

# Existence of Solutions for a Fractional and Non-Local Elliptic Operator

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## Abstract

In this paper, we consider a fractional and p-laplacian elliptic equation. In order to study this problem, we apply the technique of Nehari manifold and fibering map, which permit treating the existence of nontrivial solutions of a fractional and nonlocal equation, satisfies the homogeneous Dirichlet boundary conditions.

**Keywords:** Nontrivial solutions; Fractional p-laplacian equation; Nehari manifold

**Classification:** 35J35, 35J50, 35J60

## Introduction

Consider the fractional and p-laplacian elliptic problem

$$\begin{cases} (-\Delta)_p^\gamma u - \alpha\phi(t)|u|^{q-2}u - \psi(t)|u|^{r-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}, \quad (1)$$

We assume that the  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  and  $\partial\Omega$  its smooth boundary,  $p \geq 2$ ,  $\gamma \in (0,1)$ ,  $1 < q < p < r < p^* = \frac{np}{n-\gamma p}$  if  $n > p$  and  $p^* = \infty$  else,  $\phi(t), \psi(t) \in C(\Omega)$ .  $\alpha > 0$

and the fractional p-laplacian operator may be defined for  $p \in (1, \infty)$  as

$$(-\Delta)_p^\gamma u(t) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(t)} \frac{|u(t) - u(z)|^{p-2} (u(t) - u(z))}{|t - z|^{n+\gamma p}} dz, \quad t \in \mathbb{R}^n. \quad (2)$$

Over the recent years, numerous scientists have been attracted by the fractional and or p-laplacian equations. In fact, a few great models have been upgraded considerably for satisfactory answers to the modelling issues. We mention as examples the fractional Navies Stokes equations [1], fractional transport equations [2] and fractional Schrödinger equations [3], integral equations of fractional order [4,5]. Generally, a large variety of applications leads to these types of equations in ecology, elasticity and finance [6-8]. Despite significant progress in the field, and because of the difficulty to find an exact solution, research projects are still ongoing.

In this paper, we will think about the partial and p-laplacian elliptic equation (1). A considerable measure has been given for to explore this type of problems as of late. We can discover comparative equations in the many works where the issue of the existence of solutions has been dealt with. For instance, in [9], a local operator issue has been treated with  $\phi(t) = \psi(t) = 1$ . In addition, in [10] we have comes a class of Kirchhoff sort having a similar right-hand-side term that in the problem (1). See likewise [11] for a late consideration of the fractional and p-laplacian elliptic issue with  $\phi = \psi = 0$ . In this case, the solution  $u$  called a  $\gamma$ -p-harmonic function. Partial Laplacian equations satisfy the homogeneous Dirichlet boundary has been as of late considered in [9,11-13], using variational techniques. The existence of solutions has been considered at Ghanmi [14] utilizing a right-hand-side term of the treated condition comprises a homogeneous map, yet at the same time positive. Moreover, Xiang et al. in [15], use non-negative weight functions with the same issue. Here, we have treated the issue with sign-changing weight functions, and we proposed another proof for the existence of solutions. In view of the disintegration of the Nehari manifold is by all accounts less demanding. The remainder of this paper

is organized as follows. In section 2, a few preliminaries are presented, in section 3 we explore the principle comes about.

## Preliminaries

We start with some preliminaries on the notation we will use in this report. See Ghanmi A, Nezza ED, Brown KJ, [16-19] for further detail.

For all  $h \in C(\Omega)$ , we consider the following properties

- $\|h\|_\infty = 1$ ;
- $h^* := \max(\pm h, 0) \neq 0$ .

For  $r \in [1, \infty]$ , we consider  $\|\cdot\|_r$  the norm of  $L^r(\Omega)$ . For all measurable functions  $u: \mathbb{R}^n \rightarrow \mathbb{R}$ , we define the Gagliardo seminorm, by

$$|u|_{\gamma,p} := \left( \int_{\mathbb{R}^{2n}} \frac{|u(t) - u(z)|^p}{|t - z|^{n+\gamma p}} dt dz \right)^{\frac{1}{p}}.$$

Following Di Nezza [16], we consider the fractional Sobolev space

$$W^{\gamma,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) : u \text{ measurable}, |u|_{\gamma,p} < \infty \right\},$$

with the norm defined by

$$\|u\|_{\gamma,p} := \left( \|u\|_p^p + |u|_{\gamma,p}^p \right)^{\frac{1}{p}}.$$

