Exponential Stability of Nonlinear Nonautonomous Multivariable Discrete Systems

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Abstract

We consider a class of nonautonomous discrete-time systems governed by semilinear vector difference equations with slowly varying linear parts. Sharp exponential stability conditions are suggested. They are formulated in terms of the eigenvalues of the coefficients and constants characterizing the nonlinearities. Our approach is based on the recent norm estimates for solutions of matrix equations.

Keywords: Discrete-time systems; Nonlinear nonautonomous systems; Exponential stability

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Introduction

The basic method for the stability analysis of nonlinear time-discrete systems is the Lyapunov functions one, cf. [1,2]. By this method many very strong results are obtained, but finding Lyapunov's functions is often connected with serious mathematical difficulties. Explicit stability conditions for nonlinear multivariable systems are established mainly in the case of autonomous linear parts, cf. [3-9].

In the present paper we investigate a class of discrete-time systems governed by semilinear vector difference equations with non-autonomous linear parts. Explicit stability conditions are suggested. They are formulated in terms of the eigenvalues of the coefficients and constants characterizing the nonlinearities. It is shown that the suggested stability conditions are sharp.

Let \( \mathbb{C} \) be a Euclidean space of \( n \)-complex vectors endowed with a scalar product \((.,.)\) and the Euclidean norm \( ||.|| = \sqrt{\langle ., . \rangle} \), \( I \) is the identity matrix and \( \omega_r = \{ h \in \mathbb{C}^n : ||h|| = r \} \) for a positive \( r \in \mathbb{R} \). Our main object is the equation

\[
\begin{align*}
\dot{u}_k &= A_k u_k + F_k(u_k) \quad (k=0,1,\ldots), \\
\text{where } &A_k(k=0,1,\ldots) \text{ are } n \times n \text{-matrices } F_k: \omega \to \mathbb{C}^n \text{ are mappings satisfying} \\
&||F_k(w)|| \leq v_k ||w|| \quad (w \in \omega) \\
&||v_k|| \text{ are given non-negative constants.}
\end{align*}
\]

The zero solution of system (1) is said to be exponentially stable if there are constants \( M \in I, \xi \in (0,1) \) and \( \delta > 0 \), such that \( ||u_k|| \leq M \xi^k ||u_0|| \) (\( k=1,2,\ldots \)) for any solution \( u_k \) of (1), provided \( ||u_0|| < \delta \).

Introduce the notations. For an \( n \times n \) matrix \( A, A' \) is the adjoint one; \( \lambda_j(A) \) \( j=1,\ldots,n \) are the eigenvalues of \( A \), counted with their multiplicities; \( ||A|| = \sup_{||h||=1} ||Ah|| / ||h|| \) is the spectral (operator) norm of \( A \); \( r(A) \) is the spectral radius. The following quantity (the departure from normality of \( A \)) plays a key role hereafter:

\[
g(A) = ||A||^2 - \sum_j |\lambda_j(A)|^2,\]

where \( |A| = (\text{Trace } AA')^{1/2} \) is the Frobenius (Hilbert-Schmidt norm) of \( A \). If \( A \) is a normal matrix: \( AA' = A'A \), then \( g(A) \).

Suppose that the matrix \( A \) is semi-normal: \( \sum_j |\lambda_j(A)|^2 = g(A) < \infty \).

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They are formulated in terms of the eigenvalues of the coefficients and constants characterizing the nonlinearities. It is shown that the suggested stability conditions are sharp.

Let the conditions (2), (3) and

\[
\dot{\mu}_k := \sum_{m} g(A_k) \omega_r(A_k),
\]

where \( \dot{\mu}_k \) are given non-negative constants.

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The following relations are checked in [10]
\[ g'(A) \leq A^2 - \text{Trace } A^2 \quad \text{and} \quad g''(A) \leq \frac{A - A^2}{2} \geq 2 |A|^2. \]

Where \( A = (A^2)/2i \). If \( A_i \) and \( A_j \) are commutating matrices, then \( g(A_i + A_j) \leq g(A_i) + g(A_j) \). By the inequality between geometric and arithmetic mean values we have
\[ \left( \frac{1}{n} \sum_{i=1}^{n} \lambda_i(A)^2 \right)^{1/2} \geq \left( \prod_{i=1}^{n} |\lambda_i(A)| \right)^{1/2}. \]

So \( g'(A) \leq A^2 - \text{m(det } A)^2 \).

**Auxiliary Results**

Let \( A \) and \( C \) be \( n \times n \) matrices and \( r_f(A) < 1 \). The equation
\[ Y - AYA = C \]
has a unique solution \( Y(A,C) \), cf. [11]. Put
\[ q(A) := \sup_{C \in \mathbb{C}^{nn}} \left\{ \frac{\|Y(A,C)\|}{\|C\|} \right\}, \]
where \( \mathbb{C}^{nn} \) is the set of all \( n \times n \) matrices.

