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Existence Results of Ordinary Differential Inclusions in Banach Algebra under Weak Topology

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Abstract

This article is devoted to investigating the existence of solutions to a coupled system of ordinary inclusions at resonance in suitable Banach algebras. We convert the system of differential inclusions to a system of fixed point problems with nonlinear multi-valued inputs. More precisely, we introduce a new class of multi-valued mappings for a weak topology.

Keywords: Differential inclusions; Nonlinear multi-valued mappings; Measure of weak noncompactness; Weakly sequentially closed graph; Weakly completely continuous; Sequentially weakly upper semi-compact

Introduction

The area devoted to the study of differential equations and differential inclusions has received much attention due to the fact that they describe many phenomenons in various fields of applied sciences like physics, control theory, chemistry, biology, and so forth. For some of these applications, one can see [1-8] and the references therein. In most papers, sufficient conditions for the existence of solutions to the corresponding problems are obtained by using classical fixed point theory. We refer, for example, to few studies [9-14]. In this work, we are mainly concerned with the existence results of solutions for the following two-dimensional quadratic problems for ordinary differential inclusion (in short QV P) occurring in some physical and biological problems:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{x(t)-k(t,x(t))}{f(t,y(t))}\right) \in G(t,y(t)) \\ y(t)-f1(t,y(t))=\left[\left(\lambda+\int_0^{\sigma(t)} p(s,x(\lambda s))ds\right)\bullet v\right] \\ (x(0),y(0))=(x_0,y_0)\in\mathbb{R}^2 \end{cases} \quad (1.1)$$

where $v \in X\backslash\{0\}$ with $\|v\|<1$; the functions f, k, p, f_1: $[0,\tau]\times X\to X$ are supposed to be weakly sequentially continuous and the multi-valued mapping G: $[0,\tau]\times X \to P(X)$ is supposed to be weakly completely continuous with respect to the second variable.

Here, $[0,\tau]\tau< 1$; is a closed and bounded interval of the real line $\mathbb{R}$ and, X is a Banach algebra satisfying certain topological conditions of sequential nature. Such systems occur in various problems of applied nature, for instance [13-18].

By a solution of the above problem, we mean a pair of continuous functions (x(t), y(t)) that satisfies the differential inclusions (1.1) and such that the function $t \to \dfrac{x(t)-k(t,x(t))}{f(t,y(t))}$ is differentiable.

The problem QV P (1.1) has not been studied in the literature before, so the results of this paper are new to the theory of differential inclusions for a weak topology.

The special cases of the problem QV P (1.1) has been discussed in previous studies [9,13] with x=y, λ=0, σ(t)=0, and f_1(t, y)=y for all t ∈

$[0,\tau]$. If more G(t, x)={g(t, x)} is a single-valued mapping, we obtain the following integral equation:

$$\begin{cases} x(t)= k(t,x(t)) + f(t,x(t))\int_0^t g(s,x(s))ds \\ x(0)= x_0\in\mathbb{R} \end{cases} \quad (1.2)$$

The integral eqn. (1.2) has been discussed in previous studies [11,12] for different aspects of the solutions under suitable conditions.

Note that the problem (1.1) may be transformed into the following fixed point problem in weak topology settings for the 2×2 block operator matrix:

$$\begin{pmatrix} A & B.B' \\ C & D \end{pmatrix} \quad (1.3)$$

where the entries are generally nonlinear multi-valued mappings. Our assumptions are as follows: A, B and B' maps a nonempty, closed, and convex (not necessarily bounded) subset S into the classes, denoted P(X), of all nonempty subsets of a Banach algebra X satisfying condition (P), C from S into X and D from X into X. The assumptions of our main results are formulated in terms of an axiomatic definition of the measure of weak- noncompactness. More precisely, we present a new class of multi-valued mappings of the form $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)$ and we use the properties of D-set-Lipschitzian for the multi-valued mappings A and B: Our results extend and generalize well-known results for a weakly sequentially continuous single-valued mappings in previous studie [15]. Again, we remove the quasi-regularity assumption on the operator B(I-D)$^{-1}$C in a study [15].

In a series of papers [7,14,15], Jeribi et al. have established some fixed point theorems for the block operator matrix (1.3), where the inputs are multi-valued and single-valued mappings based on the convexity of the bounded domain, on the well-known Schauders fixed point theorem, and also on the properties of the inputs (cf.weakly

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sequentially continuous, lower semi-continuous,…). The obtained results are then applied to a coupled system of perturbed functional differential inclusions of initial and boundary value problems in order to prove the existence of solutions under a Caratheodory condition.

In this direction, Ben Amar et al. have established in a study [16] some existence theorems for a two- dimensional mixed boundary problem, based on a new generalized Schauder's fixed point theorem, in terms of completely continuous, and on Krasnoselskii's fixed point theorem for the block operator matrix (1.3) acting on Lp×Lp space for p ∈]1,+∞[in the case where B'=1: It is important to mention that the theoretical study was based on the existence of a solution of nonlinear differential inclusions of the respective forms:

$$x \in Ax.Bx + Cx, x \in S \tag{1.4}$$

where A;B and C are multi-valued mappings and S is a nonempty, closed and convex subset of a Banach algebra X, and obtained several valuable results in Banach algebras. They were mainly based on the convexity of the bounded domain and the properties of the operators A, B and C (cf. weakly 1-set- contractive [17-19], D-set-Lipschitzian, sequentially weakly upper semicompact, weakly completely continuous, weakly condensing and the potential tool of the axiomatic measures of non-compactness [20-23]. Since the weak topology is the practice setting and it is natural to investigate the fixed point problems occurring problems dealing with physics, it turns out that the above mentioned results cannot be easily applied. However, because of the lacks of stability of convergence for the product sequences under the weak topology, the authors in a study [24] have introduced a new class of Banach algebras satisfying the condition denoted (P):

$$(P) \begin{cases} \textit{For any sequence } \{x_n\} \textit{ and } \{y_n\} \textit{ of } X \textit{ such that } x_n \to x \textit{ and } y_n \to y \\ \textit{then } x_n \cdot y_n \to x \cdot y; \textit{denote weak convergence} \end{cases}$$

and they have established some new variants of fixed point results based on the notion of weak sequential continuity. This notion seems to be the most comfortable in use. Note that, if X is a Banach algebra satisfying the condition (P) then, according to the Dobrakov's theorem, C(K,X) is also a Banach algebra satisfying the condition (P), where K is a compact Hausdorff space.

The content of this paper is organized in four sections. In the next section, we give some preliminary results needed in the sequel. In Section 3 and 4; we will refine the fixed point theorems established in previous studies [7,14,15] for the 2×2 block operator matrix (1.3) by using arguments of weak topology, which consist of multi-valued mappings acting on nonempty, closed, and convex subsets in Banach algebras. The main results of this section are Theorem 5.6 and Theorem 6.1. In Section 5; we will use Theorem 5.6 in order to discuss the existence of solutions for the problem (1.1).

Auxiliary Facts and Results

Throughout this section, X denotes a Banach algebra satisfying the condition (P), endowed with the norm k k and with the zero element 0_X. For any r > 0,B_r denotes the closed ball in X centered at the zero with radius r and D(G) denotes the domain of an operator G.B(X) is the collection of all nonempty bounded subsets of X and W(X) is the subset of B(X) consisting of all nonempty weakly compact subsets of X. In the remainder, → denotes the weak convergence and → denotes the strong convergence in X. Let us recall that the notion of measure of weak noncompactness β was introduced on B(X) by De Blasi [22] in the following way:

B(S)=inf{r>0:there exists K ∈ W(X) such that $S \subseteq K + B_r$

For convenience we recall some basic properties of needed below [21,22,25].

Lemma 4.1

Let S1 and S2 be two elements of B(X): Then the following conditions are satisfied:

(1) $S_1 \subseteq S_2$ implies $\beta_1(S_1) \leq \beta_2(S_2)$,

(2) β(S_1)=0 if and only if $\overline{S_1^w} \in W(X)$, where $\overline{S_1^w}$ is the weak closure of S_1,

(3) $\beta(\overline{S_1^w}) = \beta(S1)$,

(4) β (S_1 ∪ S_2)=max {β (S_1), β(S_2)g,

(5) β (λS_1)=|λ|β(S_1), for all $\lambda \in \mathbb{R}$

(6) β(co(S_1))=(S_1), where co(S1) denotes the convex hull of S_1,

(7) β(S_1 + S_2)≤ (S_1) +(S_2),

(8) If $(S_n)_n$ is a decreasing sequence of bounded and weakly closed subsets of X with $\lim_{n \to \infty} \beta(S_n) = 0$ (Sn)=0; then M∞=∩$^{+\infty}_{n=1}$ S_n is nonempty and $\overline{S_1^w}$ is a weakly compact subset of X.

