\frac{\partial^2 \vec{u}_i}{\partial t^2}=v\nabla^2\vec{u}_i$$

which is also known as "*The Simple Wave* equation" [6]. As we stated earlier $\nabla^2=\Delta$ so we can use this to change notation to

$$\frac{\partial^2 \vec{u}_i}{\partial t^2}=v\Delta\vec{u}_i.$$

"This Simple Wave equation can be solved in three dimensions with the initial conditions

$$\vec{u}(x,y,z,0)=\phi(x,y,z) fort\geq 0$$

and

$$\frac{\partial \vec{u}}{\partial t}(x,y,z,0)=\psi(x,y,z)$$

Where $(x,y,z)\in R^3$ and how this method to a solution satisfies *Huygen's Principle*. This method can also be used to solve this wave equation in two-dimensions. To solve this problem in three-dimensions we start with an easier one first. "Let $\vec{u}=0$ so that,

$$\vec{u}(x,y,z,0)=0$$
$$\frac{\partial \vec{u}}{\partial t}(x,y,z,0)=\psi(x,y,z)$$

where Δ is the Laplacian Operator stated earlier in the paper. This problem can be solved by a *Fourier Transform* and has a solution

$$\vec{u}(x,y,x,t)=t\overline{\psi}$$

where $\overline{\vec{u}_1}$ is the *average* of the initial disturbance $\vec{u}_1$ over the sphere of radius vt centered at (x,y,z). The symbol $\overline{\psi}$ yields the *Fourier Transform*" [7]. "The verbal interpretation of this solution is that *initial* disturbance $\vec{u}_t$ radiates outward spherically (viscosity v) at each point, so that after so many seconds, the point (x,y,z) will be influenced by those initial disturbances on a sphere (of radius vt) around that point. Now let $\vec{u}=\phi$ and $\vec{u}_t=0$" [7]. "A famous theorem developed by *Stokes* says all we have to do to solve this problem is change the initial conditions to $\vec{u}=o$, $\vec{u}_t=\phi$ and then differentiate this solution with respect to time. So we solve our given simple wave problem to get $\vec{u}=t\phi$ and then differentiate with respect to time. This gives us the solution to the simple wave problem which is,

$$\vec{u}=\frac{\partial}{\partial t}[t\bar{\phi}].$$

For the one-dimensional wave equation the solution of the switch(shift) is

$$\vec{u}(x,t)=\frac{1}{2v}\int_{x-vt}^{x+vt}\phi(s)ds.$$

Differentiating this equation will yield,

$$\vec{u}_t(x,t)=\frac{1}{2}[\phi(x+vt)+\phi(x-vt)]$$

which is the solution to our given wave equation. Knowing this we have the solution to the three-dimensional simple wave equation where $\vec{u}=\phi$ and $\vec{u}_t=\psi$ initially which is just

$$\vec{u}(x,y,x,t)=t\bar{\psi}+\frac{\partial}{\partial t}[t\bar{\phi}]$$

where $\bar{\phi}$ and $\bar{\psi}$ are averages of the functions φ and ψ. This generalization is known as the *Poisson's formula* for the *free-wave equations* in three dimensions" [1].

If a general solution can be formed, then a specific one can be made with a large amount of computer time. With the given stationary and non-stationary equations, we derive them all with our stream function for both stationary and non-stationary equations. We will start at the derivation of

$$\frac{d\vec{u}_1}{dt}=v\Delta\vec{u}_1-\frac{\partial p}{\partial x_1}.$$

Since we have our defined stream function for the stationary equations, we get

$$\frac{d}{dt}(\frac{\partial w}{\partial y})=v[\frac{\partial^2}{\partial x_1^2}(\frac{\partial w}{\partial y})]-\frac{\partial p}{\partial x}.$$

If we integrate both side with respect to *x*, we will see that

$$w(x)dydt=vw(x_1^2)dy-p(x).$$

Now integrating both sides with respect to *y* and *t*, we will get

$$w(x,y,t)=w(x_1^2,y,t)-p(x,y,t).$$

Going back to our given non-stationary equations from the beginning and we solve for the acceleration with the force field equaling zero such that,

$$\frac{\partial\vec{u}_i}{\partial t}=-\vec{u}_j\sum_{j=1}^{n}\frac{\partial\vec{u}_i}{\partial x_j}+v\Delta\vec{u}_i-\frac{\partial p}{\partial x_i}.$$

Integrating both sides with respect to *x* and *t*, we arrive at

$$\vec{u}_1(x,t)=-\hat{u}_1\vec{u}_1(x_1,t)+v\vec{u}_1(x_1,t)-p(x_1,t).$$

We could describe this as a viscous velocity with a unit vector $\hat{u}_1$, where j=1. Going back to our equation where we solved for our acceleration. Now, we will substitute yet again our stream function *w* where we have our stationary functions with the relative velocity *a* $\vec{u}_1=\frac{\partial w}{\partial y}$ such that,

$$\frac{\partial}{\partial t}\frac{\partial w}{\partial y}=-\vec{u}_1\frac{\partial}{\partial x_1}\frac{\partial w}{\partial y}+v\frac{\partial^2}{\partial x^2}\frac{\partial w}{\partial y}-\frac{\partial p}{\partial x_i},$$

then

$$\frac{\partial^2 w}{\partial y\partial t}=-\hat{u}_1\frac{\partial^2 w}{\partial x_1\partial y}+v\frac{\partial^3 w}{\partial x^2\partial y}-\frac{\partial p}{\partial x_1}.$$

Now integrating both sides with respect to two space dimensions and time, we get

$$w(x,y,t)=\hat{u}_1 w(x,y,t)+vw(x,y,t)dx-p(x,y,t).$$

Since the pressure p is being eliminated and the stream function exists, the velocity $\vec{u}$ exists [11,12].

Conclusion

Turbulence is a specifc topic in fluid mechanics which is a vital part of the course when it comes to reallife situations. In two and three dimensional systems of equations and some initial conditions, if the smooth solutions exist, they have bounded kinetic energy. In three space dimensions and time, given an initial velocity vector, there exists a velocity field and scalar pressure field which are both smooth and globally defined that solve the Navier-Stokes equations.

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