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Explicit Calculations of Tensor Product Coefficients for E_7

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Abstract

We propose a new method to calculate coupling coefficients of E_7 tensor products. Our method is based on explicit use of E_7 characters in the definition of a tensor product.

When applying Weyl character formula for E_7 Lie algebra, one needs to make sums over 2903040 elements of E7 Weyl group. To implement such enormous sums, we show we have a way which makes their calculations possible. This will be accomplished by decomposing an E_7 character into 72 participating A7 characters.

Keywords: Coupling coefficients; Lie algebra; Irreducible representations; Subdominants; Tensor coupling coefficients

Introduction

Let $G_7=E_7,A_7$ and Λ, Λ' be two dominant weights of G_7 where $R(\Lambda)$ and $R(\Lambda')$ are corresponding irreducible representations. For general terms, we follow the book of Humphreys [1] as ever.

Tensor product of these two irreducible representations is defined by,

$$R(\Lambda)\otimes R(\Lambda')=R(\Lambda+\Lambda')+\sum_{\lambda\in S(\lambda+\lambda')} t(\lambda<\Lambda+\Lambda')\ (\lambda) \qquad (I.1)$$

where $S(\lambda+\lambda')$ is the set of $\Lambda+\Lambda'$ subdominants and $t(\lambda<\Lambda+\Lambda')$ s are tensor coupling coefficients. Though Steinberg formula is the best known way, a natural way to calculate tensor coupling coefficients is also to solve the equation

$$Ch(\Lambda)\otimes Ch(\Lambda')=Ch(\Lambda+\Lambda')+\sum_{\lambda\in S(\lambda+\lambda')} t(\lambda<\Lambda+\Lambda')\ (\lambda) \qquad (I.2)$$

for tensor coupling coefficients. $Ch(\lambda)$ here is the character of an irreducible representation R(λ) which corresponds to a dominant weight λ and it is defined by the famous Weyl Character formula:

$$Ch(\lambda^+)=\frac{A(\lambda^{++})}{A(\rho_{G_7})} \qquad (I.3)$$

where for a weight μ in general

$$A(\mu)\equiv\sum_{\sigma\in W(G_7)}\varepsilon(\sigma)\,e^{\sigma(\mu)} \qquad (I.4)$$

$W(G_7)$ is the Weyl Group of G_7 and each and every element σ is the so-called Weyl reflection while $\varepsilon(\sigma)$ denotes its sign and $e^{\sigma(\lambda^{++})}$) 's here are known as formal exponentials.Thoroughout this work, we assume λ^{++} λ^{++} denotes a strictly dominant weight defined for a dominant λ^+ by

$$\lambda^{++}\equiv\rho_{G_7}+\lambda^+ \qquad (I.5)$$

where ρ_{G_7} is the Weyl vector of G_7.

The crucial fact here is that

$$\|W(E_7)\|=2903040 \qquad (I.6)$$

where $\|S\|$ denotes order of set S. It is easy to see then to implement the sum in (I.4) would not be realizable explicitly. We, instead, propose 72 specifically chosen Weyl reflections which give us A_7 dominant weights participating within the same E_7 Weyl orbit $W(\Lambda^+)$ for any E_7 dominant weight Λ^+. As it is shown in the next section, this makes the evaluation of (I.4) realizable for E_7 but in terms of 72 A_7 characters and hence easily implementable.

A_7 Decomposition of E_7 Lie Algebra

For i=1,2…,7, let λ_i's and α_i's be respectively the fundamental dominant weigths and simple roots of A_7 Lie algebra with the following Dynkin diagram (Figure 1).

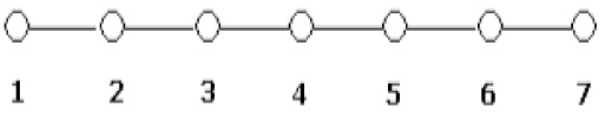

Figure 1: Dynkin diagram 1.

where $\rho_{A_7}=\lambda_1+\ldots+\lambda_7$ is A_7 Weyl vector and Λ_i's be fundamental dominant weights of E_7 Lie algebra in according with the following Dynkin diagram, (Figure 2).

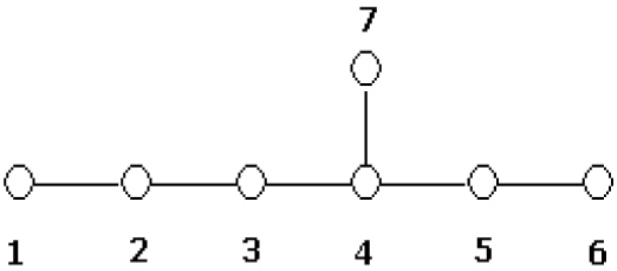

Figure 2: Dynkin diagram 2.

where $\rho_{E_7}=\Lambda_1+\ldots+\Lambda_7$ is A_7 Weyl vector. We suggest following relations allows us to embed A_7 subalgebra into E_7 algebra:

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$$\Lambda_1 = \lambda_2$$
$$\Lambda_2 = \lambda_1 + \lambda_3$$
$$\Lambda_3 = 2\,\lambda_3$$
$$\Lambda_4 = 2\,\lambda_3 + \lambda_6$$
$$\Lambda_5 = \lambda_3 + \lambda_5$$
$$\Lambda_6 = \lambda_4$$
$$\Lambda_7 = \lambda_3 + \lambda_7 \qquad (II.1)$$

This essentially means that

$$\frac{\|W(E_7)\|}{\|W(A_7)\|} = 72 \qquad (II.2)$$

which tells us that there are at most 72 A_7 dominant weights inside a Weyl orbit $W(\Lambda^+)$. Note here that it is exactly 72 when Λ^+ is a strictly dominant weight. From the now on, $W(\mu)$ will always denotes the Weyl orbit of a weight μ.

As the main point of view of this work, we present in appendix, 72 Weyl reflections to give 72 A_7 dominant weights participating in the same E_7 Weyl orbit $W(\Lambda^+)$ when they are exerted on the dominant weight Λ^+. To this end, the Weyl reflections with respect to simple roots α_i will be called simple reflections σ_i. We extend multiple products of simple reflections trivially by

$$\sigma_{i_1,i_2}(\lambda) \equiv \sigma_{i_1}(\sigma_{i_2}(\lambda)).$$

For s=1,…72, $\Sigma(s)$'s are 72 Weyl reflections mentioned above. As will also be seen by their definitions that,

1) $\varepsilon(\sigma(s))$=+1 s=1,2,…,36

2) $\varepsilon(\sigma(s))$=−1 s=37,38,…,72

Calculating Tensor Coupling Coefficients

Let us proceed in the instructive example

$$R(\Lambda_3) \otimes R(\Lambda_4) = R(\Lambda_3 + \Lambda_4) + \sum_{j=1}^{39} m(j)(\theta_j) \qquad (III.1)$$

of (I.1). One can see that there are 39 sub-dominant weights θ_j of $\Lambda_3+\Lambda_4$:

$$\theta_1 = \Lambda_3 + \Lambda_4$$
$$\theta_2 = \Lambda_1\theta_3 = \Lambda_7$$
$$\theta_4 = 3\,\Lambda_1$$
$$\theta_5 = \Lambda_1 + 2\,\Lambda_2$$
$$\theta_6 = 2\,\Lambda_1 + \Lambda_3$$
$$\theta_7 = \Lambda_1 + 2 * \Lambda_5$$
$$\theta_8 = \Lambda_1 + \Lambda_4 + \Lambda_6$$
$$\theta_9 = 2\,\Lambda_2 + \Lambda_7$$
$$\theta_{10} = \Lambda_2 + \Lambda_3 + \Lambda_6$$
$$\theta_{11} = \Lambda_2 + \Lambda_5 + \Lambda_7$$
$$\theta_{12} = \Lambda_3 + 2\,\Lambda_6$$
$$\theta_{13} = 3\,\Lambda_7$$
$$\theta_{14} = 3\,\Lambda_1 + \Lambda_6$$
$$\theta_{15} = \Lambda_1 + \Lambda_2 + 2\,\Lambda_6$$
$$\theta_{16} = \Lambda_1 + 3\,\Lambda_6$$
$$\theta_{17} = \Lambda_1 + \Lambda_6 + 2\,\Lambda_7$$
$$\theta_{18} = \Lambda_1 + \Lambda_2 + \Lambda_5$$
$$\theta_{19} = \Lambda_1 + \Lambda_3 + \Lambda_7$$
$$\theta_{20} = 2\,\Lambda_1 + \Lambda_6 + \Lambda_7$$
$$\theta_{21} = \Lambda_4 + \Lambda_7$$
$$\theta_{22} = \Lambda_5 + \Lambda_6 + \Lambda_7$$
$$\theta_{23} = \Lambda_1 + \Lambda_2$$
$$\theta_{24} = \Lambda_1 + \Lambda_6$$
$$\theta_{25} = 2\,\Lambda_1 + \Lambda_7$$
$$\theta_{26} = \Lambda_1 + 2\,\Lambda_7$$
$$\theta_{27} = \Lambda_2 + \Lambda_3$$
$$\theta_{28} = \Lambda_3$$
$$\theta_{29} = \Lambda_3 + \Lambda_5$$
$$\theta_{30} = 2\,\Lambda_6 + \Lambda_7$$
$$\theta_{31} = \Lambda_1 + \Lambda_5 + \Lambda_6$$
$$\theta_{32} = \Lambda_1 + 2\,\Lambda_6$$
$$\theta_{33} = \Lambda_6 + \Lambda_7$$
$$\theta_{34} = \Lambda_1 + \Lambda_4$$
$$\theta_{35} = \Lambda_1 + \Lambda_5$$
$$\theta_{36} = \Lambda_1 + \Lambda_2 + \Lambda_6$$
$$\theta_{37} = \Lambda_2 + \Lambda_6 + \Lambda_7$$
$$\theta_{38} = \Lambda_2 + \Lambda_7$$
$$\theta_{39} = \Lambda_3 + \Lambda_6$$
$$\theta_{40} = \Lambda_5 + \Lambda_7$$

To this end, we should care about specialization of formal exponentials [2]. Let us consider the so-called Fundamental Weights μ_I which are defined for (I=1,…8) as in the following [3]:

$\alpha_i \equiv \mu_i - \mu_{i+1}$, ($i$=1,…7.) ($III.2$)

α_i's here are A_7 simple roots mentioned above and the best way to calculate A_7 and hence E_7 characters is to use the specialization in terms of parameters $u_I \equiv e^{\mu_I}$ which are subjects of the condition $\mu_1+\mu_2+\ldots+\mu_8=0$ or

$u_1 u_2 \ldots u_8 = 1$.