We consider, thereafter, the closed subspace

$$S := \left\{ u \in W^{\gamma,p}(\mathbb{R}^n) : u(t) = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\},$$

with the norm  $\|\cdot\| = |\cdot|_{\gamma,p}$ . It is easy to verify that  $(S, \|\cdot\|)$  is a uniformly convex Banach space and that the embedding  $S \rightarrow L^r(\Omega)$  is continuous for all  $1 \leq r \leq p_\gamma^*$ , and compact for all  $1 \leq r < p_\gamma^*$ . The dual space of  $(S, \|\cdot\|)$  is denoted by  $(S', \|\cdot\|_*)$ , and  $\langle \cdot, \cdot \rangle$  denotes the usual duality between  $S$  and  $S^*$

We define a weak solutions by,

**Definition 2.1:** A function  $u$  is a weak solution of (1) in  $S$ ; if for every  $v \in S$  we have:

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$$\int_{\mathbb{R}^{2n}} \frac{|u(t) - u(z)|^{p-2} (u(t) - u(z))(v(t) - v(z))}{|t - z|^{n+pq}} dt dz =$$

$$\alpha \int_{\Omega} \phi(t) |u(t)|^{q-2} u(t) v(t) dt + \int_{\Omega} \psi(t) |u(t)|^{r-2} u(t) v(t) dt.$$

The energy functional associated to the problem (1) is given by

$$\mathcal{E}_{\alpha}(u) = \frac{1}{p} \|u\|^p - \frac{\alpha}{q} \int_{\Omega} \phi(t) |u(t)|^q dt - \frac{1}{r} \int_{\Omega} \psi(t) |u(t)|^r dt.$$

The functional  $\mathcal{E}_{\alpha}$  is frechet differentiable. We have  $\langle \mathcal{E}'_{\alpha}(u), u \rangle = 0$ , if  $u$  is a weak solution in  $S$  of (1). Then, the weak solutions of (1) are critical points of the functional  $\mathcal{E}_{\alpha}$ . The energy functional  $\mathcal{E}_{\alpha}$  is unbounded below on the space  $S$ . Besides, this will certainly require the construction of an additional subset  $\mathcal{F}_{\alpha}$  of  $S$ , where the functional  $\mathcal{E}$  is bounded. To accomplish this end, we will study the following Nehari manifold to ensure that a solution exists

$$\mathcal{F}_{\alpha} := \{u \in S : \langle \mathcal{E}'_{\alpha}(u), u \rangle = 0\}.$$

Then,  $u \in \mathcal{F}_{\alpha}$  if and only if

$$\|u\|^p - \alpha \int_{\Omega} \phi(t) |u(t)|^q dt - \int_{\Omega} \psi(t) |u(t)|^r dt = 0. \tag{3}$$

The aim in the following to provide an existence result.

**Theorem 2.2:** *If  $f$  and  $g$  satisfying  $(\mathcal{P}_1 - \mathcal{P}_2)$ . Then, there exists  $\alpha_0 > 0$  such that for all  $0 < \alpha < \alpha_0$ , problem (1) has at least two nontrivial solutions.*

The proof of the last theorem comprises basically of a simple few stages.

**Lemma 2.3:**  $\mathcal{E}_{\alpha}$  is coercive and bounded bellow on  $\mathcal{F}_{\alpha}$ .

**Proof:** Let  $u \in \mathcal{F}_{\alpha}$ , then, we have

$$\begin{aligned} \mathcal{E}_{\alpha}(u) &= \left(\frac{1}{p} - \frac{1}{r}\right) \|u\|^p - \frac{\alpha}{q} \int_{\Omega} \phi(t) |u(t)|^q dt - \frac{1}{r} \int_{\Omega} \psi(t) |u(t)|^r dt \\ &= \left(\frac{1}{p} - \frac{1}{r}\right) \|u\|^p - \alpha \left(\frac{1}{q} - \frac{1}{r}\right) \int_{\Omega} \phi(t) |u(t)|^q dt \\ &\geq c_1 \|u\|^p - c_2 \|u\|^q. \end{aligned}$$

Hence,  $\mathcal{E}_{\alpha}$  is bounded bellow and coercive on  $\mathcal{F}_{\alpha}$ .

We define fiber maps  $F_u : [0, \infty) \rightarrow \mathbb{R}$  according Drabek P and Brown [17,20] by,

$$F_u(s) = \mathcal{E}_{\alpha}(su).$$

These fiber maps  $F_u$  Act as an important use in the proof because the Nehari manifold is closely linked to the behavior for them.