A correspondence G: X→ P_p(X) is called a multi-valued operator or a multi-valued mapping on X into X. A point x ∈ X is called a fixed point of G if x ∈ G(x). For the sake of convenience:

$G(S) = \bigcup_{\infty \in S} G(x)$ for all subsets S of X:

P(X)={ S⊂X, S is nonempty }.

P_{bd}(X)={ S⊂X, S is nonempty and bounded}.

P_{cv}(X)={S⊂X, S is nonempty and convex}

P_{cl}(X)={ S⊂X S is nonempty and closed }.

P_{bd,cv,cl}(X)={ S⊂X, S is nonempty, closed, bounded and convex}.

For any S ∈ P_{bd,cl}(X), we denote by $\|S\| p = \sup\{\|s\|, s \in S\}$

The function d_H: P_{cl}(X)×P_{cl}(X)→R_+ defined by the formula

$$d_H(S,S') = \max\left\{\sup_{\infty \in S} d(x,S'), \sup_{y \in S'} d(y,S)\right\}$$

is called a Hausdorff metric on X [26].

Definition 4.1

A multi-valued map G:X→X is called D-Lipschitzian if there exists a continuous and nondecreasing function ∅:ℝ⁺→ℝ⁺, such that $d_H(G_x, G_y) \leq \phi(\|x - y\|)$, for all x, y ∈ X, where ϕ(0)=0:

Moreover, if ϕ(r) < r, r > 0, then G is called a nonlinear D-contraction.

Definition 4.2

A multi-valued mapping G is said to have a weakly sequentially closed graph, if for every weakly convergent sequence $(x_n)_n$ of D(G) to a point x, and for every weakly convergent sequence y_n ∈ G(x_n) to a point y, we have y ∈ G(x).

Definition 4.3

A multi-valued mapping G is weakly completely continuous if G is weakly compact and has a weakly sequentially closed graph.

Definition 4.4

A multi-valued mapping G is sequentially weakly upper semicompact in S (s.w.u.sco. for short) if for any weakly convergent sequence $(x_n)_n$ in D(G) and for any arbitrary $y_n \in G(x_n)$, the sequence $(y_n)n$ has a weakly convergent subsequence.

If G is a single-valued mapping, then G is sequentially weakly upper semicompact if for any weakly convergent sequence $(x_n)_n$, the sequence $(Gx_n)_n$ has a weakly convergent subsequence in X. We say that G is weakly sequentially continuous if for every weakly convergent sequence $(x_n)_n$ of D(G) to a point x, we have $Gx_n \to G_x$.

Definition4.5

A multi-valued mapping $G : D(G) \subset P(X)$ is called D-set- Lipschitz (with respect to) if there exists a continuous nondecreasing function: ψ :$R^+ \to R^+$ with ψ (0)=0 such that $\beta(G(M)) \leq \psi$ β (M)) for all bounded sets M of X such that $G(M) \in P_{bd}(X)$. In the special case, when (r)=kr, k > 0, the map G is called k-set-Lipschitz mapping, and if k < 1, then the multi-valued mapping G is called a k-set-contraction mapping. Moreover, if (r) < r, then G is called a nonlinear D-set-contraction mapping on X. Finally, we say that G is a β-condensing multi-valued mapping if G is bounded and (G(M)) < (M) for any bounded subset M of D(G) with (M) >0 [27].

Remark 4.1: If $G : D(G) \subset X \to P_{bd,cl}(X)$ is sequentially weakly upper semicompact and is D- Lipschitzian, then G is D-set-Lipschitz with respect to In the particular case when G is a weakly sequentially continuous and D-Lipschitzian single-valued mapping, then G is a D-set-Lipschitz mapping ([28], Lemma 1.2). In the sequel, we will use the following results which were established in a study [29].

Theorem 4.1

Let S be a nonempty, closed, and convex subset of a Banach space. Assume that G: S $\to P_{cv}(S)$ has a weakly sequentially closed graph such that G(S) is weakly relatively compact. Then G has a fixed point ([29], Theorem 4.2]).

Before stating the generalization of Theorem 4.1, we give a useful definition.

Definition 4.6

A multi-valued mapping G is β-condensing on S if for any bounded subset M of S with (M) > 0, we have G(M) is bounded and (G(M)) < (M).

Theorem 4.2

Let S be a closed and convex subset of a Banach space. Assume that G: S $\to P_{cv}(S)$ is β-condensing and has a weakly sequentially closed graph. In addition, suppose that G(S) is bounded, Then G has a fixed point ([29], Theorem 4.3]).

Multi-valued Fixed Point Theory

In this section, we generalize and extend well-known results for weakly sequentially continuous single- valued mappings in a study [15]. In particular, we remove the assumption of boundedness of the domain.

Theorem 5.1

Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, $B' : S \to P(X)$, C:S $\to$X and D: X$\to$X are five multi-valued mappings satisfying the following properties:

(i) A, B and B' are D-set-Lipschitzians with D-functions ØA, ØB and B' respectively,

(ii) A, B and B' are s.w.u.sco. with weakly sequentially closed graphs,

(iii) C is weakly sequentially continuous and is D-Lipschitzian with D-function ØC,

(iv) D is weakly sequentially continuous and is a contraction with a constant k,

(v) A(S), B(S) and B' (S) are bounded and $C(S) \subset (I-D)(S)$,

(vi) Ax + B(I-D)⁻¹Cx. $B'(I-D)^{-1} Cx \in P_{cv}(S)$

Then, the operator matrix (1.3) has a fixed point in S×S whenever $\phi_A(r) + \phi_B$ (2-C) (r)-B0

Proof: Let $\{\theta_n, n \in N\}$ be a weakly convergent sequence of S to a point $\theta \in S$. In this case, the set $\{\theta_n, n \in N\}$ is relatively weakly compact. Using both Proposition 3:1 in a study [30] and the following equality

$$(I-D)^{-1}C=C + D(I-D)^{-1}C \qquad (2.1)$$

and knowing the weak sequential continuity of the maps C and D; we infer that

$$\beta(\{I-D)^{-1}C(\theta_n), n \in N\}) \leq \beta(\{D(I-D)^{-1}C(\theta_n), n \in \mathbb{N}\})$$

$$\leq k\beta(\{D(I-D)^{-1}C(\theta_n), n \in \mathbb{N}\})$$

This shows that $\beta(\{D(I-D)^{-1}C(\theta_n), n \in \mathbb{N}\})=0$ Hence, $(\{D(I-D)^{-1}C(\theta_n), n \in \mathbb{N}\})$ is relatively weakly compact, and so there exists a subsequence (θ_{nk}) of $\{\theta_n, n \in N\}$ such that

$$(I-D)^{-1}C(\theta_{nk}) \to \gamma$$

Going back to eqn. (2.1) yields $\gamma=C\theta + D\gamma$. Now we claim that(I-D)⁻¹C$(\theta_{nk}) \to \gamma$.

Otherwise, then there exists a subsequence (θ_{nj}) of $(\theta_n)_n$ such that $(I-D)^{-1}C(\theta_{nj})$ V^w, for some weak neighborhood V^w of (I-D)-1C(θ): Arguing as before, we may extract a subsequence (θ_{njk}) of (θ_{nj}) such that

$$(I-D)-1C(\theta_{njk}) \to (I-D)-1C(\theta),$$

which is absurd. Hence, the operator inverse (I-D)⁻¹C is weakly sequentially continuous. Again, using Lemma 2:1 in a study [24] and rewording the above discussion, one obtains that

$$\beta\left((I-D)^{-1}C(M)\right) \leq \frac{1}{1-k}\phi_C\left(\beta(M)\right) (M)),C$$

for each bounded subset M of S with $\beta(M) > 0$.