To exemplify (I.3) for E_7, we would like to give detailed calculation of $Ch(\Lambda_3+\Lambda_4)$. By applying 72 specifically chosen Weyl reflections on strictly dominant weight $\rho_{E_7} + \Lambda_3 + \Lambda_4$, one can see we have the following decompositions:

$$A(\rho_{E_7} + \Lambda_3 + \Lambda_4) = \sum_{k=1}^{36} Ch(\nu_k) - \sum_{k=37}^{72} Ch(\nu_k) \qquad (III.3)$$

where

$$\nu_1 = \lambda_1 + \lambda_2 + 11\lambda_3 + \lambda_4 + \lambda_5 + 2\lambda_6 + \lambda_7$$
$$\nu_2 = 2\lambda_1 + 2\lambda_2 + 8\lambda_3 + \lambda_4 + \lambda_5 + 5\lambda_6 + \lambda_7$$
$$\nu_3 = 5\lambda_1 + \lambda_2 + 7\lambda_3 + \lambda_4 + 3\lambda_5 + 2\lambda_6 + 3\lambda_7$$
$$\nu_4 = 4\lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 3\lambda_5 + 4\lambda_6 + 3\lambda_7$$
$$\nu_5 = 2\lambda_1 + 4\lambda_2 + 4\lambda_3 + \lambda_4 + 5\lambda_5 + \lambda_6 + 5\lambda_7$$
$$\nu_6 = 5\lambda_1 + 2\lambda_2 + 5\lambda_3 + 2\lambda_4 + 2\lambda_5 + 3\lambda_6 + 4\lambda_7$$
$$\nu_7 = 5\lambda_1 + 2\lambda_2 + 5\lambda_3 + \lambda_4 + 4\lambda_5 + 3\lambda_6 + 2\lambda_7$$
$$\nu_8 = 7\lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 2\lambda_5 + 2\lambda_6 + 5\lambda_7$$
$$\nu_9 = 7\lambda_1 + \lambda_2 + 5\lambda_3 + 2\lambda_4 + 3\lambda_5 + 2\lambda_6 + 3\lambda_7$$
$$\nu_{10} = 2\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 6\lambda_7$$
$$\nu_{11} = 3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 2\lambda_4 + 4\lambda_5 + 2\lambda_6 + 6\lambda_7$$
$$\nu_{12} = 3\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 + 6\lambda_5 + 2\lambda_6 + 4\lambda_7$$
$$\nu_{13} = 6\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 2\lambda_5 + 4\lambda_6 + 5\lambda_7$$
$$\nu_{14} = 6\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 + 3\lambda_7$$
$$\nu_{15} = \lambda_1 + 6\lambda_2 + \lambda_3 + 4\lambda_4 + 2\lambda_5 + \lambda_6 + 8\lambda_7$$
$$\nu_{16} = 4\lambda_1 + 4\lambda_2 + 3\lambda_3 + \lambda_4 + 4\lambda_5 + \lambda_6 + 7\lambda_7$$
$$\nu_{17} = 4\lambda_1 + 4\lambda_2 + 2\lambda_3 + 2\lambda_4 + 5\lambda_5 + \lambda_6 + 5\lambda_7$$
$$\nu_{18} = \lambda_1 + 6\lambda_2 + \lambda_3 + \lambda_4 + 8\lambda_5 + \lambda_6 + 2\lambda_7$$

$$\nu_{19} = 7\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 + 3\lambda_5 + 3\lambda_6 + 4\lambda_7$$
$$\nu_{20} = 8\lambda_1 + 2\lambda_2 + 2\lambda_3 + 4\lambda_4 + \lambda_5 + 5\lambda_6 + \lambda_7$$
$$\nu_{21} = 10\lambda_1 + \lambda_2 + 3\lambda_3 + 3\lambda_4 + \lambda_5 + 4\lambda_6 + 2\lambda_7$$
$$\nu_{22} = 11\lambda_1 + \lambda_2 + \lambda_3 + 6\lambda_4 + \lambda_5 + 2\lambda_6 + \lambda_7$$
$$\nu_{23} = \lambda_1 + 4\lambda_2 + \lambda_3 + 4\lambda_4 + 3\lambda_5 + \lambda_6 + 9\lambda_7$$
$$\nu_{24} = 4\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 5\lambda_5 + \lambda_6 + 8\lambda_7$$
$$\nu_{25} = 4\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 6\lambda_5 + \lambda_6 + 6\lambda_7$$
$$\nu_{26} = \lambda_1 + 4\lambda_2 + \lambda_3 + \lambda_4 + 9\lambda_5 + \lambda_6 + 3\lambda_7$$
$$\nu_{27} = 2\lambda_1 + 5\lambda_2 + \lambda_3 + 3\lambda_4 + 2\lambda_5 + 2\lambda_6 + 9\lambda_7$$
$$\nu_{28} = 5\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 + 5\lambda_5 + 2\lambda_6 + 6\lambda_7$$
$$\nu_{29} = 9\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 + 6\lambda_6 + 2\lambda_7$$
$$\nu_{30} = \lambda_1 + 7\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6 + 10\lambda_7$$
$$\nu_{31} = 10\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 6\lambda_6 + \lambda_7$$
$$\nu_{32} = 13\lambda_1 + \lambda_2 + \lambda_3 + 4\lambda_4 + \lambda_5 + 3\lambda_6 + \lambda_7$$
$$\nu_{33} = 2\lambda_1 + 2\lambda_2 + \lambda_3 + 5\lambda_4 + \lambda_5 + \lambda_6 + 12\lambda_7$$
$$\nu_{34} = \lambda_1 + 5\lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5 + \lambda_6 + 11\lambda_7$$
$$\nu_{35} = 16\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6 + \lambda_7$$
$$\nu_{36} = \lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + 15\lambda_7$$
$$\nu_{37} = 3\lambda_1 + \lambda_2 + 9\lambda_3 + \lambda_4 + \lambda_5 + 4\lambda_6 + \lambda_7$$
$$\nu_{38} = 2\lambda_1 + \lambda_2 + 8\lambda_3 + \lambda_4 + \lambda_5 + 6\lambda_6 + \lambda_7$$
$$\nu_{39} = 4\lambda_1 + 2\lambda_2 + 6\lambda_3 + \lambda_4 + 3\lambda_5 + 3\lambda_6 + 3\lambda_7$$
$$\nu_{40} = 6\lambda_1 + \lambda_2 + 6\lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 4\lambda_7$$
$$\nu_{41} = 6\lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 4\lambda_5 + 2\lambda_6 + 2\lambda_7$$
$$\nu_{42} = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + \lambda_4 + 5\lambda_5 + 2\lambda_6 + 5\lambda_7$$
$$\nu_{43} = 5\lambda_1 + \lambda_2 + 5\lambda_3 + 2\lambda_4 + 2\lambda_5 + 4\lambda_6 + 4\lambda_7$$
$$\nu_{44} = 5\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 4\lambda_5 + 4\lambda_6 + 2\lambda_7$$
$$\nu_{45} = 3\lambda_1 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4 + 4\lambda_5 + \lambda_6 + 6\lambda_7$$
$$\nu_{46} = 3\lambda_1 + 4\lambda_2 + 3\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 4\lambda_7$$
$$\nu_{47} = 6\lambda_1 + 2\lambda_2 + 5\lambda_3 + \lambda_4 + 2\lambda_5 + 3\lambda_6 + 5\lambda_7$$
$$\nu_{48} = 6\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4 + 3\lambda_5 + 3\lambda_6 + 3\lambda_7$$
$$\nu_{49} = 8\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 3\lambda_5 + 2\lambda_6 + 4\lambda_7$$
$$\nu_{50} = 9\lambda_1 + \lambda_2 + 3\lambda_3 + 4\lambda_4 + \lambda_5 + 4\lambda_6 + \lambda_7$$
$$\nu_{51} = 3\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4 + 5\lambda_5 + \lambda_6 + 7\lambda_7$$
$$\nu_{52} = 3\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 7\lambda_5 + \lambda_6 + 5\lambda_7$$
$$\nu_{53} = \lambda_1 + 5\lambda_2 + \lambda_3 + 4\lambda_4 + 2\lambda_5 + 2\lambda_6 + 8\lambda_7$$
$$\nu_{54} = 4\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 + 4\lambda_5 + 2\lambda_6 + 7\lambda_7$$
$$\nu_{55} = 4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 + 5\lambda_5 + 2\lambda_6 + 5\lambda_7$$
$$\nu_{56} = \lambda_1 + 5\lambda_2 + \lambda_3 + \lambda_4 + 8\lambda_5 + 2\lambda_6 + 2\lambda_7$$
$$\nu_{57} = 7\lambda_1 + \lambda_2 + 4\lambda_3 + \lambda_4 + 3\lambda_5 + 4\lambda_6 + 4\lambda_7$$
$$\nu_{58} = 8\lambda_1 + \lambda_2 + 2\lambda_3 + 4\lambda_4 + \lambda_5 + 6\lambda_6 + \lambda_7$$
$$\nu_{59} = 2\lambda_1 + 6\lambda_2 + \lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 9\lambda_7$$
$$\nu_{60} = 5\lambda_1 + 4\lambda_2 + 2\lambda_3 + \lambda_4 + 5\lambda_5 + \lambda_6 + 6\lambda_7$$
$$\nu_{61} = 9\lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 + 5\lambda_6 + 2\lambda_7$$
$$\nu_{62} = 12\lambda_1 + \lambda_2 + \lambda_3 + 5\lambda_4 + \lambda_5 + 2\lambda_6 + 2\lambda_7$$
$$\nu_{63} = 11\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 5\lambda_6 + \lambda_7$$
$$\nu_{64} = \lambda_1 + 2\lambda_2 + \lambda_3 + 6\lambda_4 + \lambda_5 + \lambda_6 + 11\lambda_7$$
$$\nu_{65} = 2\lambda_1 + 4\lambda_2 + \lambda_3 + 3\lambda_4 + 3\lambda_5 + \lambda_6 + 10\lambda_7$$
$$\nu_{66} = 5\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 7\lambda_7$$
$$\nu_{67} = \lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 + 11\lambda_5 + \lambda_6 + \lambda_7$$
$$\nu_{68} = \lambda_1 + 6\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 10\lambda_7$$
$$\nu_{69} = 10\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 7\lambda_6 + \lambda_7$$
$$\nu_{70} = 15\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5 + \lambda_6 + \lambda_7$$
$$\nu_{71} = \lambda_1 + 3\lambda_2 + \lambda_3 + 4\lambda_4 + \lambda_5 + \lambda_6 + 13\lambda_7$$
$$\nu_{72} = \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + 16\lambda_7$$