For  $u \in S$ , we can denote that  $tu \in \mathcal{F}_{\alpha}$  if and only if  $F'_u(s) = 0$ . Thus, we consider the follow parts  $\mathcal{F}_{\alpha}$  into three parts corresponding to relative minima, relative maxima and points of inflection.

$$\mathcal{F}_{\alpha}^+ = \{su \in S : F'_u = 0, F''_u > 0\}, \mathcal{F}_{\alpha}^- = \{su \in S : F'_u = 0, F''_u < 0\}, \text{ and } \mathcal{F}_{\alpha}^0 = \{su \in S : F'_u = F''_u = 0\}.$$

We need to define  $m_u : [0, \infty) \rightarrow \mathbb{R}$  by

$$m_u(s) = s^{p-q} \|u\|^p - s^{r-q} \int_{\Omega} \psi(t) |u|^r dt.$$

Clearly, for  $s > 0$ ,  $s_u \in \mathcal{F}_{\alpha}$  if and only if  $s$  is a solution of

$$m_u(s) = \alpha \int_{\Omega} \phi(s) |u(t)|^q dt.$$

We consider the following subsets

$$\mathcal{V}^+ = \{u \in S : I_{\phi} > 0\}; \mathcal{V}^- = \{u \in S : I_{\phi} < 0\}; \mathcal{W}^+ = \{u \in S : I_{\psi} > 0\} \text{ and } \mathcal{W}^- = \{u \in S : I_{\psi} < 0\}.$$

$$\text{with } I_{\phi} = \int_{\Omega} \phi(t) |u(t)|^q dt \text{ and } I_{\psi} = \int_{\Omega} \psi(t) |u(t)|^r dt.$$

For studying the fiber map  $F_u$  correspond to the sign of  $I_{\phi}$  and  $I_{\psi}$ , then, four possible cases can occur:

- If  $u \in \mathcal{W}^+ \cap \mathcal{V}^+$ , then,  $F_u(0) = 0$  and  $F'_u(t) > 0, \forall t > 0$  which implies that  $F_u$  is strictly increasing, this resulting the absence of critical points.

- $u \in \mathcal{W}^+ \cap \mathcal{V}^-$  (or  $u \in \mathcal{W}^- \cap \mathcal{V}^+$ ). As we have  $m_u(s_1) = 0$ . Here the only critical point of  $F_u$  is  $s_1$ , which is a absolute minimum point. Hence  $s_1 u \in \mathcal{F}_{\alpha}^+$ .

- $u \in \mathcal{W}^- \cap \mathcal{V}^+$ , it exists  $\mu_0 > 0$  such that for  $\alpha \in (0, \mu_0)$ ,  $F_u$  has exactly one relative minimum  $s_1$  and one relative maxima  $s_2$ . Thus  $s_1 u \in \mathcal{F}_{\alpha}^+$  and  $s_2 u \in \mathcal{F}_{\alpha}^-$ .

We have the following result:

**Corollary 2.4:** *If  $\alpha < \mu_0$ , then, there exists  $\delta_1 > 0$  such that  $\mathcal{E}_{\alpha}(u) > \delta_1$  for all  $u \in \mathcal{F}_{\alpha}^-$ .*

*Proof.* Let  $u \in \mathcal{F}_{\alpha}^-$ , then  $F_u$  has a positive absolute maximum at  $T = 1$  and  $\int_{\Omega} \phi(t) |u(t)|^q dt > 0$ . Thus, if  $\alpha < \mu_0$ , then we have

$$\begin{aligned} \mathcal{E}_{\alpha}(u) &= F_u(1) = F_u(T) \\ &\geq \delta^p \left( \delta^{\frac{p-q}{p}} - \alpha c \right) > 0, \end{aligned}$$

the value of  $\delta$  is given in Lemma 2.5.

**Lemma 2.5:** *There exists  $\delta_0 > 0$  such that for  $\alpha \in (0, \mu_0)$ ,  $F_u$  take positive value for all non-zero  $u \in S$ . Moreover, if*

$$u \in \mathcal{F}^+ \cap \mathcal{V}^+, \text{ then, } F_u \text{ has exactly two critical points.}$$

*Proof.* Let  $u \in S$ , define

$$M_u(s) = \frac{s^p}{p} \|u\|^p - \frac{s^r}{r} \int_{\Omega} \psi(t) |u|^r dt.$$

