Ashdin Publishing
Journal of Physical Mathematics
Vol. 3 (2011), Article ID P110505, 5 pages
doi:10.4303/jpm/P110505

Research Article

Extended Lie Derivatives and a New Formulation of $D = 11$ Supergravity

Leonardo Castellani[1,2]

[1] *Department of Sciences and Advanced Technologies, Eastern Piedmont University, Via Bellini 25/G, 15100 Alessandria, Italy*
[2] *National Institute of Nuclear Physics (INFN), Via Giuria 1, 10125 Torino, Italy*
Address correspondence to Leonardo Castellani, leonardo.castellani@mfn.unipmn.it

Received 25 March 2011; Accepted 10 May 2011

Abstract Introducing an extended Lie derivative along the dual of A, the 3-form field of $D = 11$ supergravity, the full diffeomorphism algebra of $D = 11$ supergravity is presented. This algebra suggests a new formulation of the theory, where the 3-form field A is replaced by bivector B^{ab}, bispinor $B^{\alpha\beta}$, and spinor-vector $\eta^{a\beta}$ 1-forms. Only the bivector 1-form B^{ab} is propagating, and carries the same degrees of freedom of the 3-form in the usual formulation, its curl $\mathcal{D}_{[\mu}B^{ab}_{\nu]}$ being related to the $F_{\mu\nu ab}$ curl of the 3-form. The other 1-forms are auxiliary, and the transformation rules on all the fields close on the equations of motion of $D = 11$ supergravity.

MSC 2010: 53Z05, 83F50

1 Introduction

Supergravity in eleven dimensions [7,8] (a particular limit of M-theory, for a review see e.g. [17,18]), can be formulated within the framework of free differential algebras (FDAs) [16,10,5,19,4,3,6], a generalization of Lie algebras which include p-form potentials. In this framework, the 3-form of $D = 11$ supergravity acquires an algebraic interpretation.

In preceding papers [6,2], it was shown how FDAs can be related to ordinary Lie algebras via extended Lie derivatives: the Lie algebra underlying the FDA of $D = 11$ supergravity was identified in [2], and found to coincide with an algebra discussed in [1] in the context of supermembranes.

In Section 2, we recall briefly the FDA formulation of $D = 11$ supergravity of [10]. In Section 3, we present the full diffeomorphism algebra of $D = 11$ supergravity on the FDA "manifold", which encodes all the symmetries of the theory. In Section 4, a new formulation is proposed, where the 3-form A is replaced by 1-form potentials with (couples of) Lorentz vector and spinor indices.

For a résumé on FDAs and the notion of extended Lie derivative along duals of p-forms we refer to [2]. Here the general theory is applied to the FDA of $D = 11$ supergravity.

2 The FDA of $D = 11$ supergravity, Bianchi identities, field equations and transformation rules

The FDA structure [10] is contained in the following curvature definitions:

$$\begin{aligned} R^{ab} &= d\omega^{ab} - \omega^{ac}\omega^{cb}, \\ R^a &= dV^a - \omega^{ab}V^b - \frac{i}{2}\bar{\psi}\Gamma^a\psi \equiv \mathcal{D}V^a - \frac{i}{2}\bar{\psi}\Gamma^a\psi, \\ \rho &= d\psi - \frac{1}{4}\omega^{ab}\Gamma^{ab}\psi \equiv \mathcal{D}\psi, \\ R(A) &= dA - \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi V^a V^b. \end{aligned} \tag{2.1}$$

The Bianchi identities are obtained by taking the exterior derivative of (2.1):

$$\begin{aligned} & dR^{ab} + R^{ac}\omega^{cb} - \omega^{ac}R^{cb} \equiv \mathcal{D}R^{ab} = 0, \\ & \mathcal{D}R^a + R^{ab}V^b - i\bar{\psi}\Gamma^a\rho = 0, \\ & \mathcal{D}\rho + \frac{1}{4}R^{ab}\Gamma^{ab}\psi = 0, \\ & dR(A) - \bar{\psi}\Gamma^{ab}\rho V^a V^b + \bar{\psi}\Gamma^{ab}\psi R^a V^b = 0. \end{aligned} \tag{2.2}$$

The superPoincaré curvatures R^{ab}, R^a, ρ (resp. the Lorentz curvature, the torsion and the gravitino curvature) are 2-forms, and $R(A)$ is a 4-form. These can be expanded on a superspace basis spanned by the vielbein V^a and the gravitino ψ. The group-geometric method of [3,4,9,11,14,15] requires the Ansatz that all "external" components of the curvatures be expressed in terms of the "spacetime" ones, spacetime meaning along the V^a vielbeins only.