Let us define the multi-valued mapping G: S$\to P_{cv}(S)$ by the formula:

$$G(x) = Ax + B(I-D)^{-1}Cx.B'(I-D)Cx \qquad (2.2)$$

Notice that G is well defined and is s.w.u.sco. In fact, consider $\{\theta_n, n \in N\}$ as a weakly convergent sequence of S to a point $\theta$and $\{z_n, n \in N\}$ as a sequence in G(θ_n). Then,

$$z_n = a_n + b_n.b_n'$$

for some $a_n \in A(\theta_n)$, $b_n \in B(I-D)^{-1}C\theta_n$ and $b_n' \in B'(I-D)^{-1}C\theta_n$. From the first part of assumption (ii), we can suppose that $a_n \to a, b_n \to b_n'$ and $b_n' \to b'$

The sequential condition (P) guarantees that,

$$z_n = a_n + bn.b_n' \to z = a + b.b' \in G(\theta)$$

This implies that G is a s.w.u.sco. multivalued mapping and consequently G has a weakly sequentially closed graph in view of the last part of assumption (ii):

Our next task is to show that G is condensing with respect to β. To do so, take an arbitrary nonempty bounded subset M of S with β(M) > 0. It is easy to verify that:

$$G(M) \subseteq A(M) + B(I-D)^{-1} C(M).B'(I-D)^{-1} C(M)$$
$$\subseteq A(S) + B(S).B'(S)$$

This implies that G(M) is bounded in view of assumption (vi); and we have:

$$\beta(G(M)) \le \beta\left(A(M) + B(I-D)^{-1} C(M) \cdot B'(I-D)^{-1} C(M)\right)$$
$$\le \|B(S)\| p\beta\left(B'(I-D)^{-1} C(M)\right) + \|B'(S)\| p\beta\left(B(I-D)^{-1} C(M)\right)$$
$$+ \beta\left(B(I-D)^{-1} C(M)\right)\beta\left(B'(I-D)^{-1} C(M)\right) + \beta(A(M))$$
$$\le \left[\|B(S)\| p\phi'_B O(2\phi_C) + \|B'(S)\| p\phi_B O(2\phi_C) + \phi_A\right](\beta(M))$$
$$+ \phi_B O(2\phi_C)(\beta(M))\phi_{B'} O(2\phi_C)(\beta(M))$$

This inequality means, in particular, that is G is β-condensing in view of assumption (i). The use of Theorem 4.2 achieves the proof. Q.E.D.

When A, B and B′ are three single valued mappings, we obtain the following result.

Corollary 5.1: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P). Assume that A, B, C, D, B′: S→X are ve weakly sequentially continuous operators satisfying the following properties:

(i) A, B, C and B′ are D-set-Lipschitzians with D-functions φ A, φ B, φ C and φ B′,

(ii) D is a contraction mapping with a constant k,

(iii) A(S),B(S) and B′ (S) are bounded and C(S) (I-D)(S),

(iv) $Ax + B(I-D)^{-1} Cx.B'(I-D)^{-1} Cx \in S$, for all x ∈ S.

Then, the operator matrix (1.3) has a fixed point in S×X whenever

$$\phi_A(r) + (\phi_B \cdot \phi_{B'})O\left(\frac{1}{1-k}\phi_C\right)(r) + \left[\|B'(S)\| p\phi B + \|B(S)\| p\phi_{B'}\right]O\left(\frac{1}{1-k}\phi_C\right)(r) < r$$

In the following result, we will use the notion of D-Lipschitzian operators.

Corollary 5.2: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, C, D, B′: S →X are five weakly sequentially continuous mapping satisfying the following properties:

(i) A, B and C are D-Lipschitzians with functions φA, φB and φC,

(ii) D is a contraction mapping with a constant k,

(iii) B′ maps bounded sets into relatively weakly compact sets,

(iv) A(S), B(S) and B′ (S) are bounded and C(S)(I-D)(S), and

(v) $Ax + B(I-D)^{-1} Cx.B'(I-D)^{-1} Cx \in S$ for all x ∈ S.

Then, the operator matrix (1.3) has, at least, one fixed point in S×S whenever

$$\|B'(S)\| p\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) \le r$$

Remark 5.1: Because every q-lipschitzian and weakly sequentially continuous mapping is a D-set- lipschitzian mapping with D-function φ(r)=qr ([30], Proposition 3.1)), then Theorem 3:4 in a study [15] follows as a consequence of Corollary 3.2. }

Remark 5.2: If we take C=1_X and D=0_X in the above result, in which 1_X and 0_X represents respectively the unit and the zero element of the Banach algebra X, then we obtain the following Corollary, which extends and generalize one of the results obtained in a study [30], Theorem 3.8.

Corollary 5.3: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the condition (P): Assume that A,B: X→ X and $B': S \to X$ are three weakly sequentially continuous mapping satisfying the following properties [31-34]:

(i) A and B are D-Lipschitzians with D-functions φA and ΦB respectively,

(ii) B′ maps bounded sets into relatively weakly compact sets,

(iii) A(S), B(S) and B′(S) are bounded, and

(iv) $x = Ax + Bx \cdot B'y, y \notin (S) \Rightarrow x \in S$

Then, the operator equation $x \in Ax + Bx \cdot B'x$, has a solution whenever $\|B'(S)\| p\phi_B(r) + \phi A(r) \le r$

Proof: Let y be fixed in S and let us define the mapping Ty: X →X by the formula

$$Ty(x) = Ax + Bx.B'y$$

Let x_1, x_2 ∈ X, The use of assumption (i) leads to

$$\|Ty(x1) - Ty(x_2)\| \le \|Ax_1 - Ax_2\| + \|Bx_1.B'y - Bx_2.B'y\|$$
$$\le \|Ax_1 - Ax_2\| + \|Bx_1 - Bx_2\|\|B'y\|$$
$$\le \phi A(\|x1 - x2\|) + \|B'(S)\|\phi_B(\|x_1 - x_2\|)$$

Now, an application of Boyd and Wong's fixed point theorem [14] yields that there is a unique point

x_y ∈ X such that $T_y(x_y)=x_y$, or equivalently

$$x_y = Ax_y + Bx_y.B'y$$

Consequently, the operator $T := \left(\frac{I-A}{B}\right)^{-1} B' : S \to X$ is well defined. Because assumption (iv) holds for all y ∈ S; we have T(S)⊆S. Using the following equality

$$T = AT + BTB' \tag{2.3}$$

combined with the assumption (iii); we obtain

$$T(S) \subset A(TS) + B(TS)B'(S)$$
$$\subset A(S) + B(S)B'(S)$$

Using the subadditivity of the De Blasi's measure of weak noncompactness, we get

$$\beta(T(S)) \le \beta(A(TS) + B(TS)B'(S))$$
$$\le \beta(AT(S)) + \beta(BT)B'(S))$$
$$\le \phi_A(\beta(T(S))) + \|B'(S)\|\phi B(\beta(T(S)))$$

If β(T(S)) 6=0; then we get a contradiction and so T(S) is a relatively weakly compact subset of S. In order to achieve the proof, we will apply Arino, Gautier and Penot's fixed point theorem [4]. Hence, we only

have to show that the operator T is weakly sequentially continuous. Indeed, consider $\{y_n\}_{n=0}^\infty$ as a sequence in S that is weakly convergent to y: From the above discussion, it is easy to see that $\{Ty_n\}_{n=0}^\infty$ is relatively weakly compact. Then, there exists a subsequence (yn_j) of $\{y_n\}_{n=0}^\infty$ such that $Ty_{nj} \rightharpoonup x$. Taking into account the weak sequential continuity of the maps A, B and B' and using the equality (2.3) to obtain x=Ax + Bx By. Now we claim that $Ty_n \to Ty$. Suppose the contrary, then there exists a subsequence (yn_j) of $\{y_n\}_{n=0}^\infty$ and a weak neighborhood V^w of Ty such that $Ty_{nj_k} \notin V^w$, for all j ∈ N: Arguing as before, we may extract a subsequence (yn_{j_k}) such that $Tyn_{jk} \to Ty$, which is a contradiction and the claim is approved. The use of Arino, Gautier and Penot's fixed point theorem achieves the proof Q.E.D.

Remark 5.3: Under the assumptions of Corollary 3.3, the fixed point set of the operator inverse $\left(\dfrac{I-A}{B}\right)^{-1} B'$ is a nonempty and relatively weakly compact subset of X. Now, we can modify some assumptions of Theorem 3.1 in order to study the same problem.

Theorem 5.2

Let S be a nonempty, closed, and convex subset of a Banach algebra X satisfying the condition (P): Assume that A, B, B':S→$P_{bd,cl,cv}$(X), C: S →X and D: X→X are ve multi-valued operators satisfying the following proprieties:

(i) A and B are D-set-Lipschitzians with D-functions φA and ΦB, respectively,

(ii) A and B are s.w.u.sco with weakly sequentially closed graphs,

(iii) C is weakly sequentially continuous and is D-Lipschitzian with D-function φC,

(iv) D is weakly sequentially continuous and is a contraction with a constant k;

(v) B0 is weakly completely continuous and C(S)⊆(I-D)(S),

(vi) Ax + B(I-D)$^{-1}$Cx B' (I-D)$^{-1}$Cx ∈ Pcv(S), for all x ∈ S,

Then, the operator matrix (1.3) has a fixed point whenever

$$\|B'(S)\|\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r)+\phi_A(r)<r$$

Proof: An argument similar to that in the proof of Theorem 5.1 leads to the weak sequential continuity of the mapping (I-D)$^{-1}$C: Let $\{x_n, n \in N\}$ be a sequence in S that is weakly convergent and let $\{z_n, n \in N\}$ be any sequence of B' (I-D^{-1}C(x_n). Since B' is weakly completely continuous, it follows that the sequence $\{z_n\}$ has a weakly convergent subsequence, and so B' (I-D)$^{-1}$C is a s.w.u.sco. multi-valued mapping.