A_7 characters $Ch(\nu_k)$'s are defined by

$$A(\rho_{A_7})\, Ch(\nu_k) = \sum_{\sigma \in W(A_7)} \varepsilon(\sigma)\, e^{\sigma(\rho_{A_7} + \nu_k)} \quad (III.4)$$

Note here that $W(A_7)$ is the permutation group of 8 objects.

To display our result here, we use the following specialization of formal exponentials with only one free parameter x:

$$u_1 = 1$$
$$u_2 = 2$$
$$u_3 = 3$$
$$u_4 = 4$$
$$u_5 = 5$$
$$u_6 = 6$$
$$u_7 = x$$
$$u_8 = 1/(720\,x) \quad (III.5)$$

In this specialization, one obtains the following one-parameter characters:

$$A(\rho_{A_7}) = -\frac{1}{2^{20}\times 3^{11}\times 5^6\, x^7}\times$$
$$(-6+x)\times(-5+x)\times(-4+x)\times(-3+x)\times(-2+x)\times$$
$$(-1+x)\times(-1+720\,x)\times(-1+1440\,x)\times(-1+2160\,x)\times$$
$$(-1+2880\,x)\times(-1+3600\,x)\times(-1+4320\,x)\times(-1+720\,x^2)$$

$$A(\rho_{E_7}) = -\frac{7^3\times 11\times 13^2\times 17^2\times 23\times 29\times 47\times 59^2\times 71\times 89\times 179\times 239\times 359}{2^{40}\times 3^{19}\times 5^9\times x^{10}}\times$$
$$(-1+6x)\times(-1+8x)\times(-1+10x)\times(-1+12x)^2\times(-1+15x)$$
$$(-1+18x)\times(-1+20x)\times(-1+24x)^2\times(-1+30x)^2\times(-1+36x)$$
$$(-1+40x)\times(-1+48x)\times(-1+60x)^2\times(-1+72x)\times(-1+90x)\times(-1+120x)$$

$$ch(\Lambda_4) = \frac{1}{2^{16}\times 3^8\times 5^4\ x^4}($$
$$2^2\times 7\times 29\times 31\times 113\times 25849 +$$
$$3\times 5^3\times 7^2\times 41\times 13469\times 25841\ x +$$
$$2^2\times 7^2\times 227\times 1997\times 1004276389\ x^2 +$$
$$2^2\times 3^4\times 5^2\times 7^4\times 17\times 41\times 13469\times 45307\ x^3 +$$
$$2^7\times 3^2\times 266944787316406807\ x^4 +$$
$$2^6\times 3^6\times 5^3\times 7^4\times 17\times 41\times 13469\times 45307\ x^5 +$$
$$2^{10}\times 3^4\times 5^2\times 7^2\times 227\times 1997\times 1004276389\ x^6 +$$
$$2^{12}\times 3^7\times 5^6\times 7^2\times 41\times 13469\times 25841\ x^7 +$$
$$2^{18}\times 3^8\times 5^4\times 7\times 29\times 31\times 113\times 25849\ x^8)$$

$$ch(\Lambda_3+\Lambda_4) = \frac{7}{2^{28}\times 3^{13}\times 5^6\ x^7}($$
$$2^5\ 7^2\times 139\times 40819\times 22523219 +$$
$$3\times 296955329011336071883\ x +$$
$$7\times 93629\times 104327\times 20612147800357\ x^2 +$$
$$2^2\times 7\times 89\times 509\times 407193532921684756441\ x^3 +$$
$$2^2\times 7\times 19\times 1447\times 73091587\times 1489316745532201\ x^4 +$$
$$2^4\times 3\times 11^2\times 37\times 1117\times 18045889\times 1661840436868789\ x^5 +$$
$$2^4\times 3^2\times 7\times 11^2\times 5237\times 130069\times 351401\times 12440163841487\ x^6 +$$
$$2^9\times 3^3\times 11\times 83\times 757\times 4830390973\times 258355213888973\ x^7 +$$
$$2^8\times 3^4\times 5\times 7\times 11^2\times 5237\times 130069\times 351401\times 12440163841487\ x^8 +$$
$$2^{12}\times 3^5\times 5^2\times 11^2\times 37\times 1117\times 18045889\times 1661840436868789\ x^9 +$$
$$2^{14}\times 3^6\times 5^3\times 7\times 19\times 1447\times 73091587\times 1489316745532201\ x^{10} +$$
$$2^{18}\times 3^8\times 5^4\times 7\times 89\times 509\times 407193532921684756441\ x^{11} +$$
$$2^{20}\times 3^{10}\times 5^5\times 7\times 9362\times 104327\times 20612147800357\ x^{12} +$$
$$2^{24}\times 3^{13}\times 5^6\times 296955329011336071883\ x^{13} +$$
$$2^{33}\times 3^{14}\times 5^7\times 7^2\times 139\times 40819\times 22523219\ x^{14})$$

$$ch(\theta_2) = \frac{1}{144\ x}(3\times 7\times 17 + 5\times 25841\ x + 2^4\times 3^3\times 5\times 7\times 17\ x^2)$$

$$ch(\theta_3) = \frac{1}{2^8\times 3^4\times 5\ x^2}($$
$$5\times 25841 + 3\times 7\times 19\times 83\times 2459\ x + 2^2\times 3^3\times 7^2\times 17\times 45307\ x^2 +$$
$$2^4\times 3^3\times 5\times 7\times 19\times 83\times 2459\ x^3 + 2^8\times 3^4\times 5^3\times 25841\ x^4)$$

$$ch(\theta_4) = \frac{1}{2^{12}\times 3^6\times 5^2\ x^3}($$
$$2\times 3^2\times 7\times 251\times 11287 +$$
$$2\times 7\times 11\times 1634509733\ x +$$
$$3^2\times 7^2\times 23\times 73\times 101\times 281\times 5783\ x^2 +$$
$$41\times 953\times 5987\times 153547507\ x^3 +$$
$$2^4\times 3^4\times 5\times 7^2\times 23\times 73\times 101\times 281\times 5783\ x^4 +$$
$$2^9\times 3^4\times 5^2\times 7\times 11\times 1634509733\ x^5 +$$
$$2^13\times 3^8\times 5^3\times 7\times 251\times 11287\ x^6)$$

$$ch(\theta_5) = \frac{7}{2^{20}\times 3^{10}\times 5^4\ x^5}($$
$$3^4\times 389\times 2621\times 326611 +$$
$$44372305108670731\ x +$$
$$2\times 3\times 7\times 888351682544432651\ x^2 +$$
$$2\times 6835359680937443668841\ x^3 +$$
$$2^3\times 3\times 17\times 313\times 2029\times 4912124122975679\ x^4 +$$
$$2^2\times 7\times 17\times 87942983\times 1353499222770007\ x^5 +$$
$$2^7\times 3^3\times 5\times 17\times 313\times 2029\times 4912124122975679\ x^6 +$$
$$2^9\times 3^4\times 5^2\times 6835359680937443668841\ x^7 +$$
$$2^{13}\times 3^7\times 5^3\times 7\times 888351682544432651\ x^8 +$$
$$2^{16}\times 3^8\times 5^4\times 44372305108670731\ x^9 +$$
$$2^{20}\times 3^{14}\times 5^5\times 389\times 2621\times 326611\ x^{10})$$