The Bianchi *identities* then become *equations* for the curvatures, whose solution [10] is as follows:

$$\begin{aligned} R^{ab} &= R^{ab}{}_{cd}\, V^c V^d + i\left(2\bar\rho_{c[a}\Gamma_{b]} - \bar\rho_{ab}\Gamma_c\right)\psi V^c \\ &\quad + F^{abcd}\,\bar\psi\Gamma^{cd}\psi + \frac{1}{24}F^{c_1c_2c_3c_4}\,\bar\psi\Gamma^{abc_1c_2c_3c_4}\psi, \end{aligned} \tag{2.3}$$

$$R^a = 0, \tag{2.4}$$

$$\rho = \rho_{ab}V^aV^b + \frac{i}{3}\left(F^{ab_1b_2b_3}\Gamma^{b_1b_2b_3} - \frac{1}{8}F^{b_1b_2b_3b_4}\Gamma^{ab_1b_2b_3b_4}\right)\psi V^a, \tag{2.5}$$

$$R(A) = F^{a_1\cdots a_4}V^{a_1}V^{a_2}V^{a_3}V^{a_4}, \tag{2.6}$$

where the spacetime components $R^{ab}{}_{cd}, \rho_{ab}, F^{a_1\cdots a_4}$ satisfy the well-known propagation equations (Einstein, gravitino and Maxwell equations):

$$R^{ac}{}_{bc} - \frac{1}{2}\delta^a_b R = 3\,F^{ac_1c_2c_3}F^{bc_1c_2c_3} - \frac{3}{8}\,\delta^a_b\,F^{c_1\cdots c_4}F^{c_1\cdots c_4}, \tag{2.7}$$

$$\Gamma^{abc}\rho_{bc} = 0, \tag{2.8}$$

$$\mathcal{D}_a F^{ab_1b_2b_3} - \frac{1}{2\cdot 4!\cdot 7!}\,\epsilon^{b_1b_2b_3a_1\cdots a_8}\,F^{a_1\cdots a_4}F^{a_5\cdots a_8} = 0. \tag{2.9}$$

In the group geometric formulation, the symmetries gauged by the superPoincaré fields V^a, ω^{ab} and ψ are seen as diffeomorphisms on the "FDA manifold", generated by the Lie derivative along the tangent vectors t_a, t_{ab}, τ dual to these 1-form fields. Thus, setting $\varepsilon = \varepsilon^a t_a + \varepsilon^{ab}t_{ab} + \epsilon\tau$, the transformation rules under local supertranslations and Lorentz rotations are generated by the Lie derivative

$$\ell_\varepsilon \equiv d\, i_\varepsilon + i_\varepsilon\, d. \tag{2.10}$$

Explicitly

$$\delta V^a = \ell_\varepsilon V^a = \mathcal{D}\varepsilon^a + \varepsilon^{ab}V^b + i\bar\epsilon\Gamma^a\psi, \tag{2.11}$$

$$\begin{aligned} \delta\omega^{ab} &= \ell_\varepsilon\omega^{ab} = \mathcal{D}\varepsilon^{ab} + 2R^{ab}{}_{cd}\,\varepsilon^c V^d + i\left(2\bar\rho_{c[a}\Gamma_{b]} - \rho_{ab}\Gamma_c\right)\left(\epsilon V^c - \psi\varepsilon^c\right) \\ &\quad - 2F^{abcd}\,\bar\psi\Gamma^{cd}\epsilon - \frac{1}{12}F^{c_1c_2c_3c_4}\,\bar\psi\Gamma^{abc_1c_2c_3c_4}\epsilon, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \delta\psi &= \ell_\varepsilon\psi = \mathcal{D}\epsilon + \frac{i}{4}\varepsilon^{ab}\Gamma_{ab}\psi + 2\rho_{ab}\varepsilon^a V^b \\ &\quad + \frac{i}{3}\left(F^{ab_1b_2b_3}\Gamma^{b_1b_2b_3} - \frac{1}{8}F^{b_1b_2b_3b_4}\Gamma^{ab_1b_2b_3b_4}\right)\left(\epsilon V^a - \psi\varepsilon^a\right), \end{aligned} \tag{2.13}$$