Then,

$$M'_u(s) = s^{p-1} \|u\|^p - s^{r-1} \int_{\Omega} \psi(t) |u|^r dt.$$

If  $\int_{\Omega} \psi(t) |u|^r dt > 0$ ,  $M_u$  reaches its maximum value at

$$T = \left( \frac{\|u\|^p}{\int_{\Omega} \psi(t) |u|^r dt} \right)^{\frac{1}{r-p}}. \text{ Moreover,}$$

$$M_u(T) = \left( \frac{1}{p} - \frac{1}{r} \right) \left( \frac{\|u\|^r}{\int_{\Omega} \psi(t) |u|^r dt} \right)^{\frac{p}{r-p}}$$

and

$$M''_u(T) = (p-r) \frac{\|u\|^{\frac{p(r-2)}{r-p}}}{\left( \int_{\Omega} \psi(t) |u|^r dt \right)^{\frac{p-2}{r-p}}} < 0.$$

For  $\leq v < p_{\alpha}^*$  we denoted by  $S_v$  be the Sobolev constant of embedding  $S \rightarrow L^v(\Omega)$ , then, by 3 we have

$$M_u(T) \geq \frac{r-p}{r p (KS_r^r)^{\frac{p}{r-p}}} = \delta, \tag{4}$$

which is independent of  $u$ . We now show that there exists  $\mu_0 > 0$  such that  $F_u(T) > 0$ . Using condition  $g$  satisfying  $(P_1 - P_2)$  and the Sobolev imbedding, we get

$$\begin{aligned} \frac{T^q}{q} \int_{\Omega} \phi(t) |u(t)|^q dt &\leq \frac{S^q}{q} \|u\|^q T^q \\ &= \frac{S^q}{q} \|u\|^q \left( \frac{\|u\|^p}{\int_{\Omega} \psi(t) |u|^r dt} \right)^{\frac{q}{r-p}} \\ &= \frac{S^q}{q} \left( \frac{\|u\|^r}{\int_{\Omega} \psi(t) |u|^r dt} \right)^{\frac{q}{r-p}} \\ &= \frac{S^q}{q} (M_u(T))^{\frac{q}{p}}, \end{aligned}$$

Thus

$$\begin{aligned} F_u(T) &= M_u(T) - \alpha \frac{T^q}{q} \int_{\Omega} \phi(t) |u(t)|^q dt \\ &\leq M_u(T) - \alpha c M_u(T)^{\frac{q}{p}} \\ &= \delta^{\frac{q}{p}} \left( \delta^{\frac{p-q}{p}} - \alpha c \right), \end{aligned}$$

where  $\delta$  is the constant given in (4).

Let

$$\mu_0 = \frac{q \delta^{\frac{p-q}{p}}}{S^q}. \tag{5}$$

Then, choice of such  $\mu_0$  completes the proof.

**Lemma 2.6:** *There exists  $\mu_1$  such that if  $0 < \alpha < \mu_1$ , then  $\mathcal{F}_\alpha^0 = \emptyset$ .*

**Proof:** Let

$$\mu_1 = \frac{r-p}{S^q (r-q)} \left( \frac{p-q}{KS_r^r (r-q)} \right)^{\frac{p-q}{r-p}},$$

where  $K$  is given by (3).

Suppose otherwise, that  $0 < \alpha < \mu_1$  such that  $\mathcal{F}_\alpha^0 \neq \emptyset$ . Then, for  $u \in \mathcal{F}_\alpha^0$ , we have

$$0 = F_u'(1) = (p-1) \|u\|^p - (r-1) \int_{\Omega} \psi(t) |u|^r dt - \alpha(q-1) \int_{\Omega} \phi(t) |u(t)|^q dt.$$

So, it follows from (3) that

$$\begin{aligned} (r-p) \|u\|^p &= \alpha(r-q) \int_{\Omega} \phi(t) |u|^q dt \\ &\leq \alpha(r-q) S^q \|u\|^q, \end{aligned}$$

and so

$$\|u\| \leq \left( S \frac{r-q}{r-p} \right)^{\frac{p}{q}}. \tag{6}$$

On the other hand, by (3) we get

$$(p-q) \|u\|^p = \alpha(r-q) \int_{\Omega} \psi(t) |u|^r dt$$

$$\leq K(r-q) S_r^r \|u\|^r,$$

then

$$\|u\| \geq \left( \frac{p-q}{KS_r^r (r-q)} \right)^{\frac{1}{r-p}}. \tag{7}$$

Combining (6) and (7) we obtain  $\alpha \geq \mu_1$ , which is a contradiction.