$$ch(\theta_6) = \frac{1}{2^{20}\times 3^{10}\times 5^4\ x^5}($$
$$2\times 3^5\times 5\times 7\times 67\times 118953227 +$$
$$2\times 29\times 191\times 1259\times 18041\times 1205779\ x +$$
$$3\times 7\times 419\times 953\times 23122934503549\ x^2 +$$
$$7\times 11\times 13\times 3881\times 9624859\times 1896757637\ x^3 +$$
$$2^2\times 3^2\times 7\times 111767\times 14341763\times 22885698919\ x^4 +$$
$$2^2\times 3^2\times 5\times 31\times 114376828127\times 589217525459\ x^5 +$$
$$2^6\times 3^4\times 5\times 7\times 111767\times 14341763\times 22885698919\ x^6 +$$
$$2^8\times 3^4\times 5^2\times 7\times 11\times 13\times 3881\times 9624859\times 1896757637\ x^7 +$$
$$2^{12}\times 3^7\times 5^3\times 7\times 419\times 953\times 23122934503549\ x^8 +$$
$$2^{17}\times 3^8\times 5^4\times 29\times 191\times 1259\times 18041\times 1205779\ x^9 +$$
$$2^{21}\times 3^{15}\times 5^6\times 7\times 67\times 118953227\ x^{10})$$

$$ch(\theta_7) = \frac{7}{2^{26}\times 3^{14}\times 5^6\ x^7}($$
$$2^2\times 3^3\times 71\times 137\times 897339719$$
$$+2^3\times 3^2\times 9427181\times 12037823647\ x +$$
$$3\times 5\times 7^3\times 6472827206445606859\ x^2 +$$
$$7\times 167\times 1217\times 3345931489\times 3986554903\ x^3 +$$
$$2^3\times 3^2\times 1913\times 287125211770483024406l1\ x^4 +$$
$$2^3\times 3^2\times 5\times 7\times 31\times 55040343315964372274l8843\ x^5 +$$
$$2^4\times 3^5\times 5^3\times 157\times 311\times 6473\times 6273427\times 25424142019\ x^6 +$$
$$2^{11}\times 3^5\times 5\times 11\times 13\times 193\times 38358559\times 321202718587369\ x^7 +$$
$$2^8\times 3^7\times 5^4\times 157\times 311\times 6473\times 6273427\times 25424142019\ x^8 +$$
$$2^{11}\times 3^6\times 5^3\times 7\times 31\times 55040343315964372274l8843\ x^9 +$$
$$2^{15}\times 3^8\times 5^3\times 1913\times 287125211770483024406l1\ x^{10} +$$
$$2^{16}\times 3^8\times 5^4\times 7\times 167\times 1217\times 3345931489\times 3986554903\ x^{11} +$$
$$2^{20}\times 3^{11}\times 5^6\times 7^3\times 6472827206445606859\ x^{12} +$$
$$2^{27}\times 3^{14}\times 5^6\times 9427181\times 12037823647\ x^{13} +$$
$$2^{30}\times 3^{17}\times 5^7\times 71\times 137\times 897339719\ x^{14})$$

$$ch(\theta_8)=\frac{7}{2^{26}\times 3^{12}\times 5^6\ x^7}($$
$$70782069982080+$$
$$2^5\times 3\times 7\times 19\times 101\times 2437\times 7883\times 45307\ x+$$
$$3^1\times 839\times 2591\times 566506158976003\ x^2+$$
$$5^2\times 127\times 743526604818728544311\ x^3+$$
$$2^2\times 5\times 83\times 263\times 6791\times 13901\times 522761\times 27245893\ x^4+$$
$$2^2\times 1439\times 6833\times 57110377\times 29440768222739\ x^5+$$
$$2^8\times 3^2\times 79\times 2833\times 5435185631\times 1421253019763\ x^6+$$
$$2^6\times 3^2\times 5\times 7\times 37577587\times 1652664511\times 109347145543\ x^7+$$
$$2^{12}\times 3^4\times 5\times 79\times 2833\times 5435185631\times 1421253019763\ x^8+$$
$$2^{10}\times 3^4\times 5^2\times 1439\times 6833\times 57110377\times 29440768222739\ x^9+$$
$$2^{14}\times 3^6\times 5^4\times 83\times 263\times 6791\times 13901\times 522761\times 27245893\ x^{10}+$$
$$2^{16}\times 3^8\times 5^6\times 127\times 743526604818728544311\ x^{11}+$$
$$2^{20}\times 3^{11}\times 5^5\times 839\times 2591\times 566506158976003\ x^{12}+$$
$$2^{29}\times 3^{13}\times 5^6\times 7\times 19\times 101\times 2437\times 7883\times 45307\ x^{13}+$$
$$2^{35}\times 3^{15}\times 5^8\times 19\times 101\times 2437\times 7883\ x^{14})$$

$$ch(\theta_9)=\frac{1}{2^{23}\times 3^{12}\times 5^5\ x^6}($$
$$5\times 11\times 1730263\times 475374719+$$
$$3\times 5\times 7\times 11\times 4933\times 15316021986509\ x+$$
$$5\times 7\times 13\times 1728312877825877278 03x^2+$$
$$3\times 7^2\times 205820307753559239132761x^3+$$
$$2^2\times 3^3\times 41\times 707606359\times 1135814759505821x^4+$$
$$2\times 3\times 5\times 7\times 41\times 1252496251\times 20514266507074489\ x^5+$$
$$2^3\times 3^3\times 5\times 67\times 16903\times 2819023\times 2271886853005801\ x^6+$$
$$2^5\times 3^3\times 5^2\times 7\times 41\times 1252496251\times 20514266507074489\ x^7+$$
$$2^{10}\times 3^7\times 5^2\times 41\times 707606359\times 1135814759505821\ x^8+$$
$$2^{12}\times 3^7\times 5^3\times 7^2\times 205820307753559239132761\ x^9+$$
$$2^{16}\times 3^8\times 5^5\times 7\times 13\times 172831287782587727803\ x^{10}+$$
$$2^{20}\times 3^{11}\times 5^6\times 7\times 11\times 4933\times 15316021986509\ x^{11}+$$
$$2^{24}\times 3^{12}\times 5^7\times 11\times 1730263\times 475374719\ x^{12})$$

$$ch(\theta_{10})=\frac{1}{2^{26}\times 3^{12}\times 5^6\ x^7}($$
$$2^2\times 3\times 5\times 7^2\times 63113\times 7700299+$$
$$3\times 7^3\times 45307\times 63113\times 7700299\ x+$$
$$3\times 5^2\times 7\times 17\times 191\times 30444055104860819\ x^2+$$
$$2^2\times 13\times 854020492656685314145399\ x^3+$$
$$2^2\times 3\times 7\times 13\times 137\times 1619903\times 40109150434199081\ x^4+$$
$$2^4\times 3\times 7\times 13\times 948542029\times 239506095510065339\ x^5+$$
$$2^4\times 3\times 7\times 11485393\times 14627879671585167600299\ x^6+$$
$$2^6\times 3^7\times 5^2\times 19\times 61\times 883\times 8053\times 44101\times 244021\times 6224627\ x^7+$$
$$2^8\times 3^3\times 5\times 7\times 11485393\times 14627879671585167600299\ x^8+$$
$$2^{12}\times 3^5\times 5^2\times 7\times 13\times 948542029\times 239506095510065339\ x^9+$$
$$2^{14}\times 3^7\times 5^3\times 7\times 13\times 137\times 1619903,1\times 40109150434199081\ x^{10}+$$
$$2^{18}\times 3^8\times 5^4\times 13\times 854020492656685314145399\ x^{11}+$$
$$2^{20}\times 3^{11}\times 5^7\times 7\times 17\times 191\times 30444055104860819\ x^{12}+$$
$$2^{24}\times 3^{13}\times 5^6\times 7^3\times 45307\times 63113\times 7700299\ x^{13}+$$
$$2^{30}\times 3^{15}\times 5^8\times 7^2\times 63113\times 7700299\ x^{14})$$

$$ch(\theta_{11})=\frac{11}{2^{26}\times 3^{14}\times 5^6\times x^7}($$
$$3^2\times 7\times 233\times 2439503305003+25158065915617451 4557\ x+$$
$$2\times 3\times 7\times 197\times 479\times 83919415489238839\ x^2+$$
$$2\times 3^5\times 7\times 257\times 13163\times 561199\times 23985981091\ x^3+$$
$$2^4\times 3\times 5^2\times 7\times 23\times 31\times 4297\times 5779\times 1802909\times 106751927\ x^4+$$
$$2^3\times 3^3\times 207797403557\times 58222225954444073\ x^5+$$
$$2^5\times 3^3\times 7\times 22924263480995135201368307741\ x^6+$$
$$2^6\times 3^5\times 29\times 10268423472246027440877336011\ x^7+$$
$$2^9\times 3^5\times 5\times 7\times 22924263480995135201368307741\ x^8+$$
$$2^{11}\times 3^7\times 5^2\times 207797403557\times 58222225954444073\ x^9+$$
$$2^{16}\times 3^7\times 5^5\times 7\times 23\times 31\times 4297\times 5779\times 1802909\times 106751927\ x^{10}+$$
$$2^{17}\times 3^{13}\times 5^4\times 7\times 257\times 13163\times 561199\times 23985981091\ x^{11}+$$
$$2^{21}\times 3^{11}\times 5^5\times 7\times 197\times 479\times 83919415489238839\ x^{12}+$$
$$2^{24}\times 3^{12}\times 5^6\times 25158065915617451 4557\ x^{13}+$$
$$2^{28}\times 3^{16}\times 5^7\times 7\times 233\times 2439503305003\ x^{14})$$