$$\delta A = \ell_\varepsilon A = -\bar\psi\Gamma^{ab}\epsilon\, V^aV^b + \bar\psi\Gamma^{ab}\psi\,\varepsilon^a V^b + 4F^{a_1\cdots a_4}\varepsilon^{a_1}V^{a_2}V^{a_3}V^{a_4}, \tag{2.14}$$

where the exterior derivatives on the fields have been expressed in terms of the curvatures (2.1), and the solutions (2.3)–(2.6) have been used. The closure of these transformations is then equivalent to the propagation equations (2.7)–(2.9), as is usual in locally supersymmetric theories.

3 The algebra of diffeomorphisms on the $D = 11$ supergravity FDA "manifold"

On a soft group manifold, that is a manifold whose vielbeins μ^A have in general nonvanishing curvatures

$$R^A = d\mu^a + \frac{1}{2}C^C_{AB}\mu^A\mu^B, \tag{3.1}$$

the algebra of diffeomorphisms is given by the commutators of Lie derivatives:

$$\left[\ell_{\varepsilon_1^A \mathbf{t}_A}, \ell_{\varepsilon_2^B \mathbf{t}_B}\right] = \ell_{[\varepsilon_1^A\partial_A\varepsilon_2^C - \varepsilon_2^A\partial_A\varepsilon_1^C - 2\varepsilon_1^A\varepsilon_2^B\mathcal{R}^C_{AB}]\mathbf{t}_C}, \tag{3.2}$$

where $\mathbf{t}_A$ are the tangent vectors dual to the 1-forms μ^A, and

$$\mathcal{R}^C_{AB} \equiv R^C_{AB} - \frac{1}{2}C^C_{AB} \tag{3.3}$$

involves the curvature components on the vielbein basis and the group structure constants. The Jacobi identities between multiple commutators of Lie derivatives are equivalent to the Bianchi identities,

$$\partial_{[B}\mathcal{R}^A_{CD]} + 2\,\mathcal{R}^A_{E[B}\mathcal{R}^E_{CD]} = 0. \tag{3.4}$$

On a soft "FDA manifold", the algebra of diffeomorphisms includes also the diffeomorphisms in the p-form directions, generated by an extended Lie derivative $\ell_{\varepsilon\mathbf{t}}$, where $\mathbf{t}$ is a "tangent vector" dual to the p-form, and ε is a $p-1$ form parameter [6,2].

In the case of $D = 11$ supergravity, all the local symmetries of the theory are given by the following FDA diffeomorphism algebra:

$$\begin{aligned}
\left[\ell_{\varepsilon_1^A\mathbf{t}_A}, \ell_{\varepsilon_2^B\mathbf{t}_B}\right] &= \ell_{(\varepsilon_1^A\partial_A\varepsilon_2^C - \varepsilon_2^A\partial_A\varepsilon_1^C - 2\varepsilon_1^A\varepsilon_2^B\mathcal{R}^C_{AB})\mathbf{t}_C} \\
&\quad + \ell_{(-\bar{\epsilon}_1\Gamma^{ab}\epsilon_2V^aV^b - \varepsilon_1^a\varepsilon_2^b\,\bar{\psi}\Gamma^{ab}\psi + 2\varepsilon_2^a\bar{\epsilon}_1\,\Gamma^{ab}\psi V^b - 2\varepsilon_1^a\bar{\epsilon}_2\,\Gamma^{ab}\psi V^b - 12\varepsilon_1^a\varepsilon_2^b\,F^{abcd}V^cV^d)\mathbf{t}},
\end{aligned} \tag{3.5}$$

$$\left[\ell_{\varepsilon^A\mathbf{t}_A}, \ell_{\varepsilon\mathbf{t}}\right] = \ell_{\zeta\mathbf{t}}, \tag{3.6}$$

$$\left[\ell_{\varepsilon_1\mathbf{t}}, \ell_{\varepsilon_2\mathbf{t}}\right] = 0, \tag{3.7}$$