By using Theorem 5.1, it is sufficient to check that the multi-valued mapping G defined in eqn. (3.2) is condensing with respect to β. In order to achieve this, let M be a bounded subset of S with (M) > 0.

Then (I-D)$^{-1}$C(M) is a bounded subset with bound $\delta=\dfrac{2}{1-k}\left(\left\|(I-D)^{-1}Ca\right\|+\|C(M)\|\right)$, for some fixed point a ∈ M, and we have

$$\beta\big(G(M)\big)\leq\beta\Big(A(M)+B(I-D)^{-1}C(M)\cdot B'(I-D)^{-1}C(M)\Big)$$

$$\leq\|B'(S)\|p\beta\Big(B(I-D)^{-1}C(M)\Big)+\beta\big(A(M)\big)$$

$$\leq\left[\|B'(S)\|p\phi B\mathrm{O}\left(\frac{1}{1-k}\phi_C\right)+\phi_A\right]\big(B(M)\big)$$

Since $\|B'(S)\|\phi_B\mathrm{O}\left(\dfrac{1}{1-k}\phi_C\right)(r)+\phi_A(r)<r$, it follows that the multi-valued mapping G is condensing with respect to β. Now, we may apply Theorem 4.2 to infer that G has, at least, one fixed point x in S and consequently, the vector y=(I-D)$^{-1}$Cx solves the problem. Q.E.D.

Corollary 5.4: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, C: S→ X, D: X→X and $B' : S' \to P_{bd,cl,cv}(X)$ be five multi-valued operators satisfying the following properties:

(i) A, B, C and B' are D-Lipschitzians with D-functions φA, φB, φC and φ B',

(ii) A, B, C and D are weakly sequentially continuous, and

(iii) D is a contraction mapping with a constant k and C(S) (I-D) (S),

(iv) B' is s.w.u.sco with weakly sequentially closed graph,

(v) Ax + B(I-D)-1Cx B' (I-D)$^{-1}$Cx ∈ Pcv(S); for all x ∈ S:

Then, the operator matrix (1.3) has a fixed point in S × S whenever

$$\|B'(S)\|p\phi_{B'}\mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r)+\|B'(S)\|p\phi_B\mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r)+\phi_A(r)<r$$

The next result extends Theorem 3:2 in a study [34] in a Banach algebra relative to the weak topology.

Theorem 5.3

Let S be a closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, B': S→P(X) are three multi-valued operators and C: S→ X and D: X→X are two weakly sequentially continuous operators satisfying the following properties:

(i) A and B are D-set-Lipschitzians with D-functions φA, φB and φ B' respectively,

(ii) A and B are s.w.u.sco. with weakly sequentially closed graphs,

(iii) C is D-Lipschitzian with D-function φC and C(S)⊆ (I-D)(S),

(iv) D is a contraction with a constant k,

(v) B' is weakly completely continuous and A(S), B(S) and B' (S) are bounded,

(vi) $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in P_{cv}(S)$

Then, the operator matrix (1.3) has a fixed point whenever

$$\|B'(S)\|\phi_B\mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r)+\phi_A(r)<r$$

Proof: An argument similar to that in the proof of Theorem 3.1 leads to the multi-valued mapping defined in eqn. (2.2) is well defined and has weakly sequentially closed graph.

Our next task is to prove that G is condensing with respect to β. For this purpose, let M ∈ B(S) with β(M) > 0, then G(M) is a bounded subset of S and we have

$$\beta\big(G(M)\big)\leq\beta\Big(A(M)+B(I-D)^{-1}C(M)\cdot B'(I-D)^{-1}C(M)\Big)$$

$$\leq\|B'(S)\|p\beta\Big(B(I-D)^{-1}C(M)\Big)+\beta\big(A(M)\big)$$

$$\leq\left[\|B'(S)\|p\phi B\mathrm{O}\left(\frac{1}{1-k}\phi_C\right)+\phi_A\right]\big(B(M)\big)$$

The result follows from Theorem 2.2.

When A, B, C and D are single valued operators, Theorem 3.3 reduces to Q.E.D

Corollary 5.5: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, C:S→XD: X →X and $B' : S \to P_{cv}(X)$ are five multi-valued operators satisfying the following properties:

(i) A, B and C are D-Lipschitzians with D-functions ϕ A, ϕ B and ϕ C respectively,

(ii) A, B, C and D are weakly sequentially continuous,

(iii) A(S), B(S) and $B'(S)$ are bounded and $C(S) \subset (I-D)(S)$,

(iv) D is a contraction mapping with a constant k,

(v) B' is weakly completely continuous,

Then, the operator matrix (1.3) has a fixed point whenever

$$\|B'(S)\| p\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r) + \phi A(r) \leq r \,.$$

Proof: According to Theorem 3.3, it is sufficient to show that G defines a multi-valued operator with a convex values, where G is defined in eqn. (2.2). Let x ∈ S be arbitrary and let $u, u' \in Gx$. Then, there is v, v' $\in B'((I-D)Cx)$ such that

$$\begin{cases} u = Ax + B(I-D)^{-1}Cx \cdot v \\ u' = Ax + B(I-D)^{-1}Cx \cdot v' \end{cases}$$

For all λ ∈ [0, 1], we have

$$\lambda u + (1-\lambda)u' = Ax + B(I-D)^{-1}Cx \cdot (\lambda v + (1-\lambda)v')$$

This implies that $\lambda u + (1-\lambda)u' \in Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx$ and consequently G has convex values on S: The use of Theorem 5.3 achieves the proof. Q.E.D.

Theorem 5.4

Let S be a closed, convex and bounded subset of a Banach algebra X satisfying the condition (P): Assume that A, B, B': S→$P_{bd,cl,cv}$(X), C: S →X and D: X →X are five multi-valued operators satisfying the following properties:

(i) A, B, C, B' and D are Lipschitzians with constants α_1, α_2, α_3, α_4 and k ∈ [0; 1],

(ii) A, B and B' have weakly sequentially closed graphs,

(iii) A, B(I-D)⁻¹C and $B'(I-D)^{-1}C$ are s.w.u.sco. and C(S)⊆(I-D)(S),

(iv) Ax + B(I-D)⁻¹Cx B' (I-D)⁻¹Cx 2 Pcv(S), for all x ∈ S.

Then, the operator matrix (1.3) has a fixed point in S ×S whenever

$$\|B'(S)\| p\frac{\alpha_2\alpha_3}{1-k} + \|B(S)\| p\frac{\alpha_3\alpha_4}{1-k} + \frac{\alpha_2\alpha_3^2\alpha_4}{(1-k)^2} diam(S) + \alpha_1 < 1$$

Proof: By using Remark 4:1 in a study [14], we infer that B(I-D)⁻¹C and B' (I-D)⁻¹C are two Lipschtzian multi-valued mapping with constants $\frac{\alpha_2\alpha_3}{1-k}$ and $\frac{\alpha_3\alpha_4}{1-k}$, respectively. Besides, taking into account that X is a Banach algebra satisfying the condition (P), and using the assumption (iii), we deduce that, the multi-valued mapping G defined in eqn. (2.2) is s.w.u.sco.

Next we will prove that G is condensing with respect to β. For this purpose, consider a subset M of S with β (M) > 0, then G(M) is bounded and we have

$$\beta(G(M)) \leq \beta(A(M)) + B(I-D)^{-1}C(M) \cdot B'(I-D)^{-1}C(M))$$
$$\leq \|B'(S)\| p\beta\left(B(I-D)^{-1}C(M)\right) + \|B(S)\| p\beta\left(B'(I-D)^{-1}C(M)\right)$$
$$+ \beta\left(B(I-D)^{-1}C(M)\right)\beta\left(B'(I-D)^{-1}C(M)\right) + \beta(A(M))$$
$$\leq \left[\|B'(S)\| p\frac{\alpha_2\alpha_3}{1-k} + \|B(S)\| p\frac{\alpha_2\alpha_3}{1-k} + \frac{\alpha_2\alpha3_3^2\alpha_4}{1-k} diam(S) + \alpha_1\right]\beta(M)$$

he use of Theorem 4.2 achieves the proof. Q.E.D.

Remark 5.4: We should consider the case where C=1_X and D=0_X, in which 1_X and 0_X represents the unit element and the zero element of the Banach algebra X. We have the following particular cases that constitute the versions of hybrid fixed point theorems under weak topology. Precisely, if we take α_4=0 and B' is weakly completely continuous in the above Theorem, then we obtain the following Corollary 3.6, which represents the new version of the fixed point result obtained in a study [31, Theorem 5.4].