**Lemma 2.7:** *Let  $u$  be a relative minimizer for  $\mathcal{E}_\alpha$  on subsets  $\mathcal{F}_\alpha^+$  or  $\mathcal{F}_\alpha^-$  of  $\mathcal{F}_\alpha$ , then  $u$  is a critical point of  $\mathcal{E}_\alpha$ .*

**Proof:** Since  $u$  is a minimizer for  $\mathcal{E}_\alpha$  under the constraint  $I_\alpha(u) := \langle \mathcal{E}_\alpha(u), u \rangle = 0$ , by the theory of Lagrange multipliers, there exists  $\mu \in \mathbb{R}$  such that  $\mathcal{E}'_\alpha(u) = \mu I'_\alpha(u)$ . Thus:

$$\langle \mathcal{E}'_\alpha(u), u \rangle = \mu \langle I'_\alpha(u), u \rangle = \mu F_u''(1) = 0,$$

but  $u \notin \mathcal{F}_\alpha^0$  and so  $F_u''(1) \neq 0$ . Hence  $\mu = 0$  completes the proof.

In the remain of this section, we assume that the parameter  $\alpha$  satisfies  $0 < \alpha < \alpha_0$ , where  $\alpha_0$  is constant. That leads us consequently to the following results on the existence of minimizers in  $\mathcal{F}_\alpha^+$  and  $\mathcal{F}_\alpha^-$  for  $\alpha \in (0, \alpha_0)$ .

**Theorem 2.8:** *We have the following results*

$\mathcal{E}_\alpha$  has reached its minimum on  $\mathcal{F}_\alpha^+$  and its maxima on  $\mathcal{F}_\alpha^-$

**Proof:** To prove the theorem we proceed in two steps

**Step 1:** Since  $\mathcal{E}_\alpha$  is bounded below on  $\mathcal{F}_\alpha$  and so on  $\mathcal{F}_\alpha^+$ , there exists a minimizing sequence  $\{u_k\} \subset \mathcal{F}_\alpha^+$  such that

$$\lim_{k \rightarrow \infty} \mathcal{E}_\alpha(u_k) = \inf_{u \in \mathcal{F}_\alpha^+} \mathcal{E}_\alpha(u).$$

As  $\mathcal{E}_\alpha$  is coercive on  $\mathcal{F}_\alpha$ ,  $\{u_k\}$  is a bounded sequence in  $S$ . Therefore, for all  $1 \leq \nu < p_s^*$  we have

$$\begin{cases} u_k \rightharpoonup u_\alpha & \text{weakly in } S \\ u_k \rightarrow u_\alpha & \text{strongly in } L^\nu(\mathbb{R}^n). \end{cases}$$

If we choose  $u \in S$  such that  $\int_{\Omega} \phi(t) |u(t)|^q dt > 0$ , then, there exists  $s_1 > 0$  such that  $s_1 u \in \mathcal{F}_\alpha^+$  and  $\mathcal{E}_\alpha(s_1 u) < 0$ . Hence,  $\inf_{u \in \mathcal{F}_\alpha^+} \mathcal{E}_\alpha(u) < 0$ .

On the other hand, since  $\{u_k\} \subset \mathcal{F}_\alpha^+$ , then we have

$$\alpha \left( \frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} \phi(t) |u_k(t)|^q dt = \left( \frac{1}{p} - \frac{1}{r} \right) \|u_k\|^p - \mathcal{E}_\alpha(u_k).$$

and so

$$\alpha \left( \frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} \phi(t) |u_k(t)|^q dt = \left( \frac{1}{p} - \frac{1}{r} \right) \|u_k\|^p - \mathcal{E}_\alpha(u_k).$$

Letting  $k$  to infinity, we get

$$\int_{\Omega} \phi(t) |u_\alpha(t)|^q dt > 0. \tag{8}$$

Next we claim that  $u_k \rightarrow u_\alpha$ . Suppose this is not true, then

$$\|u_\alpha\|^p < \liminf_{k \rightarrow \infty} \|u_k\|^p.$$

Since  $F_{u_\alpha}'(s_1) = 0$  it follows that  $F_{u_k}'(s_1) > 0$  for sufficiently large  $k$ . So, we must have  $s_1 > 1$  but  $s_1 u_\alpha \in \mathcal{F}_\alpha^+$  and so

$$\mathcal{E}_\alpha(s_1 u_\alpha) < \mathcal{E}_\alpha(u_\alpha) \leq \lim_{k \rightarrow \infty} \mathcal{E}_\alpha(u_k) = \inf_{u \in \mathcal{F}_\alpha^+} \mathcal{E}_\alpha(u),$$

which is a contradiction. It leads to  $u_k \rightarrow u_\alpha$  and so  $u_\alpha \in \mathcal{F}_\alpha^+$ , since  $\mathcal{F}_\alpha^0 = \emptyset$ . Finally,  $u_\alpha$  is a minimizer for  $\mathcal{E}_\alpha$  on  $\mathcal{F}_\alpha^+$ .