where the indices $A, B, C, \ldots$ run on the Lie algebra directions (corresponding to the vielbein, gravitino and spin connection 1-forms), that is $A = a, \alpha, ab$ (Lorentz). The quantities $\mathcal{R}^C_{AB}$ defined in (3.3) are given by the solutions for R^{ab}, R^a, ρ in (2.3)–(2.5) and by the superPoincaré structure constants encoded in the first three lines of (2.1). The two-form parameters ε and ζ in (3.6) are

$$\varepsilon \equiv \varepsilon_{ab}V^aV^b + 2\varepsilon_{a\beta}V^a\psi^\beta + \varepsilon_{\alpha\beta}\psi^\alpha\psi^\beta, \tag{3.8}$$

$$\begin{aligned}
\zeta \equiv\ & \varepsilon^A\big(\mathcal{D}_A\varepsilon_{cd}\big)V^cV^d - 2\varepsilon^c\big(\mathcal{D}\varepsilon_{cd}\big)V^d + 2i\varepsilon_{cd}\bar{\epsilon}\Gamma^c\psi V^d + i\varepsilon_{cd}\varepsilon^d\bar{\psi}\Gamma^c\psi \\
&+ 2\varepsilon^A\big(\mathcal{D}_A\varepsilon_{c\alpha}\big)V^c\psi^\alpha - 2\varepsilon^c\big(\mathcal{D}\varepsilon_{c\alpha}\big)\psi^\alpha - 2\varepsilon^\alpha\big(\mathcal{D}\varepsilon_{c\alpha}\big)V^c + 2i\varepsilon_{c\alpha}\bar{\epsilon}\Gamma^c\psi\psi^\alpha \\
&+ i\varepsilon_{c\alpha}\epsilon^\alpha\bar{\psi}\Gamma^c\psi - 2\varepsilon_{c\alpha}\varepsilon^c\rho^\alpha - 4\varepsilon_{c\alpha}\big(\rho^\alpha_{Ab}\varepsilon^AV^b + \rho^\alpha_{A\beta}\varepsilon^A\psi^\beta\big)V^c \\
&+ \varepsilon^A\big(\mathcal{D}_A\varepsilon_{\alpha\beta}\big)\psi^\alpha\psi^\beta - 2\varepsilon^\alpha\big(\mathcal{D}\varepsilon_{\alpha\beta}\big)\psi^\beta + 2\varepsilon_{\alpha\gamma}\big(\rho^\alpha_{Ab}\varepsilon^AV^b + \rho^\alpha_{A\beta}\varepsilon^A\psi^\beta\big)\psi^\gamma - 2\varepsilon_{\alpha\beta}\epsilon^\alpha\rho^\beta
\end{aligned} \tag{3.9}$$

(ε_{ab} here not to be confused with the Lorentz rotation parameter of Section 2). The Lie derivative along $\mathbf{t}$, the dual of the 3-form A, is a particular case of the extended Lie derivatives along p-forms B^i (i being a G-representation index) introduced in [6,2], the fields in this general setting being the G Lie algebra 1-forms μ^A supplemented by the p-form B^i. The extended Lie derivative is given by

$$\ell_{\varepsilon^i\mathbf{t}_i} \equiv i_{\varepsilon^i\mathbf{t}_i}d + d\,i_{\varepsilon^i\mathbf{t}_i}, \tag{3.10}$$

the contraction operator $i_{\varepsilon^i\mathbf{t}_i}$ being defined by its action on a generic form $\omega = \omega_{i_1\cdots i_nA_1\cdots A_m}B^{i_1}\wedge\cdots B^{i_n}\wedge\mu^{A_1}\wedge\cdots\mu^{A_m}$ as

$$i_{\varepsilon^j\mathbf{t}_j}\omega = n\,\varepsilon^j\omega_{ji_2\cdots i_nA_1\cdots A_m}B^{i_2}\wedge\cdots B^{i_n}\wedge\mu^{A_1}\wedge\cdots\mu^{A_m}, \tag{3.11}$$

where ε^j is a $(p-1)$-form. Thus the contraction operator still maps p-forms into $(p-1)$-forms. Note that (i) $i_{\varepsilon^j\mathbf{t}_j}$ vanishes on forms that do not contain at least one factor B^i; (ii) the extended Lie derivative commutes with d and satisfies the Leibnitz rule.