Corollary 5.6: Let S be a closed, convex and bounded subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, $B' : S \to P_{bd,cl,cv}(S)$ are three multi-valued operators two operators satisfying the following properties:

(i) A and B are Lipschitzians with Lipschitz constants q_1 and q_2 respectively,

(ii) A and B are s.w.u.sco. with weakly sequentially closed graphs,

(iii) B' is weakly completely continuous,

(iv) $Ax + Bx \cdot B'x \in P_{cv}(S)$, for all x ∈ S:

Then, the operator equation $x \in Ax + Bx \cdot B'x$ has a solution in S whenever $q_1 + Mq_2 < 1$.

New Class of Multivalued Mapping

Before stating the main result, we need the following lemma [32-35]:

Lemma 6.1

Let S be a nonempty, closed, bounded, and convex subset of a Banach algebra X satisfying the condition (P): Suppose that A, B, $B' : S \to P_{cv}(X)$, C: S→X and D: X→X are five multi-valued operators satisfying the following properties:

(i) A and B are D-set-Lipschitzians with D-functions ϕ_A and ϕ_B respectively,

(ii) A and B(I-D)⁻¹C are s.w.u.sco. with weakly sequentially closed graphs,

(iii) C and D are Lipschitzian with constants q and k respectively with q + k < 1,

(iv) C and D are weakly sequentially continuous and $C(S) \subseteq (I-D)(S)$,

(v) $x \in Ax + B(I-D)^{-1}Cx \cdot B(I-D)^{-1}Cy \subseteq S$, for all x, y ∈ S.

Then, the multivalued operator $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)$ exists on $B'(I-D)^{-1}C(S)$, whenever $\dfrac{Mq}{1-k}\phi_B(r) + \phi_A(r) < r$ for r > 0, where $M = \left\|B'(I-D)^{-1}C(S)\right\|$

Proof: Let y be a fixed point in S and let $z \in B'(I-D)^{-1}C(y)$. Let us define the multi-valued mapping $\varphi_z : S \to P_{cv}(S)$ by the formula

$$\varphi_z(x) = Ax + B(I-D)^{-1}Cx \cdot z$$

Notice that this operator has a weakly sequentially closed graph. Indeed, let $\{u_n, n \in N\}$ be any sequence in S weakly converging to a point u and let $v_n \in \varphi_z(u_n)$ such that $v_n \to v$. Then, there is $\theta_n \in A(u_n)$ and $\xi_n \in B(I-D)^{-1}C(u_n)$ such that $v_n = \theta_n + \xi_n.z$. The use of both assumption (iv) and assumption (v) ensures that there exists a weakly converging subsequence (θ_{nk}) to a point $\theta \in A(u)$ and there exist a weakly converging subsequence (ξ_{nk}) to a point $\xi \in B(I-D)^{-1}C(u)$.

Using the sequential condition (P); we obtain

$$v_n = \theta n + \xi n.z \to \theta + \xi.z \in \varphi_z(u).$$

Consequently, the multi-valued mapping φ_z has a weakly sequentially closed graph.

Taking into account the weak sequential continuity of the maps C and D and using the equality (2.1) we get, for any bounded subset M of S with $\beta(M) > 0$

$$\beta((I-D)^{-1}C(M)) \leq \beta(C(M)) + \beta(D(I-D)^{-1}C(M))$$

$$\leq q\beta(M) + k\beta((I-D)^{-1}C(M))$$

This implies that

$$\beta\left((I-D)^{-1}C(M)\right) < \frac{q}{1-k}\beta(M)$$

Besides, it is easy to see that $\varphi_z(M)$ is bounded and we have

$$\beta\left((I-D)^{-1}C(M)\right) \leq \frac{q}{1-k}\beta(M)$$

$$\leq \left[\phi A + \frac{qM}{1-k}\phi B\right](\beta(M))$$

$$< \beta(M)$$

Thus φ_z is condensing with respect to β. Accordingly, the multi-valued mapping φ_z has a fixed point by using Arino, Gautier and Penot's fixed point theorem and so there is $x \in S$ such that

$$x \in \varphi_z(x) = Ax + B(I-D)^{-1}Cx.z.$$

Or equivalently

$$z \in \left(\frac{I-A}{B(I-D)^{-1}C}\right)(x)$$

Then

$$z \in \left(\frac{I-A}{B(I-D)^{-1}C}\right)(x) \cap B'(I-D)^{-1}Cx \cdot z$$

So, the multi-valued mapping $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ exists on $B'(I-D)^{-1}C(S)$. Q.E.D.

In the following result, we will combine Theorem 5.1 and Lemma 6.1.

Theorem 6.1

Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, B': S $\to$P(X), C: S$\to$X and D: X$\to$X are five multi-valued operators satisfying the following properties:

(i) A and B are D-set-Lipschitzians with functions ϕ_A and ϕ_B respectively,

(ii) A, B and B' have weakly sequentially closed graphs,

(iii) C is D-Lipschitzian with D-function ϕ_C,

(iv) D is a contraction mapping with a constant k,

(v) C and D are weakly sequentially continuous and $C(S) \subseteq (I-D)(S)$,

(vi) A, B and B' are s.w.u.sco. and A(S), B(S) and $B'(S)$ are bounded

(vii) $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$, for all y $\in$ S,

(viii) $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C$ is condensing with respect to β,

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Proof: Let us define the multi-valued mapping T: S$\to$P$_{cv}$(S) by the formula

$$T(x) = \left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(x)$$

From Lemma 4.1, it follows that this operator is well defined and has a sequentially closed graph. Indeed,

let $\{x_n, n \in N\}$ be any sequence in S weakly converging to a point x and let $y_n \in T(x_n)$ such that $y_n \to y$. In this case, the set $\{x_n, n \in N\}$ is relatively weakly compact and we have

$$y_n \in Ay_n + B(I-D)^{-1}Cy_n \cdot B'(I-D)^{-1}Cx_n|$$

Thus, there is $a_n \in A(y_n)$, $b_n \in B(I-D)^{-1}Cy_n$ and $b'_n \in B'(I-D)^{-1}Cx_n$ such that $y_n = a_n + b_n \cdot b'_n$ n:

Using the equality $(I-D)^{-1}C = C + D(I-D)^{-1}C$ and the fact that C and D are weakly sequentially continuous, we obtain

$$\beta\left(\left\{(I-D)^{-1}C(x_n), n \in \mathbb{N}\right\}\right) \leq \beta\left(\left\{C(x_n), n \in \mathbb{N}\right\}\right) + \beta\left(\left\{D(I-D)^{-1}C(x_n), n \in \mathbb{N}\right\}\right)$$

$$\leq \phi C\left(\beta\left(\left\{x_n, n \in \mathbb{N}\right\}\right)\right) + k\beta\left(\left\{(I-D)^{-1}C(x_n), n \in \mathbb{N}\right\}\right)$$

and therefore

$$\beta\left(\left\{(I-D)^{-1}C(x_n), n \in \mathbb{N}\right\}\right) \leq \frac{1}{1-k}\phi_C\left(\beta\left(\left\{(x_n), n \in \mathbb{N}\right\}\right) + \beta\right) \text{ n 2 Ng)):}$$

This inequality means, in particular, that the set $(I-D)^{-1}C(x_n)$, n $\in$ N is relatively weakly compact.

Then, there exists a subsequence (x_{nk}) such that $(I-D)^{-1}C(x_{nk}) \to \gamma$:

Going back to eqn. (2.1) yields $\gamma = D(\gamma) + Cx$. Accordingly we have $(I-D)^{-1}C(x_{nk}) \to (I-D)^{-1}C(x)$.

Now a standard argument shows that the operator $(I-D)^{-1}C$ is weakly sequentially continuous [36,37]. From both assumption (i) and assumption (vii); it follows that there exists a subsequence (y_{nk}) of $\{y_n, n \in N\}$such that

$$y_{nk} = a_{nk} + b_{nk}b'_{nk} \to a + b \cdot b' \in Ay + B(I-D)^{-1}Cy \cdot B'(I-D)^{-1}Cx$$

for some a $\in$ A$_y$, b $\in$ B(I-D)$^{-1}$Cy and $b' \in B'(I-D)^{-1}Cx$.

Besides, taking into account assumptions (iii) and (vi), we get

$$T(S) \subseteq AT(S) + B(I-D)^{-1}CT(S) \cdot B'(I-D)^{-1}C(S)$$

$$\subseteq A(S) + B(I-D)^{-1}C(S) \cdot B'(I-D)^{-1}C(S)$$

$$\subseteq A(S) + B(S) \cdot B'(S)$$

Now, by applying Theorem 4.2, we deduce that T has, at least one, fixed point in S. Hence, the vector y=(I-D)$^{-1}$Cx D) xsolve the problem. Q.E.D.