**Step 2:** Let  $u \in \mathcal{F}_\alpha^-$ , then from corollary 2.4, there exists  $\delta_1 > 0$  such that  $\mathcal{E}_\alpha(u) \geq \delta_1$ . So, there exists a minimizing sequence  $\{u_k\} \subset \mathcal{F}_\alpha^-$  such that

$$\lim_{k \rightarrow \infty} \mathcal{E}_\alpha(u_k) = \inf_{u \in \mathcal{F}_\alpha^-} \mathcal{E}_\alpha(u) > 0. \tag{9}$$

On the other hand, since  $\mathcal{E}_\alpha$  is coercive,  $\{u_k\}$  is a bounded sequence in  $S$ . Therefore, for all  $1 \leq \nu < p_s^*$  we have

$$\begin{cases} u_k \rightharpoonup v_\alpha & \text{weakly in } S \\ u_k \rightarrow v_\alpha & \text{strongly in } L^\nu(\mathbb{R}^n). \end{cases}$$

Since  $u \in \mathcal{F}_\alpha^-$ , then we have

$$\mathcal{E}_\alpha(u_k) = \left(\frac{1}{p} - \frac{1}{q}\right) \|u_k\|^p + \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega \psi(t) |u_k|^r dt. \tag{10}$$

Letting  $k$  to infinity, it follows from (9) and (10) that

$$\int_\Omega \psi(t) |v_\alpha|^r dt > 0. \tag{11}$$

### Conclusion

Hence,  $v_\alpha \in \mathcal{F}_\alpha^+$  and so  $F_{v_\alpha}$  has a absolute maximum at some point  $T$  and consequently,  $Tv_\alpha \in \mathcal{F}_\alpha^-$ . on the other hand,  $u_k \in \mathcal{F}_\alpha^-$  implies that  $1$  is a absolute maximum point for  $F_{u_k}$  i.e.

$$\mathcal{E}_\alpha(su_k) = F_{u_k}(s) \leq F_{u_k}(1) = \mathcal{E}_\alpha(u_k). \tag{12}$$

Next we claim that  $u_k \rightarrow u_\alpha$ . Suppose it is not true, then

$$\|u_\alpha\|^p < \liminf_{k \rightarrow \infty} \|u_k\|^p,$$

it follows from (12) that

$$\begin{aligned} \mathcal{E}_\alpha(Tv_\alpha) &= \frac{T^p}{p} \|v_\alpha\|^p - \frac{T^r}{r} \int_\Omega \psi(t) |v_\alpha|^r dt - \alpha \frac{T^q}{q} \int_\Omega \phi(t) |v_\alpha|^q dt \\ &< \inf_{k \rightarrow \infty} \left( \frac{T^p}{p} \|u_k\|^p - \frac{T^r}{r} \int_\Omega \psi(t) |u_k|^r dt - \alpha \frac{T^q}{q} \int_\Omega \phi(t) |u_k|^q dt \right) \\ &\leq \lim_{k \rightarrow \infty} \mathcal{E}_\alpha(Tu_k) \leq \lim_{k \rightarrow \infty} \mathcal{E}_\alpha(u_k) = \inf_{u \in \mathcal{F}_\alpha^-} \mathcal{E}_\alpha(u), \end{aligned}$$

which is a contradiction. Hence,  $u_k \rightarrow v_\alpha$  and so  $v_\alpha \in \mathcal{F}_\alpha^-$ , since  $\mathcal{F}_\alpha^0 = \emptyset$ .

Now, Let us proof Theorem 1.1: By the Lemmas 2.5, 2.6, 2.7 and the theorem 2.8 the problem (1) has two weak solution  $u_\alpha \in \mathcal{F}_\alpha^+$  and  $v_\alpha \in \mathcal{F}_\alpha^-$ . On the other hand, from (8) and (11), this solutions are nontrivial. Since  $\mathcal{F}_\alpha^- \cap \mathcal{F}_\alpha^+ = \emptyset$ , then,  $u_\alpha$  and  $v_\alpha$  are distinct.

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