$$ch(\theta_{12})=\frac{1}{2^{24}\times 3^{12}\times 5^6\times x^7}($$
$$2^4\times 3^3\times 5^2\times 7^2\times 41\times 13469+$$
$$2^2\times 3^3\times 5\times 7^3\times 41\times 13469\times 45307\ x+$$
$$3\times 7^3\times 41\times 13469\times 115259\times 666353\ x^2+$$
$$5\times 2999\times 75703\times 67759950443299\ x^3+$$
$$2^2\times 3\times 7\times 41\times 73^2\times 367\times 2736080070054181\ x^4+$$
$$2^2\times 3^2\times 7\times 76859693519146030344078 49\ x^5+$$
$$2^4\times 3^3\times 5^2\times 7^2\times 373\times 571\times 973787\times 1054751382911\ x^6+$$
$$2^6\times 3^4\times 5\times 37\times 223\times 311\times 3767\times 12319481\times 1298488019\ x^7+$$
$$2^8\times 3^5\times 5^3\times 7^2\times 373\times 571\times 973787\times 1054751382911\ x^8+$$
$$2^{10}\times 3^6\times 5^2\times 7\times 76859693519146030344078 49\ x^9+$$
$$2^{14}\times 3^7\times 5^3\times 7\times 41\times 73^2\times 367\times 2736080070054181\ x^{10}+$$
$$2^{16}\times 3^8\times 5^5\times 2999\times 75703\times 67759950443299\ x^{11}+$$
$$2^{20}\times 3^{11}\times 5^5\times 7^3\times 41\times 13469\times 115259\times 666353\ x^{12}+$$
$$2^{26}\times 3^{15}\times 5^7\times 7^3\times 41\times 13469\times 45307\ x^{13}+$$
$$2^{32}\times 3^{17}\times 5^9\times 7^2\times 41\times 13469\ x^{14})$$

$$ch(\theta_{13})=\frac{11}{2^{24}\times 3^{12}\times 5^5\ x^6}($$
$$3238985337918907+$$
$$3^4\times 5\times 7\times 841610704258279\ x+$$
$$7^2\times 13\times 1482820942905285871\ x^2+$$
$$3^3\times 7\times 193\times 467\times 5690719\times 2565345073\ x^3+$$
$$2^2\times 3^2\times 626862641769013044421771\ x^4+$$
$$2^4\times 3^3\times 5\times 7\times 791349944357515288887903\ x^5+$$
$$2^6\times 3^4\times 31\times 149\times 181\times 9313661572349696183\ x^6+$$
$$2^8\times 3^5\times 5^2\times 7\times 79134994435751528887903\ x^7+$$
$$2^{10}\times 3^6\times 5^2\times 626862641769013044421771\ x^8+$$
$$2^{12}\times 3^9\times 5^3\times 7\times 193\times 467\times 5690719\times 2565345073\ x^9+$$
$$2^{16}\times 3^8\times 5^4\times 7^2\times 13\times 1482820942905285871\ x^{10}+$$
$$2^{20}\times 3^{14}\times 5^6\times 7\times 841610704258279\ x^{11}+$$
$$2^{24}\times 3^{12}\times 5^6\times 3238985337918907\ x^{12})$$

$$ch(\theta_{14}) = \frac{1}{2^{18}\times 3^8\times 5^4\ x^5}\,($$
$$2^3\times 3^2\times 5\times 7\times 251\times 11287 +$$
$$2\times 3^2\times 7^2\times 251\times 11287\times 45307\ x +$$
$$2\times 7^2\times 13\times 47\times 5077\times 280370003\ x^2 +$$
$$1305047\times 32776442887619\ x^3 +$$
$$7\times 29\times 67792850454653521231\ x^4 +$$
$$2^2\times 5\times 59\times 557\times 739\times 1838570856221483\ x^5 +$$
$$2^4\times 3^2\times 5\times 7\times 29\times 67792850454653521231\ x^6 +$$
$$2^8\times 3^4\times 5^2\times 1305047\times 32776442887619\ x^7 +$$
$$2^{13}\times 3^6\times 5^3\times 7^2\times 13\times 47\times 5077\times 280370003\ x^8 +$$
$$2^{17}\times 3^{10}\times 5^4\times 7^2\times 251\times 11287\times 45307\ x^9 +$$
$$2^{23}\times 3^{12}\times 5^6\times 7\times 251\times 11287\ x^{10})$$

$$ch(\theta_{15}) = \frac{7}{2^{24}\times 3^{12}\times 5^6\times x^7}\,($$
$$2^{10}\times 3^3\times 5^3\times 51977 +$$
$$2^8\times 3^3\times 5^2\times 7\times 45307\times 51977\ x + +$$
$$2^6\times 3\times 5\times 7\times 51977\times 115259\times 666353\ x^2 +$$
$$3^2\times 861317\times 4060866137260903\ x^3 +$$
$$3\times 5\times 41\times 53\times 16481\times 24852500017071257\ x^4 +$$
$$2^3\times 223\times 10353543146161257526472 09\ x^5 +$$
$$2^6\times 3^2\times 5\times 495553698751\times 90995390639929\ x^6 +$$
$$2^7\times 3^2\times 5\times 7\times 163\times 227\times 99661\times 33171175875735791\ x^7 +$$
$$2^{10}\times 3^4\times 5^2\times 495553698751\times 90995390639929\ x^8 +$$
$$2^{11}\times 3^4\times 5^2\times 223\times 10353543146161257526472 09\ x^9 +$$
$$2^{12}\times 3^7\times 5^4\times 41\times 53\times 16481\times 24852500017071257\ x^{10} +$$
$$2^{16}\times 3^{10}\times 5^4\times 861317\times 4060866137260903\ x^{11} +$$
$$2^{26}\times 3^{11}\times 5^6\times 7\times 51977\times 115259\times 666353\ x^{12} +$$
$$2^{32}\times 3^{15}\times 5^8\times 7\times 45307\times 51977\ x^{13} +$$
$$2^{38}\times 3^{17}\times 5^{10}\times 51977\ x^{14})$$

$$ch(\theta_{16}) = \frac{1}{2^{22}\times 3^{10}\times 5^6\times x^7}\,($$
$$2^6\times 3^3\times 5^3\times 7\times 17 +$$
$$2^4\times 3^3\times 5^2\times 7^2\times 17\times 45307\ x +$$
$$2^2\times 3\times 5\times 7^2\times 17\times 115259\times 666353\ x^2 +$$
$$3\times 7^3\times 17\times 37\times 221303\times 196292779\ x^3 +$$
$$5\times 7\times 29\times 147145853\times 90252382399\ x^4 +$$
$$2^2\times 5\times 86214774415429023276727\ x^5 +$$
$$2^4\times 3^4\times 5^2\times 7\times 863\times 4003\times 33749\times 4576169137\ x^6 +$$
$$2^6\times 3^3\times 5^2\times 23\times 43\times 7907\times 14463096839915333\ x^7 +$$
$$2^8\times 3^6\times 5^3\times 7\times 863\times 4003\times 33749\times 4576169137\ x^8 +$$
$$2^{10}\times 3^4\times 5^3\times 86214774415429023276727\ x^9 +$$
$$2^{12}\times 3^6\times 5^4\times 7\times 29\times 147145853\times 90252382399\ x^{10} +$$
$$2^{16}\times 3^9\times 5^4\times 7^3\times 17\times 37\times 221303\times 196292779\ x^{11} +$$
$$2^{22}\times 3^{11}\times 5^6\times 7^2\times 17\times 115259\times 666353\ x^{12} +$$
$$2^{28}\times 3^{15}\times 5^8\times 7^2\times 17\times 45307\ x^{13} +$$
$$2^{34}\times 3^{17}\times 5^{10}\times 7\times 17\ x^{14})$$

$$ch(\theta_{17}) = \frac{7}{2^{26}\times 3^{12}\times 5^6\times x^7}\,($$
$$2^2\times 3^2\times 5\times 1920917020339 +$$
$$3^2\times 7\times 45307\times 1920917020339\ x +$$
$$2^2\times 11\times 1699\times 20848837\times 3497751463\ x^2 +$$
$$2^3\times 3\times 13\times 31\times 47\times 2517821\times 2229533970737\ x^3 +$$
$$5\times 23\times 239\times 22442053352210346516283\ x^4 +$$
$$2^3\times 3\times 11\times 24418648739531173988158 9177\ x^5 +$$
$$2^8\times 3^3\times 11^2\times 3229\times 1352409142907613715103\ x^6 +$$
$$2^8\times 3^2\times 5\times 11\times 99368214071063405675670 9367\ x^7 +$$
$$2^{12}\times 3^5\times 5\times 11^2\times 3229\times 1352409142907613715103\ x^8 +$$
$$2^{11}\times 3^5\times 5^2\times 11\times 24418648739531173988158 9177\ x^9 +$$
$$2^{12}\times 3^6\times 5^4\times 23\times 239\times 22442053352210346516283\ x^{10} +$$
$$2^{19}\times 3^9\times 5^4\times 13\times 31\times 47\times 2517821\times 2229533970737\ x^{11} +$$
$$2^{22}\times 3^{10}\times 5^5\times 11\times 1699\times 20848837\times 3497751463\ x^{12} +$$
$$2^{24}\times 3^{14}\times 5^6\times 7\times 45307\times 1920917020339\ x^{13} +$$
$$2^{30}\times 3^{16}\times 5^8\times 1920917020339\ x^{14})$$

$$ch(\theta_{18}) = \frac{1}{2^{22}\times 3^{12}\times 5^5\times x^6}\,($$
$$2^2\times 3^3\times 5\times 11\times 36319\times 4281313 +$$
$$2^4\times 3\times 5\times 7\times 11^2\times 46549\times 679298969\ x +$$
$$3^2\times 5^2\times 7\times 11\times 41\times 10712225499148973\ x^2 +$$
$$3\times 5^2\times 7\times 569\times 476209471\times 24269673079\ x^3 +$$
$$2^7\times 5^2\times 7\times 717343331\times 35775113699503\ x^4 +$$
$$2^8\times 3^2\times 7\times 20327\times 3472171\times 37024507616807\ x^5 +$$
$$2^4\times 3^2\times 3301\times 32532331436342 11446037993\ x^6 +$$
$$2^{12}\times 3^4\times 5\times 7\times 20327\times 3472171\times 37024507616807\ x^7 +$$
$$2^{15}\times 3^4\times 5^4\times 7\times 717343331\times 35775113699503\ x^8 +$$
$$2^{12}\times 3^7\times 5^5\times 7\times 569\times 476209471\times 24269673079\ x^9 +$$
$$2^{16}\times 3^{10}\times 5^6\times 7\times 11\times 41\times 10712225499148973\ x^{10} +$$
$$2^{24}\times 3^{11}\times 5^6\times 7\times 11^2\times 46549\times 679298969\ x^{11} +$$
$$2^{26}\times 3^{15}\times 5^7\times 11\times 36319\times 4281313\ x^{12})$$