Returning to the FDA of $D = 11$ supergravity, since A is a 3-form in the identity representation of the superPoincaré Lie algebra, parameters in the extended Lie derivative (along the dual $\mathbf{t}$ of A) are 2-forms carrying no representation index, and are explicitly given for the algebra of FDA diffeomorphisms in (3.9) and (3.8).

The action of the extended Lie derivative on the basic fields is simply as follows:

$$\ell_{\varepsilon\mathbf{t}}\mu^A = 0, \quad \ell_{\varepsilon\mathbf{t}}A = d\varepsilon$$

with $\mu^A = V^a, \omega^{ab}, \psi$. Using these rules together with the variations (2.11)–(2.14) (generated by the usual Lie derivative) leads to the diffeomorphism algebra given in (3.5)–(3.7). As discussed in [2] for the general case, the algebra of FDA diffeomorphisms closes provided that the FDA Bianchi identities hold. Therefore, if we use in (3.5) and (3.6) the solutions (2.3)–(2.5) for the curvatures, the algebra (3.5)–(3.7) closes on the $D = 11$ field equations (2.7)–(2.9).

Note that the commutator of two ordinary Lie derivatives, computed on the 3-form A, *does not close* on the usual (3.2) diffeomorphism algebra, but develops an extra piece, that is the second line in the "extended" diffeomorphism algebra of (3.5), containing the extended Lie derivative.

4 A new formulation of $D = 11$ supergravity

The idea is to reinterpret the extended Lie derivative $\ell_{\varepsilon\mathbf{t}}$ of the $D = 11$ FDA in terms of ordinary Lie derivatives along new tangent vectors $\mathbf{t}_{ab}$, $\mathbf{t}_{a\beta}$, $\mathbf{t}_{\alpha\beta}$, via the following identification:

$$\ell_{\varepsilon\mathbf{t}} = \ell_{\varepsilon^{ab}V^aV^b\mathbf{t}+\varepsilon^{a\beta}V^a\psi^\beta\mathbf{t}+\varepsilon^{\alpha\beta}\psi^\alpha\psi^\beta\mathbf{t}} \equiv \ell_{\varepsilon^{ab}\mathbf{t}_{ab}+\varepsilon^{a\beta}\mathbf{t}_{a\beta}+\varepsilon^{\alpha\beta}\mathbf{t}_{\alpha\beta}}.$$

The 0-forms ε^{ab}, $\varepsilon^{a\beta}$, $\varepsilon^{\alpha\beta}$, that is the coefficients of the expansion on the superspace basis of the 2-form parameter ε in the extended Lie derivative, are reinterpreted as parameters of ordinary Lie derivatives along the new tangent vectors $\mathbf{t}_{ab} = V^aV^b\mathbf{t}$, $\mathbf{t}_{a\beta} = V^a\psi^\beta\mathbf{t}$, $\mathbf{t}_{\alpha\beta} = \psi^\alpha\psi^\beta\mathbf{t}$.

This is possible when the set

$$\ell_{\varepsilon^A\mathbf{t}_A},\ \ell_{\varepsilon^{ab}\mathbf{t}_{ab}},\ \ell_{\varepsilon^{a\beta}\mathbf{t}_{a\beta}}, \ell_{\varepsilon^{a\beta}\mathbf{t}_{a\beta}} \tag{4.1}$$

closes on a diffeomorphism algebra similar to the one in (3.2), that is a diffeomorphism algebra of an ordinary group manifold. If this is the case, the new operators can be seen as *bona fide* Lie derivatives, generating ordinary diffeomorphisms along new directions.

Now (3.5) indeed is of the form (3.2), and the extra piece on the right-hand side simply defines new curvatures and structure constants in $\mathcal{R}^C_{AB}$ of (3.3). However, the other commutations (3.6) contain terms with exterior (covariant) derivatives of the parameters ε^{ab}, $\varepsilon^{a\beta}$, $\varepsilon^{\alpha\beta}$, not amenable to the form of the derivative terms in (3.2). These parameters (associated with the new directions) will therefore be taken to be covariantly constant in the arguments that follow. This is the price to pay if we want to interpret the $D = 11$ diffeomorphism algebra (3.5)-(3.7) as an algebra of ordinary Lie derivatives.