Due to Corollary 3 from [12],
\[ q(A) \leq \sum_{i=1}^{\infty} \frac{\|g'(A)\|^{m_i}}{(m_i)!} \left( \sum_{j=0}^{m_i} \frac{1}{j!} \right)^{m_j}. \]

Recall that \( \frac{1}{(m_j)!} = 0 \) for \( m < j \). From (7) it follows
\[ q(A) \leq \sum_{i=1}^{\infty} \frac{\|g'(A)\|^2 \|A^2\|^{m_i}}{(m_i)!} \left( \sum_{j=0}^{m_i} \frac{1}{j!} \right)^{m_j}. \]

Now taking into account that
\[ \sum_{i=1}^{\infty} m_i! \frac{1}{(m_j)!} \frac{1}{j!} = 0 \]
we get
\[ q(A) \leq \sum_{i=1}^{\infty} \frac{\|g'(A)\|^2 \|A^2\|^{1-\gamma}}{(1-\gamma)^{1-\gamma}}. \]

**Proof of Theorem 1**

Let \( Q_k \) be a solution of the equation
\[ Q_k - A \hat{Q}_k A_k = I. \]

First, let \( r_f = \text{max} \). Then from (1) and (2) we have
\[ (Q_k - A \hat{Q}_k A_k) = (Q_k - A \hat{Q}_k A_k) + (Q_k - A \hat{Q}_k A_k) + (Q_k - A \hat{Q}_k A_k) \leq \left| Q_k \right| (2 \| A \| v_i + v_j) \| A_k \| \| u_i \| ^2, \]
where
\[ (Q_k - A \hat{Q}_k A_k) + (Q_k - A \hat{Q}_k A_k) - (Q_k - A \hat{Q}_k A_k) \leq \left| Q_k \right| (2 \| A \| v_i + v_j) \| A_k \| \| u_i \| ^2. \]

If
\[ \left| Q_k - A \hat{Q}_k A_k \right| + \left| Q_k \right| (2 \| A \| v_i + v_j) \| A_k \| \| u_i \| ^2 < 1, \]
Then
\[ (Q_k - A \hat{Q}_k A_k) \leq \| Q_k \| (2 \| A \| v_i + v_j) \| u_i \| ^2. \]
But \( Q_k \geq A \). So
\[ (u_i, u_j) \leq \| Q_k \| (2 \| A \| v_i + v_j) \| u_i \| ^2. \]

Now by a small perturbation we arrive at the following result.

**Lemma 1**

Let the conditions (2) with \( r_f \) and (10) hold. Then the zero solution to (1) is (globally) exponentially stable. According to (11), under (10) there is a constant \( \gamma \) independent of \( u_i \), such that \( \| u_i \| \leq \gamma \| u_i \| \). So taking \( \| u_i \| \leq \gamma \| u_i \| \) we can remove the condition \( r_f(A) < 1 \).

Furthermore, the equation
\[ Y - A Y A_k = C \]
has a unique solution \( Y(A_k, C) \) for any \( C \). Put
\[ q(A_k) := \sup_{C \in \mathbb{C}^{nn}} \left\{ \frac{\|Y(A_k,C)\|}{\|C\|} \right\}. \]

To estimate \( \|Q_k - Q_k^{*}\|_2 < \|Q_k - Q_k^{*}\| \), note that
\[ Q_k - Q_k^{*} = A \hat{Q}_k A_k - A \hat{Q}_k A_k. \]

Denote \( Y_k = Q_k - Q_k^{*} \) and \( Y_k \geq A \hat{Q}_k A_k \). Then
\[ Y_k \geq A \hat{Q}_k A_k = C_k, \]
where \( C_k := A \hat{Q}_k A_k + \Delta_k \).

Due to (13), \( \|Q_k - Q_k^{*}\|_2 \leq \|Q_k - Q_k^{*}\|_2 \leq \|Q_k - Q_k^{*}\|_2 \leq q(A_k) \Delta_k \| Q_k \| \| A_k \| \| A_k \|. \]

So
\[ \|Q_k - Q_k^{*}\|_2 \leq \|Q_k - Q_k^{*}\|_2 \leq \|Q_k - Q_k^{*}\|_2 \leq \|Q_k - Q_k^{*}\|_2 \leq \|Q_k - Q_k^{*}\|_2 \leq q(A_k) \Delta_k \| Q_k \| \| A_k \| \| A_k \|. \]

Thus, Lemma 1 implies

**Corollary 2**

Let
\[ \frac{q^2(A_k)}{q^2(A_k) + q^2(A_k) + q(A_k) \| \hat{Q}_k \|_2 \| 2 \Delta_k \| v_i + v_j} < 1. \]

Then the zero solution to (1) is exponentially stable.

Proof of Theorem 1: From (7) it follows \( q(A_k) \leq \mu_k \). Now the previous corollary implies the required result.

**Example**

Let \( A_k (k = 0, 1, \ldots) \) be normal matrices. Then condition (5) takes the form
\[ \frac{1}{(1 - r_f(A)) \| A \|} \| A \| \| A \| \| A \| \| A \| + \frac{1}{1 - r_f(A)} (2 \| A \| v_i + v_j) < 1. \]

In particular, if \( A_k \equiv A \) is constant, then due to (4) from (14) it follows
\[ \frac{1}{1 - r_f(A)} (2 \| A \| v_i + v_j) < 1. \]

where \( \nu := \sup_{i} 2 \| \hat{A} \| \). But \( \| A \| = r_f(A) \). So \( 2 \| A \| (2 \| A \| v_i + v_j) < 1 - r_f(A) \) and therefore, the exponential stability test under the consideration is
\[ r_f(A) < 1. \]

This condition shows that Theorem 1 is sharp. Indeed, take \( F_{\omega, i} = i \nu \). Then condition (15) is necessary for the exponential stability. So Theorem 1 is really sharp.

**References**