$$ch(\theta_{19}) = \frac{7}{2^{24}\times 3^{12}\times 5^5\ x^6}\,($$
$$11\times 19\times 54132425994601 +$$
$$2^2\times 3\times 1873\times 801641\times 1539041989\ x +$$
$$2^3\times 5\times 3391\times 157884472828805717\ x^2 +$$
$$3\times 5\times 41\times 421\times 26894683525927701469\ x^3 +$$
$$2^3\times 5296547\times 40807969\times 561290628359\ x^4 +$$
$$2^2\times 3^2\times 5\times 7\times 18542831\times 2641233933522645623\ x^5 +$$
$$2^4\times 3^4\times 11699291\times 71109193\times 2011720082303\ x^6 +$$
$$2^6\times 3^4\times 5^2\times 7\times 18542831\times 2641233933522645623\ x^7 +$$
$$2^{11}\times 3^4\times 5^2\times 5296547\times 40807969\times 561290628359\ x^8 +$$
$$2^{12}\times 3^7\times 5^4\times 41\times 421\times 26894683525927701469\ x^9 +$$
$$2^{19}\times 3^8\times 5^5\times 3391\times 157884472828805717\ x^{10} +$$
$$2^{22}\times 3^{11}\times 5^5\times 1873\times 801641\times 1539041989\ x^{11} +$$
$$2^{24}\times 3^{12}\times 5^6\times 11\times 19\times 54132425994601\ x^{12})$$

$$ch(\theta_{20}) = \frac{7}{2^{22}\times 3^{10}\times 5^5\times x^6}($$
$$2^3\times 5\times 11\times 1634509733 +$$
$$2\times 7\times 11\times 45307\times 1634509733\, x +$$
$$3^1\times 5^2\times 188673113\times 883791547\, x^2 +$$
$$2^5\times 821\times 5573\times 41867209636027\, x^3 +$$
$$3^4\times 269\times 27680957\times 2625874917649\, x^4 +$$
$$2^3\times 3^2\times 13\times 29\times 71\times 74073603909992156759\, x^5 +$$
$$2^5\times 3^4\times 5\times 1489\times 291506349305759397989\, x^6 +$$
$$2^7\times 3^4\times 5\times 13\times 29\times 71\times 7407360390999215675 9\, x^7 +$$
$$2^8\times 3^8\times 5^2\times 269\times 27680957\times 2625874917649\, x^8 +$$
$$2^{17}\times 3^6\times 5^3\times 821\times 5573\times 41867209636027\, x^9 +$$
$$2^{16}\times 3^9\times 5^6\times 188673113\times 883791547\, x^{10} +$$
$$2^{21}\times 3^{10}\times 5^5\times 7\times 11\times 45307\times 1634509733\, x^{11} +$$
$$2^{27}\times 3^{12}\times 5^7\times 11\times 1634509733\, x^{12})$$

$$ch(\theta_{21}) = \frac{7}{2^{24}\times 3^{12}\times 5^5\, x^6}($$
$$2^3\times 11\times 1091\times 12756480773 +$$
$$3\times 5\times 7\times 4954321\times 8800968989\, x +$$
$$3^2\times 7\times 106275811\times 430270161227\, x^2 +$$
$$2^3\times 3\times 5\times 7^2\times 13\times 3137\times 25147\times 246251\times 502781\times x^3 +$$
$$2^2\times 3^3\times 82567\times 255832037\times 34487228383\, x^4 +$$
$$2^5\times 3^3\times 7\times 23\times 28753\times 45439\times 24160314328237\, x^5 +$$
$$2^7\times 3^5\times 5\times 7\times 23\times 31\times 1867\times 102273597906161063\, x^6 +$$
$$2^9\times 3^5\times 5\times 7\times 23\times 28753\times 45439\times 24160314328237\, x^7 +$$
$$2^{10}\times 3^7\times 5^2\times 82567\times 255832037\times 34487228383\, x^8 +$$
$$2^{15}\times 3^7\times 5^4\times 7^2\times 13\times 3137\times 25147\times 246251\times 502781\, x^9 +$$
$$2^{16}\times 3^{10}\times 5^4\times 7\times 106275811\times 430270161227\, x^{10} +$$
$$2^{20}\times 3^{11}\times 5^6\times 7\times 4954321\times 8800968989\, x^{11} +$$
$$2^{27}\times 3^{12}\times 5^6\times 11\times 1091\times 12756480773\, x^{12})$$

$$ch(\theta_{22}) = \frac{1}{2^{24}\times 3^{12}\times 5^4\, x^7}($$
$$2^2\times 3^2\times 5^2\times 7\times 23\times 443\times 110339 +$$
$$3^2\times 5\times 7^2\times 23\times 443\times 45307\times 110339\, x +$$
$$2\times 5\times 7\times 79\times 107166946015042819\, x^2 +$$
$$2\times 3\times 7^2\times 36713\times 34478412780970831\, x^3 +$$
$$3^3\times 7\times 8069\times 484086291493596614 53\, x^4 +$$
$$2^3\times 3^2\times 13\times 17\times 4099\times 26525879\times 4080242157239\, x^5 +$$
$$2^4\times 3^3\times 7\times 1277014026590062232215 10827\, x^6 +$$
$$2^6\times 3^4\times 5\times 401\times 125717793427846612 1552321\, x^7 +$$
$$2^8\times 3^5\times 5\times 7\times 1277014026590062232215 10827\, x^8 +$$
$$2^{11}\times 3^6\times 5^2\times 13\times 17\times 4099\times 26525879\times 4080242157239\, x^9 +$$
$$2^{12}\times 3^9\times 5^3\times 7\times 8069\times 484086291493596614 53\, x^{10} +$$
$$2^{17}\times 3^9\times 5^4\times 7^2\times 36713\times 34478412780970831\, x^{11} +$$
$$2^{21}\times 3^{10}\times 5^4\times 7\times 79\times 107166946015042819\, x^{12} +$$
$$2^{24}\times 3^{14}\times 5^7\times 7^2\times 23\times 443\times 45307\times 110339\, x^{13} +$$
$$2^{30}\times 3^{16}\times 5^9\times 7\times 23\times 443\times 110339\, x^{14})$$

$$ch(\theta_{23}) = \frac{1}{2^{12}\times 3^6\times 5^2\, x^3}($$
$$2^6\times 3\times 5\times 7\times 51977 +$$
$$3^2\times 37\times 967\times 991\times 1277\, x +$$
$$3\times 5\times 7^3\times 37\times 67\times 199\times 58321 x^2 +$$
$$2^3\times 11\times 1040903040186 61\, x^3 +$$
$$2^4\times 3^3\times 5^2\times 7^3\times 37\times 67\times 199\times 58321\, x^4 +$$
$$2^8\times 3^6\times 5^2\times 37\times 967\times 991\times 1277\, x^5 +$$
$$2^{18}\times 3^7\times 5^4\times 7\times 51977\, x^6)$$

$$ch(\theta_{24}) = \frac{7}{2^{10}\times 3^4\times 5^2\, x^3}($$
$$2^2\times 3\times 5\times 17 + 3\times 7\times 17\times 45307\, x +$$
$$5\times 7103\times 192949\, x^2 +$$
$$2^2\times 5\times 29\times 541\times 1456541\, x^3 +$$
$$2^4\times 3^2\times 5^2\times 7103\times 192949\, x^4 +$$
$$2^8\times 3^5\times 5^2\times 7\times 17\times 45307\, x^5 +$$
$$2^{14}\times 3^7\times 5^4\times 17\, x^6)$$

$$ch(\theta_{25}) = \frac{1}{2^{16}\times 3^8\times 5^3\, x^4}($$
$$2\times 7\times 11\times 1634509733 +$$
$$3\times 5\times 7\times 11149\times 240779587\, x +$$
$$2^3\times 97\times 5019793\times 33822727\, x^2 +$$
$$3^3\times 7^2\times 19\times 1429\times 862136910847\, x^3 +$$
$$2^2\times 3^5\times 7\times 11\times 127\times 190523\times 803051407\, x^4 +$$
$$2^4\times 3^5\times 5\times 7^2\times 19\times 1429\times 862136910847\, x^5 +$$
$$2^{11}\times 3^4\times 5^2\times 97\times 5019793\times 33822727\, x^6 +$$
$$2^{12}\times 3^7\times 5^4\times 7\times 11149\times 240779587\, x^7 +$$
$$2^{17}\times 3^8\times 5^4\times 7\times 11\times 1634509733\, x^8)$$

$$ch(\theta_{26}) = \frac{1}{2^{20}\times 3^{10}\times 5^4\, x^5}($$
$$3^2\times 7\times 1920917020339 +$$
$$2^6\times 1932627887748647\, x +$$
$$2^2\times 3\times 7\times 687598205349751411\, x^2 +$$
$$5\times 7^2\times 673\times 77746343167071049\, x^3 +$$
$$2^2\times 3^2\times 7\times 41\times 4513\times 8209\times 397543\times 6033427\, x^4 +$$
$$2^4\times 3^2\times 42256259\times 5662271823471397\, x^5 +$$
$$2^6\times 3^4\times 5\times 7\times 41\times 4513\times 8209\times 397543\times 6033427\, x^6 +$$
$$2^8\times 3^4\times 5^3\times 7^2\times 673\times 77746343167071049\, x^7 +$$
$$2^{14}\times 3^7\times 5^3\times 7\times 687598205349751411\, x^8 +$$
$$2^{22}\times 3^8\times 5^4\times 1932627887748647\, x^9 +$$
$$2^{20}\times 3^{12}\times 5^5\times 7\times 1920917020339\, x^{10})$$