Corollary 6.1: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, B' B: S→P(X), C: S→X and D: X→X are multi-valued operators satisfying:

(i) A and B are D-set-Lipschitzians with functions ϕ_A and ϕ_B respectively,

(ii) A, B and B' have weakly sequentially closed graphs,

(iii) C is D-Lipschitzian with D-function ϕ_C and $C(S) \subseteq (I-D)(S)$,

(iv) D is a contraction mapping with a constant k;

(v) C and D are weakly sequentially continuous,

(vi) A, B and B' are s.w.u.sco. and A(S), B(S) and $B'(S)$ are bounded,

(vii) $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$ for all y ∈ S,

(viii) $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C$ is weakly compact.

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Corollary 6.2: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B, $B': S \to P(X)$, C: S→X and D: X→X are five multi-valued operators satisfying:

(i) A, B and C are D-Lipschitzians with D-functions ϕ_A, ϕ_B and ϕ_C respectively,

(ii) A, B and B' have weakly sequentially closed graphs,

(iii) D is a contraction mapping with a constant k,

(iv) C and D are weakly sequentially continuous with $C(S) \subseteq (I-D)(S)$,

(v) A, B and B' are s.w.u.sco. such that A(S), B(S) and $B'(S)$ are bounded,

(vi) $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$ for all y ∈ S,

(vii) $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C$ is condensing with respect to β.

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

In the particular case when A, B and B are single-valued weakly sequentially continuous mappings, we get the following corollary which extends and improves

Corollary 6.3: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the condition (P): Assume that A, C: S→X, and $B, B', D: X \to X$ are five weakly sequentially continuous operators satisfying the following properties:

(i) A and B are D-Lipschitzians with D-functions ϕ_A and ϕ_B respectively,

(ii) D is a contraction mapping with a constant k and $C(S) \subseteq (I-D)(S)$,

(iii) A(S), B(S) and $B'(S)$ are bounded,

(iv) $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$ is condensing with respect to β,

(v) $B'(I-D)^{-1}C(S) \subset \left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1}(S)$

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Theorem 6.2

Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P):

Assume that $A, B, B': S \to P_{cv}(X), C: S \to X$ and D: X→X are five multi-valued operators satisfying:

(i) A and B are D-set-Lipschitzians with functions ϕ_A and ϕ_B respectively,

(ii) A and B have weakly sequentially closed graphs,

(iii) C is D-Lipschitzian with D-function ϕ_C and C(S)→(I-D)(S),

(iv) D is a contraction mapping with a constant k;

(v) C and D are weakly sequentially continuous,

(vi) B' is weakly completely continuous,

(vii) A and B are s.w.u.sco. and A(S);B(S) and $B'(S)$ are bounded

(viii) $B'(I-D)^{-1}C(S) \subset \left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1}(S)$, and for all y ∈ S, we have

$$\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$$

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Proof: From Lemma 4.1, it follows that the multi-valued mapping $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ is well defined on $B'(I-D)^{-1}C(S)$. Proceeding as in the proof of Theorem 4.1, we obtain that the multi-valued mapping T defined in eqn. (3.1) has a weakly sequentially closed graph and the set T(S) is bounded. Next we will prove that T is condensing with respect to β. For this purpose, let S_1 be a bounded subset of S with (S_1) > 0. We first show that $\left\|\overline{B'(I-D)^{-1}C(S_1)}^w\right\| p \le \|B'(S)\| p$

Let $x \in \overline{B'(I-D)^{-1}C(S_1)}^w$, then there exists $\{x_n, n \in N\} \subset B'(I-D)^{-1}C(S_1)$ such that x$_n$→x.

Since $\|x\| \le \liminf \|x_n\|$, we obtain that

$$\|x\| \le \|B'(I-D)^{-1}C(S_1)\| p \le \|B'(S)\| p = M$$

Using the equality $T = AT + B(I-D)^{-1}CT B'(I-D)^{-1}C$, we infer that

$$\beta\left(T(M)\right) \le \beta\left(\overline{B'(I-D)^{-1}CT(M) \cdot B'(I-D)^{-1}C(M)}^{\,w}\right) + \beta\left(AT(M)\right)$$

$$M\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)\beta\left(T(M)\right) + \phi_A \beta\left(T(M)\right)$$

This shows that T(M) is a relatively weakly compact subset of S and consequently T is condensing with respect to β. The result follows from Theorem 4.2. Q.E.D.

Corollary 6.4: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A, B:S→$P_{cl,bd,cv}$(X), $B' : S \to P(S), C : S \to X$ and D: X→X are five multivalued mappings satisfying the following properties:

(i) A, B and C are D-Lipschitzians with functions ϕ_A, ϕ_B and ϕ_C respectively,

(ii) A and B have weakly sequentially closed graphs,

(iii) C and D are weakly sequentially continuous with C(S)→(I→D)(S),

(iv) D is a contraction mapping with a constant k;

(v) B' is weakly completely continuous,

(vi) A and B are s.w.u.sco. and A(S), B(S) and $B'(S)$ are bounded

(vii) $B'(I-D)^{-1}C(S) \subset \left[\dfrac{I-A}{B(I-D)^{-1}C}\right]^{-1}(S)$ and for all y ∈ S, we have

$$\left[\frac{I-A}{B(I-D)^{-1}C}\right]^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$$

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

An improved version of Corollary 4.4 under a weaker assumption (v) therefore is provided in the following fixed point theorem.

Corollary 6.5: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P):

Assume that $A, B, C, B' : S \to X$ and D: X→X are five weakly sequentially continuous operators satisfying the following properties:

(i) A, B and C are D-Lipschitzians with functions ϕ_A, ϕ_B and ϕ_C respectively,

(ii) D is a k-contraction for some k ∈ [0, 1],

(iii) A(S), B(S) and $B'(S)$ are bounded and $C(S) \subseteq (I-D)(S)$

(iv) B' is maps bounded sets into relatively weakly compact sets,

(v) $x = Ax + B(I-D)^{-1}Cx.B'(I-D)Cy; y \in S \Rightarrow x \in S$,

Then, the block operator matrix (1.3) has a fixed point whenever

$$M\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Remark 6.1: Like in the proof of Corollary 4.5, we verify that $\left[\dfrac{I-A}{B(I-D)^{-1}C}\right]^{-1} B'(I-D)^{-1}C$ relatively weakly compact single-valued mapping. Thus, Corollary 4.5 improves and extends Theorem 6.1 in a study [33] without the condition of quasi-regularity on the operator B(I-D)⁻¹C.

Taking A=M_0 a weakly compact convex subset of X in Theorem 6.2, we obtain the following result.

Corollary 6.6: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P):

Assume that $B : S \to P_{cv}(X), B' : S \to P(S)C : S \to X$ and D: X→X are five multi-valued operators satisfying:

(i) B is D-set-Lipschitzians with function ϕ_B,

(ii) C is D-Lipschitzian with D-function ϕ_C,

(iii) B and B0 have weakly sequentially closed graphs,

(iv) D is a contraction mapping with a constant k,

(v) C and D are weakly sequentialy continuous and $C(S) \subseteq (I-D)(S)$,

(vi) B and B' are s.w.u.sco. such that B(S) and $B'(S)$ is bounded

(vii) $B'(I-D)^{-1}C(S) \subset \left[\dfrac{I-M_0}{B(I-D)^{-1}C}\right]^{-1}(S)$, and for all y ∈ S, we have

$$\left[\frac{I-M_0}{B(I-D)^{-1}C}\right]^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$$

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Corollary 6.7: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P).

Assume that B: S→P_{cv}(X), $B' : S \to P(S,)C : S \to X$ and D: X→X are five multi-valued operators satisfying:

(i) B(S) is bounded and B is D-set-Lipschitzians with function ϕ_B,

(ii) B is s.w.u.sco. and has a weakly sequentially closed graphs,

(iii) C is D-Lipschitzian with D-function ϕ_C,

(iv) D is a contraction mapping with a constant k,

(v) C and D are weakly sequentially continuous with $C(S) \subseteq (I-D)(S)$,

(vi) B' is weakly completely continuous and B0(S) are bounded

(vii) $B'(I-D)^{-1}C(S) \subset \left[\dfrac{I-M_0}{B(I-D)^{-1}C}\right]^{-1}(S)$ and for all y ∈ S, we have

$$\left[\frac{I-M_0}{B(I-D)^{-1}C}\right]^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$$

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B \mathrm{O}\left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r$$

Corollary 6.8: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P):

Assume that B: S→$P_{cl,bd,cv}$(X), $B' : S \to P(S,)C : S \to X$ and D: X →X are five multi-valued operators satisfying:

(i) B and C are D-Lipschitzians with functions φB and φC, respectively

(ii) B and B' are s.w.u.sco. with weakly sequentially closed graphs,

(iii) B(S) and $B'(S)$ are bounded,

(iv) D is a contraction mapping with a constant k

(v) C and D are weakly sequentially continuous such that $C(S) \subseteq (I-D)(S)$,

(vi) $\left(\dfrac{I-M_0}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C$ is condensing with respect to β,

(vii) $B'(I-D)^{-1}C(S) \subset \left(\dfrac{I-M_0}{B(I-D)^{-1}C}\right)^{-1}(S)$ and for all y ∈ S, we have

$$\left(\frac{I-M_0}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$$

Then the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) < r$$

In particular, if we take $M_0 = 0_X$, we obtain the following:

Corollary 6.9: Let S be a nonempty, closed and convex subset of a Banach algebra X satisfying the sequential condition (P).