$$ch(\theta_{27}) = \frac{1}{2^{20}\times 3^9\times 5^4\, x^5}($$
$$7^2\times 63113\times 7700299 +$$
$$5^2\times 7\times 1699\times 195399775663\, x +$$
$$2^3\times 7\times 875675732223561979\, x^2 +$$
$$2^3\times 3\times 103\times 367\times 5791\times 125219\times 13800671\, x^3 +$$
$$2^4\times 7\times 233\times 95531\times 6117073\times 43889063\, x^4 +$$
$$2^4\times 3^2\times 7\times 31\times 7905004010787912 17977\, x^5 +$$
$$2^8\times 3^2\times 5\times 7\times 233\times 95531\times 6117073\times 43889063\, x^6 +$$
$$2^{11}\times 3^5\times 5^2\times 103\times 367\times 5791\times 125219\times 13800671\, x^7 +$$
$$2^{15}\times 3^6\times 5^3\times 7\times 875675732223561979\, x^8 +$$
$$2^{16}\times 3^8\times 5^6\times 7\times 1699\times 195399775663\, x^9 +$$
$$2^{20}\times 3^{10}\times 5^5\times 7^2\times 63113\times 7700299 x^{10})$$

$$ch(\theta_{28}) = \frac{7}{2^{12}\times 3^6\times 5^2\ x^3}\ (\\ 3\times 7\times 41\times 13469 + 5\times 25841\times 172357\ x + \\ 2^2\times 3\times 7\times 17\times 151\times 229\times 76837\ x^2 + \\ 2^2\times 3^2\times 13\times 41\times 8271569177\ x^3 + \\ 2^6\times 3^3\times 5\times 7\times 17\times 151\times 229\times 76837\ x^4 + \\ 2^8\times 3^4\times 5^3\times 25841\times 172357\ x^5 + \\ 2^{12}\times 3^7\times 5^3\times 7\times 41\times 13469\ x^6)$$

$$ch(\theta_{29}) = \frac{1}{2^{22}\times 3^{12}\times 5^5\ x^6}\ (\\ 3^2\times 7\times 37\times 87365639293 + \\ 3\times 7\times 17\times 25158739\times 163675601\ x + \\ 2\times 7^2\times 293207\times 571321\times 153212833\ x^2 + \\ 2\times 3\times 5^2\times 7^2\times 19\times 4966016564031243349\ x^3 + \\ 2^2\times 3^2\times 79\times 383\times 1049\times 29339 times 2479950110851\ x^4 + \\ 2^2\times 3^3\times 7\times 259650710821\times 25089495454453 x^5 + \\ 2^7\times 3^4\times 7^2\times 333504308440846597412509\ x^6 + \\ 2^6\times 3^5\times 5\times 7\times 259650710821\times 25089495454453\ x^7 + \\ 2^{10}\times 3^6\times 5^2\times 79\times 383\times 1049\times 29339 times 2479950110851\ x^8 + \\ 2^{13}\times 3^7\times 5^5\times 7^2\times 19\times 4966016564031243349\ x^9 + \\ 2^{17}\times 3^8\times 5^4\times 7^2\times 293207\times 571321\times 153212833\ x^{10} + \\ 2^{20}\times 3^{11}\times 5^5\times 7\times 17\times 25158739\times 163675601\ x^{11} + \\ 2^{24}\times 3^{14}\times 5^6\times 7\times 37\times 87365639293\ x^{12})$$

$$ch(\theta_{30}) = \frac{1}{2^{20}\times 3^{10}\times 5^5\ x^6}\ (\\ 2^4\times 3^2\times 5^3\times 25841 + \\ 2^2\times 3^2\times 5^2\times 7\times 25841\times 45307\ x + \\ 5\times 7\times 25841\times 115259\times 666353\ x^2 + \\ 3\times 7\times 55717\times 2804293\times 13315567\ x^3 + \\ 2^3\times 3^2\times 86399\times 959723737277659\ x^4 + \\ 2^6\times 3^3\times 5\times 7\times 19\times 372825737\times 955373927\ x^5 + \\ 2^8\times 3^4\times 5\times 11\times 59\times 1627\times 137055035733221\ x^6 + \\ 2^{10}\times 3^5\times 5^2\times 7\times 19\times 372825737\times 955373927\ x^7 + \\ 2^{11}\times 3^6\times 5^2\times 86399\times 959723737277659\ x^8 + \\ 2^{12}\times 3^7\times 5^3\times 7\times 55717\times 2804293\times 13315567\ x^9 + \\ 2^{16}\times 3^8\times 5^5\times 7\times 25841\times 115259\times 666353\ x^{10} + \\ 2^{22}\times 3^{12}\times 5^7\times 7\times 25841\times 45307\ x^{11} + \\ 2^{28}\times 3^{14}\times 5^9\times 25841\ x^{12})$$

$$ch(\theta_{31}) = \frac{1}{2^{20}\times 3^{10}\times 5^5\ x^6}\ (\\ 2^2\times 3^2\times 5\times 71\times 1558283 + \\ 3^2\times 7\times 71\times 45307\times 1558283\ x + \\ 2\times 3\times 7^2\times 1499\times 3758088575183\ x^2 + \\ 2\times 5^2\times 7^2\times 11261\times 185753\times 172860353\ x^3 + \\ 3^2\times 5^2\times 13\times 29\times 241069\times 7378760342027\ x^4 + \\ 2^3\times 3^3\times 5\times 7\times 11\times 136666703950440834769\ x^5 + \\ 2^5\times 3^4\times 5^2\times 7^2\times 1216883197\times 110293780367\ x^6 + \\ 2^7\times 3^5\times 5^2\times 7\times 11\times 136666703950440834769\ x^7 + \\ 2^8\times 3^6\times 5^4\times 13\times 29\times 241069\times 7378760342027\ x^8 + \\ 2^{13}\times 3^6\times 5^5\times 7^2\times 11261\times 185753\times 172860353\ x^9 + \\ 2^{17}\times 3^9\times 5^4\times 7^2\times 1499\times 3758088575183\ x^{10} + \\ 2^{20}\times 3^{12}\times 5^5\times 7\times 71\times 45307\times 1558283\ x^{11} + \\ 2^{26}\times 3^{14}\times 5^7\times 71\times 1558283\ x^{12})$$

$$ch(\theta_{32}) = \frac{7}{2^{16}\times 3^8\times 5^4\ x^5}\ (\\ 2^4\times 3^3\times 5^2\times 17 + \\ 2^2\times 3^3\times 5\times 7\times 17\times 45307\ x + \\ 3\times 7\times 17\times 115259\times 666353\ x^2 + \\ 5\times 2512868354279147\ x^3 + \\ 2^2\times 3^2\times 5\times 1061\times 218249\times 31482709\ x^4 + \\ 2^4\times 3^2\times 5^2\times 7\times 2069\times 634759\times 1884341\ x^5 + \\ 2^6\times 3^4\times 5^2\times 1061\times 218249\times 31482709\ x^6 + \\ 2^8\times 3^4\times 5^3\times 2512868354279147\ x^7 + \\ 2^{12}\times 3^7\times 5^3\times 7\times 17\times 115259\times 666353\ x^8 + \\ 2^{18}\times 3^{11}\times 5^5\times 7\times 17\times 45307\ x^9 + \\ 2^{24}\times 3^{13}\times 5^7\times 17\ x^{10})$$

$$ch(\theta_{33}) = \frac{1}{2^{14}\times 3^6\times 5^3\ x^4}\ (\\ 2^2\times 5^2\times 25841 + 5\times 7\times 25841\times 45307\ x + \\ 3\times 7^2\times 19\times 83\times 2459\times 45307\ x^2 + \\ 2^3\times 3\times 7^2\times 43\times 53841866849\ x^3 + \\ 2^5\times 3^4\times 5\times 71\times 109\times 401\times 1543\times 1789\ x^4 + \\ 2^7\times 3^3\times 5\times 7^2\times 43\times 53841866849 x^5 + \\ 2^8\times 3^5\times 5^2\times 7^2\times 19\times 83\times 2459\times 45307\ x^6 + \\ 2^{12}\times 3^6\times 5^4\times 7\times 25841\times 45307\ x^7 + \\ 2^{18}\times 3^8\times 5^6\times 25841\ x^8)$$

$$ch(\theta_{34}) = \frac{1}{2^{20}\times 3^{10}\times 5^4\ x^5}(\\ 2^5\times 3\times 7\times 19\times 101\times 2437\times 7883 + \\ 3^3\times 7\times 17\times 43\times 670034705677\ x + \\ 3\times 5\times 7^2\times 1093\times 72068616146713\ x^2 + \\ 2^2\times 5\times 7^2\times 144593\times 92161130410753\ x^3 + \\ 2^2\times 3^3\times 7\times 4127\times 8389\times 31543\times 1323118663\ x^4 + \\ 2^7\times 3^2\times 11\times 11943616469\times 264210902899\ x^5 + \\ 2^6\times 3^5\times 5\times 7\times 4127\times 8389\times 31543\times 1323118663\ x^6 + \\ 2^{10}\times 3^4\times 5^3\times 7^2\times 144593\times 92161130410753\ x^7 + \\ 2^{12}\times 3^7\times 5^4\times 7^2\times 1093\times 72068616146713\ x^8 + \\ 2^{16}\times 3^{11}\times 5^4\times 7\times 17\times 43\times 670034705677\ x^9 + \\ 2^{25}\times 3^{11}\times 5^5\times 7\times 19\times 101\times 2437\times 7883\ x^{10})$$

$$ch(\theta_{35}) = \frac{1}{2^{14}\times 3^8\times 5^3\ x^4}\ (\\ 3^2\times 71\times 1558283 + \\ 3\times 7^2\times 11\times 71\times 57706391\ x + \\ 5\times 7\times 71\times 569\times 15083\times 158143\ x^2 + \\ 3^2\times 5\times 7^2\times 11\times 76091\times 256751687\ x^3 + \\ 2^6\times 3^2\times 5\times 238247\times 29004591751\ x^4 + \\ 2^4\times 3^4\times 5^2\times 7^2\times 11\times 76091\times 256751687\ x^5 + \\ 2^8\times 3^4\times 5^3\times 7\times 71\times 569\times 15083\times 158143\ x^6 + \\ 2^{12}\times 3^7\times 5^3\times 7^2\times 11\times 71\times 57706391\ x^7 + \\ 2^{16}\times 3^{10}\times 5^4\times 71\times 1558283\ x^8)$$

$$ch(\theta_{36}) = \frac{1}{2^{18}\times 3^8\times 5^4\ x^5}\ (\\ 2^8\times 3\times 5^2\times 7\times 51977 + \\ 2^6\times 3\times 5\times 7^2\times 45307\times 51977\ x + \\ 3\times 7\times 13\times 577\times 696629\times 1185337\ x^2 + \\ 5\times 509\times 547\times 37532952421247\ x^3 + \\ 2^3\times 7\times 255247\times 401008447405121\ x^4 + \\ 2^4\times 5\times 53\times 2861\times 1461797\times 14111392523\ x^5 + \\ 2^7\times 3^2\times 5\times 7\times 255247\times 401008447405121\ x^6 + \\ 2^8\times 3^4\times 5^3\times 509\times 547\times 37532952421247\ x^7 + \\ 2^{12}\times 3^7\times 5^3\times 7\times 13\times 577\times 696629\times 1185337\ x^8 + \\ 2^{22}\times 3^9\times 5^5\times 7^2\times 45307\times 51977\ x^9 + \\ 2^{28}\times 3^{11}\times 5^7\times 7\times 51977^1\ x^{10})$$

$$ch(\theta_{37}) = \frac{1}{2^{22}\times 3^{10}\times 5^5\ x^6}\ (\\ 2^2\times 5\times 7\times 31\times 193\times 3624787 + \\ 7^2\times 31\times 193\times 45307\times 3624787\ x + \\ 2\times 31\times 79\times 6043\times 672439\times 3360061\ x^2 + \\ 3\times 5\times 7^2\times 40277\times 1031399\times 973746491\ x^3 + \\ 2^2\times 3\times 5\times 7\times 29\times 43\times 97\times 1375243\times 57552822853\ x^4 + \\ 2^4\times 3\times 7^2\times 239\times 45007\times 10354439038651493\ x^5 + \\ 2^6\times 3^6\times 5\times 11\times 107\times 163\times 208977919833364987\ x^6 + \\ 2^8\times 3^3\times 5\times 7^2\times 239\times 45007\times 10354439038651493\ x^7 + \\ 2^{10}\times 3^5\times 5^3\times 7\times 29\times 43\times 97\times 1375243\times 57552822853\ x^8 + \\ 2^{12}\times 3^7\times 5^4\times 7^2\times 40277\times 1031399\times 973746491\ x^9 + \\ 2^{17}\times 3^8\times 5^4\times 31\times 79\times 6043\times 672439\times 3360061\ x^{10} + \\ 2^{20}\times 3^{10}\times 5^5\times 7^2\times 31\times 193\times 45307\times 3624787\ x^{11} + \\ 2^{26}\times 3^{12}\times 5^7\times 7\times 31\times 193\times 3624787\ x^{12})$$

$$ch(\theta_{38}) = \frac{7}{2^{16}\times 3^8\times 5^3\ x^4}\ (\\ 31\times 193\times 3624787 + \\ 2\times 3\times 661\times 39827\times 198031\ x + \\ 3^4\times 5\times 9371\times 3530299103\ x^2 + \\ 2^2\times 3\times 5\times 91297\times 232912445897\ x^3 + \\ 2^4\times 3^3\times 370597\times 315220711859\ x^4 + \\ 2^6\times 3^3\times 5^2\times 91297\times 232912445897\ x^5 + \\ 2^8\times 3^8\times 5^3\times 9371\times 3530299103\ x^6 + \\ 2^{13}\times 3^7\times 5^3\times 661\times 39827\times 198031\ x^7 + \\ 2^{16}\times 3^8\times 5^4\times 31\times 193\times 3624787\ x^8)$$

$$ch(\theta_{39}) = \frac{1}{2^{18}\times 3^8\times 5^4\times x^5}\ (\\ 2^2\times 3\times 5\times 7^2\times 41\times 13469 + \\ 3\times 7^3\times 41\times 13469\times 45307\ x + \\ 5\times 7\times 19813\times 26321\times 2574193\ x^2 + \\ 2^2\times 3\times 7\times 41946929\times 2853739787\ x^3 + \\ 2^2\times 3\times 7\times 11\times 875056131284689657\ x^4 + \\ 2^4\times 3^2\times 5^2\times 8352802228682522537\ x^5 + \\ 2^6\times 3^3\times 5\times 7\times 11\times 875056131284689657\ x^6 + \\ 2^{10}\times 3^5\times 5^2\times 7\times 41946929\times 2853739787\ x^7 + \\ 2^{12}\times 3^6\times 5^4\times 7\times 19813\times 26321\times 2574193\ x^8 + \\ 2^{16}\times 3^9\times 5^4\times 7^3\times 41\times 13469\times 45307\ x^9 + \\ 2^{22}\times 3^{11}\times 5^6\times 7^2\times 41\times 13469\ x^{10})$$

$$ch(\theta_{40}) = \frac{7}{2^{18}\times 3^{10}\times 5^4\times x^5}\ (\\ 3^2\times 5\times 23\times 443\times 110339 + \\ 5\times 29\times 31\times 113\times 25841\times 25849\ x + \\ 3\times 7\times 19\times 29\times 31\times 83\times 113\times 2459\times 25849\ x^2 + \\ 3^2\times 7\times 419\times 2364953\times 600815491\ x^3 + \\ 2^{10}\times 3^3\times 11\times 23\times 8912381\times 41333989\ x^4 + \\ 2^4\times 3^4\times 61\times 67\times 17529680553917933\ x^5 + \\ 2^{14}\times 3^5\times 5\times 11\times 23\times 8912381\times 41333989\ x^6 + \\ 2^8\times 3^6\times 5^2\times 7\times 419\times 2364953\times 600815491\ x^7 + \\ 2^{12}\times 3^7\times 5^3\times 7\times 19\times 29\times 31\times 83\times 113\times 2459\times 25849\ x^8 + \\ 2^{16}\times 3^8\times 5^5\times 29\times 31\times 113\times 25841\times 25849\ x^9 + \\ 2^{20}\times 3^{12}\times 5^6\times 23\times 443\times 110339\ x^{10})$$

Now, one can see that the characters above fulfill the following equation:

$$ch(\Lambda_3)\times ch(\Lambda_4) = ch(\Lambda_3+\Lambda_4) + \\ ch(\theta_2) + ch(\theta_3) + ch(\theta_4) + ch(\theta_5) + \\ ch(\theta_6) + ch(\theta_7) + ch(\theta_8) + ch(\theta_9) + \\ ch(\theta_{10}) + ch(\theta_{11}) + ch(\theta_{12}) + ch(\theta_{13}) + \\ ch(\theta_{14}) + ch(\theta_{15}) + ch(\theta_{16}) + ch(\theta_{17}) + \\ 2\ ch(\theta_{18}) + 2\ ch(\theta_{19}) + 2\ ch(\theta_{20}) + 2\ ch(\theta_{21}) + 2\ ch(\theta_{22}) + \\ 3\ ch(\theta_{23}) + 3\ ch(\theta_{24}) + 3\ ch(\theta_{25}) + 3\ ch(\theta_{26}) + \\ 3\ ch(\theta_{27}) + 3\ ch(\theta_{28}) + 3\ ch(\theta_{29}) + 3\ ch(\theta_{30}) + \\ 4\ ch(\theta_{31}) + 4\ ch(\theta_{32}) + 4\ ch(\theta_{33}) + \\ 5\ ch(\theta_{34}) + 5\ ch(\theta_{35}) + 5\ ch(\theta_{36}) + 5\ ch(\theta_{37}) \\ +5\ ch(\theta_{38}) + 5\ ch(\theta_{39}) + 5\ ch(\theta_{40}) \qquad (III.6)$$

Conclusion

One should note however that, the 1-parameter specialization (III.5) above is not enough to find all the tensor coupling coefficients completely so we saw that at least 3-parameters specializations will be sufficient, which we used the following one.

$$u_1 = 1$$
$$u_2 = 2$$
$$u_3 = 3$$
$$u_4 = 4$$
$$u_5 = x$$
$$u_6 = y$$
$$u_7 = z$$
$$u_8 = 1/(24\ x\ y\ z)$$

References

1. Humphreys JE (1972) Introduction to Lie Algebras and Representation Theory, Springer-Verlag.
2. Kac V (1982) Infinite Dimensional Lie Algebras, Cambridge University Press.
3. Karadayi HR, Gungormez M (1999) Fundamental Weights, Permutation Weights and Weyl Character Formula. J Phys A 32: 1701-1707.

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