Assume that B: $S \to P_{cv}(X)$, $B'(S) \to P(S,) C : S \to X$ and D: X→X are five multi-valued operators satisfying:

(i) B is D-set-Lipschitzians with function φB with weakly sequentially closed graph,

(ii) B is s.w.u.sco. and B(S) is bounded

(iii) C is D-Lipschitzian with D-function φC,

(iv) D is a k-contraction for some k ∈ [0, 1],

(v) C and D are weakly sequentially continuous with $C(S) \subseteq (I-D)(S)$,

(vi) B' is weakly completely continuous such that B'(S) is bounded,

(vii) $B'(I-D)^{-1}C(S) \subset \left(\dfrac{I-M_0}{B(I-D)^{-1}C}\right)^{-1}(S)$ and for all y ∈ S, we have

$$\left(\frac{I-M_0}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}C(y) \in P_{cv}(S)$$

Then, the operator matrix (1.3) has a fixed point whenever

$$M\phi_B O\left(\frac{1}{1-k}\phi_C\right)(r) < r$$

Remark 6.2: If we take $C = 1_X$, $D = 0_X$ and $B = M_0$ a weakly compact convex subset of X in Theorem 6.2, in which 1X and 0X represents the unit element and the zero element of the Banach algebra X, then we obtain the following Corollary 4.10 and Corollary 4.11, which represents the new version of the fixed point result obtained in a study [9].

Corollary 6.10: Let S be a closed and convex subset of a Banach algebra X satisfying the sequential condition (P): Assume that A: X→ $P_{cv}(X)$ and $B' : (S) \to P(S)$ are two multi-valued operators satisfying the following properties

(i) A is D-set-contraction with D-function ϕ_A,

(ii) A and B' are s.w.u.sco with weakly sequentially closed graphs,

(iii) A(S) and B'(S) are bounded,

(iv) $\left(\dfrac{I-A}{M_0}\right)^{-1} B'$ is condensing with respect to β,

(v) $x \in Ax + M_0 \cdot B'y, y \in S \Rightarrow x \in S$, and for all y ∈ S, we have

$$\left(\frac{I-A}{M_0}\right)^{-1} B'(y) \in P_{cv}(X)$$

Then the operator equation $x \in Ax + M_0 \cdot B'x$ has, at least, one solution in S.

Corollary 6.11: Let S be a closed and convex subset of a Banach algebra X satisfying the sequential condition (P).

Assume that A: X→$P_{cv}(X)$ and $B' : S \to P(S))$ are two multi-valued operators satisfying the following properties

(i) A is D-set-contraction with D-function ϕ_A,

(ii) A is s.w.u.sco with weakly sequentially closed graph,

(iii) A(S) and B'(S) are bounded,

(iv) B' is weakly completely continuous,

(v) $x \in Ax + M_0 \cdot B'y, y \in S \Rightarrow x \in S$ and for all y ∈ S, we have

$$\left(\frac{I-A}{M_0}\right)^{-1} B'(y) \in P_{cv}(X)$$

Then the operator equation $x \in Ax + M_0 \cdot B'x$ has, at least, one solution in S.

Nonlinear Functional Integro-Differential Inclusions

Let X be a Banach algebra satisfying the sequential condition (P). The purpose of this section is to illustrate the applicability of Theorem 5.2 in order to study, in the space C([0,τ],X), 0 <τ < 1, of all continuous functions, the existence of solutions for the following nonlinear functional integral inclusion QIV P (1.1).

We need the following definition in the sequel.

Definition 7.1

A mapping g: [0,τ] →X is said to be scalarly measurable if for every φ in the topological dual space X* the function $\varphi O g$ is measurable.

Under the following hypotheses, we could reach the solution of (1.1):

$(\mathcal{H}_0)$ The mappings f,k: [0, τ]×X→X are such that:

(a) f and k are weakly sequentially continuous,

(b) f is q_1-contraction with respect to the second variable,

(c) k is D-Lipschitzian with D-function ϕ_k with respect to the second variable,

(H_1) The function p: [0,τ]×X→R is such that:

(a) p is weakly sequentially continuous and is D-Lipschitzian with D-function ϕ_p,

(b) $\|p(t,x)\| \leq \lambda r_0$, for x ∈ C([0,τ],X) such that $\|x\|_\infty \leq r_0$.

(H_2) The function f_1: [0,τ]×X→X is such that:

(a) f_1 is weakly sequentially continuous with respect to the second variable,

(b) f_1 is a Φf_1-nonlinear contraction with respect to the second variable,

(c) $\|f_1(t,0)\| + \lambda\|v\| = \lambda$ for all t ∈ [0,τ], and

(d) φ(r) < (1-λ)r, for all r > 0.

$(\mathcal{H}_3)$ The function b: [0,τ]→R_+ is continuous and nonnegative.

$(\mathcal{H}_4)$ There exists a scalarly measurable function v with v(t) ∈ G(t, x(t)), for all t ∈ [0,τ], and for each x∋C([0, τ],X).

$(\mathcal{H}_5)$ G is weakly completely continuous with respect to the second variable.

$(\mathcal{H}_6)$ For each real number r > 0, there exists a function $h_r \in L^1(J,R)$ such that

$$\|G(t,x)\|p \le h_r(t)a \cdot e \quad \text{for } t \in J \text{ for all } x \in R \text{ with} |x|<r \ (\mathcal{H}_7)$$ There exists $r_0 > \dfrac{1}{1-\tau}$

such that $\phi k(r_0) + K + F\|h_{r0}\|L^1 + q_1 r_0 \|h_{r0}\|\infty \le r_0$

For any multi-valued function $G: [0,\tau] \times X \to X$ and for each $x \in C(J,X)$; we denote $\int_0^t G(s,x(s))ds = \left\{\int_0^t v(s)ds \, v \text{ is Pettis integrable} , \ v(t) \in G(t,x(t))\right\}$, v(t) $\in$ G(t,x(t))

Then, we have the following lemma due to Musial [31].

Lemma 7.1

Let f: $[0,\tau] \to X$ be a function satisfying the following conditions:

(i) There is a sequence of Pettis integrable functions $(f_n)_n$ weakly convergent to f,

(ii) There exists $h \in L_1([0,\tau])$ such that for each $\varphi \in X^*$ and each n $\in$ N, we have $\left|\langle\varphi,f\rangle\right| \le h$

Then f is Pettis integrable and $\int_0^\tau f_n(s)ds$ converges weakly to $\int_0^\tau f(s)ds$.

Theorem 7.1

Under assumptions $(\mathcal{H}_0)$-$(\mathcal{H}_7)$, the problem (1.1) has a solution in $Br_0 \times Br_0$.

Proof: Let S be the closed ball Br_0 on $C([0,\tau],X)$ centered at origin of radius r_0. Observe that the above problem (1.1) may be written abstractly in the following form

$$\begin{cases} x(t) \in Ax(t) + By(t) \cdot B'y(t) \\ y(t) = Cx(t) + Dy(t) \end{cases}$$

Where

$$\begin{cases} Ax(t) = k(t,x(t)) \\ Bx(t) = f(t,x(t)) \\ Cx(t) = \left(\lambda + \int_0^{\sigma(t)} p(s,x(\lambda s))ds\right) \cdot v \\ Dx(t) = f_1(t,x(t)) \\ B'x(t) = \dfrac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v(s)ds \end{cases} \quad (3.1)$$

for all t $\in$ J.

We shall show that A, B, C, D and B' satisfy all conditions of Theorem 5.2. This will be achieved in a series of following steps:

Step I: A, B, C and D define four weakly sequentially continuous single valued operators on $C([0,\tau],X)$ and $B': S \to Pcl,bd,cv(C([0,\tau],X))$. The claim regarding the operators A, B, C and D is immediate. In fact, using both assumption $(_0)(b)$ and assumption $(\mathcal{H}_0)(c)$ combined with the fact that ϕ_k is continuous satisfying $\phi_k(0)=0$, we obtain Ax, Bx $\in$ C([0,τ],X). Let $\{x_n, n \in N\}$ be a weakly convergent sequence of S to a point x. In this case $\{x_n, n \in N\}$ is bounded and so we can apply the Dobrakov's theorem [20] in order to get $(t, x_n(t)) \to (t, x(t))$ in $[0,\tau] \times X$.

Using assumption $(\mathcal{H}_0)$, we obtain

$(Ax_n)(t) \to (Ax)(t)$ in X. (3.2)

Going back to eqn. (3.2) and to the fact that $\{Ax_n, n \in N\}$ is a bounded subset of X shows that A is a weakly sequentially continuous mapping. By using the same argument, we conclude that B, C and D are also three weakly sequentially continuous operators. Now, let us demonstrate that B' has convex values on $C([0,\tau],X)$. For this purpose, let x $\in$ C([0,τ],X) be arbitrary and let u_1, $u_2 \in B'x$. Then, there exists two Pettis integrable mappings v_1, v_2 with $v_1(s), v_2(s) \in G(s, x(s))$ such that $u_1(t) = \dfrac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v_1(s)ds; t \in [0.\tau]$

and $u_2(t) = \dfrac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v_1(s)ds; t \in [0.\tau]$

So, for any $\lambda \in [0, 1]$, we have

$$\lambda u_1(t) + (1-\lambda)u_2(t) = \dfrac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v(s)ds$$

where v(s)=$\lambda v_1(s) + (1-\lambda)v_2(s) \in$ G(s, x(s)) for all s $\in$ J.

As a result, $\lambda u_1 + (1-\lambda)u_2(s) \in B'x$ and B' defines a multi-valued map with convex values. Again, for each $u \in B'x$, there is a Pettis integrable v with v(s) $\in$ G(s, x(s)) such that

$$u(t) = \dfrac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v(s)ds$$

Let $t, t' \in [0,\tau]$ with $u(t) \ne u(t')$, then there exists $\varphi \in (C([0,\tau],X))^*$ with $\|\varphi\|=1$

$$\|u(t) - u(t')\| = \varphi\left(u(t) - u(t')\right)$$

From assumption $(\mathcal{H}_6)$, it follows that

$$\|u(t) - u(t')\| \le \int_t^{t'} h_r(s)ds$$

Consequently $B'x \in P\left(C([0,\tau],X)\right)$.

Step II: A, B and C are D-Lipschitzians on $C([0,\tau];=,X)$. To see this, for all x,y $\in$ C([0,τ],X); we have

$$\|Cx - Cy\| \le \|v\| \sup_{t\in[0,\tau]} \int_0^{\sigma(t)} \|p(s,x(\lambda s)) - p(s,y(\lambda(s)))\| ds$$
$$\le \|v\| \sup_{t\in[0,\tau]} \int_0^{\sigma(t)} \phi p(\|x(\lambda s) - y(\lambda s)\|) ds$$
$$\le \|v\| \sup_{t\in[0,\tau]} (\|x - y\|_\infty)$$
$$\le \|v\| \tau \phi_p(\|x - y\|_\infty)$$

This shows that C is D-Lipschitzian with D-function $\psi(r) = \|v\|\tau\phi p(r)$, and it is similarly clear that A and B are D-Lipschitzian with D-functions $\phi_A=\phi_k$ and $\phi_B(r)=q_1 r$. Rewording the above discussion, one obtains that A, B and C are D-set-Lipschitzians with D-functions ϕ_A, ϕ_B and respectively.

Step III: $C(S) \subset (I - D)(S)$. Since f_1 is a nonlinear contraction operator with respect to the second variable, it follows that the operator inverse $(I-D)^{-1}$ is well-defined on $C([0,\tau],X)$ in view of Theorem 1 in a study [14]. Let (x, y) $\in$ S$\times$C([0,τ],X)

such that

$$y(t) = \left(\lambda + \int_0^{\sigma(t)} p(t,x(\lambda s))ds\right) v + f_1(t, y(t)), t \in [0,\tau]$$

Since y ∈ C([0,τ],X), there is t*2 [0,τ] such that $\|y\|_\infty = \|y(t*)\|$ and then

$$\|y\|_\infty \le \left|\lambda + \int_0^{\sigma(t^+)} p(t*,x(\lambda s))ds\right|\|v\| + \|f1(t*,y(t*)) - f_1(t*,0)\| + \|f_1(t*,0)\|$$

$$\le (\lambda + T\lambda r_0)\|v\| + \phi(\|y(t*)\|) + \lambda(1 - \|v\|)$$

This implies that $\|y\|_\infty \le \tau r_0\|v\| + 1 \le r_0$, and so $(I-D)^{-1}C(S) \subseteq S$

Step IV: B' has a weakly sequentially closed graph. To see this, let {x_n, n ∈ N} be any sequence in S weakly converging to a point x and let $y_n \in B'(x_n)$ such that $y_n \to y$. For this case, {$x_n(s)$, s ∈ [0,τ]} is a bounded subset of X and there exists a sequence of Pettis integrable mapping $v_n \in F([0,τ],X)$ with $v_n(s) \in G(s, x_n(s))$, for each s ∈ [0,φ] such that

$$y_n(t) = \frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v_n(s)ds$$

Since G(t,), t ∈ [0,τ] is weakly completely continuous, the set {$v_n(s)$, n ∈ N} is relatively weakly compact, for all s ∈ [0,τ] and consequently the set {v_n, n ∈ N} is relatively weakly compact in view of Tychono 's theorem. Therefore, there exists a subsequence (v_{nk}) of (v_n) such that $v_{nk} \to v$. From assumption ($\mathcal{H}_6$), it follows that v is Pettis integrable and we have

$$\frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v_n(s)ds \to \frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v_n(s)ds$$

Since G(t,) has a weakly sequentially closed graph, we infer that v(s) ∈ G(s, x(s)) and

$$y(t) = \frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v(s)ds \in B'x(t)$$

Step V: B0(S) is a relatively weakly compact subset of X.

Let {y_n, n ∈ N} be any sequence in $B'(S)$, then there is a sequence {x_n, n ∈ N} of S such that $y_n \in B'(x_n)$. So, there exists a sequence of Pettis integrable mapping $v_n(s) \in G(s, x_n(s))$, s ∈ [0,τ] such that

$$y_n(t) = \frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v(s)ds$$

Since {x_n, n ∈ N} is bounded, there is a renamed subsequence such that $v_n \to v$: The use of the Dobrokov's theorem [20] ensures that $B(S)$ is a relatively weakly compact subset of X.

Conclusion

To end the proof, it remains to verify the assumption (vi) of Theorem 3.2. First, we begin by showing that $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx$ is a convex subset of C([0,τ],X). To see this, let x ∈ C([0,τ],X) and $z, z' \in X$ such that $z, z' \in Ax + B(I-D)^{-1}Cx.B'(I-D)^{-1}Cx$.

Since $(I-D)^{-1}C(S) \subseteq S$; there exists a unique y ∈ S such that

$$z(t) = f(t,y(t))\left[\frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v(s)ds\right] + k(t,x(t))$$

And

$$z'(t) = f(t,y(t))\left[\frac{x_0 - k(0,x_0)}{f(0,y_0)} + \int_0^t v'(s)ds\right] + k(t,x(t))$$

where v, v' are two Pettis integrable with $v(s), v'(s) \in G(s, y(s))$, for all s ∈ [0,τ].

Now, for all λ ∈ [0,1], we have

$$\lambda z(t) + (1-\lambda)z'|(t) = f(t,y(t))\int_0^t [\lambda v(s) + (1-\lambda)v'(s)]ds + k(t,x(t))$$

From assumption ($\mathcal{H}_5$), it follows that

$$\lambda z(t) + (-\lambda)z' \in Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx$$

$$Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in Pcv\big(C([0,τ], X)\big)$$

This implies that

$$Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in P_{cv}\big(C([0,τ], X)\big),$$

for each x ∈ C([0,τ],X). Second, if x ∈ S and $z \in Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx$, there is a unique y ∈ S and a Pettis integrable v with v(s) ∈ G(s,y(s)) such that

$$y(t) = k(t,x(t)) + f(t,y(t)) \cdot \int_0^t v(s)ds; t \in J$$

Therefore, we have

$$|y(t)| \le |k(t,x(t))| + |f(t,y(t))| \cdot \int_0^t |v(s)|ds$$

$$\le |k(t,x(t)) - k(t,0)| + \big(|f(t,y(t)) - f(t,0)| + |f(t,0)|\big)\|h\|L^1$$

$$\le \phi_k(|x(t)|) + K + [q_1|y(t)| + F]\|h_{r_0}\|L^1$$

$$\le \phi_k(|x(t)|) + K + [q_1|y(t)| + F]\|h_{r_0}\|L^1$$

This implies that

$$\|y\|_\infty \le \frac{\phi k(r0) + K + F\|h_{r0}\|L^1}{1 - q_1\|h_{r0}\|_\infty}$$

Thus $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx$ is a convex subset in S, for each x ∈ S.

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