In other words, the algebra (3.5)–(3.7) with $\mathcal{D}\varepsilon_{cd} = \mathcal{D}\varepsilon_{c\alpha} = \mathcal{D}\varepsilon_{\alpha\beta} = 0$ can be considered the diffeomorphism algebra of a manifold, whose vielbeins are V^a, ω^{ab}, ψ, B^{ab}, $B^{\alpha\beta}$ and $\eta^{a\beta}$ (the last three being the vielbeins dual to the tangent vectors $\mathbf{t}_{ab}$, $\mathbf{t}_{a\beta}$, $\mathbf{t}_{\alpha\beta}$).

Comparing (3.5)-(3.7) with the general form (3.2), we deduce the new curvature components that satisfy the Bianchi identities (3.4) implied by (3.2): R^{ab}, ρ, R^a remain unchanged as given in (2.3)–(2.5), whereas the solutions for the curvatures T^{ab}, $T^{\alpha\beta}$, $\Sigma^{a\beta}$ of the new potentials B^{ab}, $B^{\alpha\beta}$, $\eta^{a\beta}$ are

$$T^{ab} = 24\,F^{ab}{}_{cd}\,V^cV^d - \frac{3}{4}\rho^\delta_{[ab}\eta_{c]\delta}V^c - \frac{1}{4}\rho^\delta_{\alpha[a}\eta_{b]\delta}\psi^\alpha - \frac{1}{2}\rho^\gamma_{ab}\psi^\delta B_{\gamma\delta}, \tag{4.2}$$

$$T^{\alpha\beta} = \frac{1}{4}\,\rho^\gamma_{a\{\alpha}B_{\beta\}\gamma}V^a, \tag{4.3}$$

$$\Sigma^{b\beta} = -i\rho^{\{\alpha}_{ab}B^{\beta\}\alpha}V^a - \frac{i}{2}\rho^\delta_{\beta[a}\eta^\delta_{b]}V^a - \frac{i}{2}\rho^\gamma_{b\{\alpha}B_{\beta\}\gamma}\psi^\alpha \tag{4.4}$$

(all contractions Lorentz invariant, position of indices not relevant).

Using the standard formula for the variation of group manifold vielbeins μ^A under diffeomorphisms:

$$\delta\mu^A = d\varepsilon^A - 2\mathcal{R}^A{}_{BC}\,\mu^B\varepsilon^C \tag{4.5}$$

we find the variations of the potentials B^{ab}, $B^{\alpha\beta}$ and $\eta^{a\beta}$:

$$\begin{aligned}\delta B^{ab} &= \bar\psi\Gamma^{ab}\epsilon - 48F^{abcd}V^c\varepsilon^d + \frac{3}{2}\rho^\delta_{[ab}\big(\varepsilon^\delta_{c]}V^c - \varepsilon^c\eta^\delta_{c]}\big)\\ &\quad + \frac{1}{2}\rho^\delta_{\alpha[a}\big(\varepsilon^\delta_{b]}\psi^\alpha + \varepsilon^\alpha\eta^\delta_{b]}\big) + \rho^\gamma_{ab}\big(\varepsilon^\delta B^{\gamma\delta} - \varepsilon^{\gamma\delta}\psi^\delta\big),\end{aligned} \tag{4.6}$$

$$\begin{aligned}\delta B^{\alpha\beta} &= 2\big(C\Gamma_{ab}\big)^{\alpha\beta}V^a\varepsilon^b - \frac{3i}{2}\big(C\Gamma_c\big)_{\{\alpha\beta}\big(\varepsilon^\gamma\eta^c_{\gamma\}} + \varepsilon^c_{\gamma\}}\psi^\gamma\big)\\ &\quad + i\big(C\Gamma_c\big)^{\alpha\beta}\big(\varepsilon^dB^{cd} - \varepsilon^{cd}V^d\big) - \frac{1}{2}\rho^\gamma_{a\{\alpha}\big(\varepsilon^\gamma{}_{\beta\}}V^a - \varepsilon^aB^\gamma{}_{\beta\}}\big),\end{aligned} \tag{4.7}$$

$$\begin{aligned}\delta\eta^{b\beta} &= 2\big(C\Gamma_{ab}\big)_{\delta\beta}\big(V^a\varepsilon^\delta - \psi^\delta\varepsilon^a\big) + 2i\big(C\Gamma^c\big)_{\alpha\beta}\big(\psi^\alpha\varepsilon^{bc} - B^{bc}\varepsilon^\alpha\big)\\ &\quad + 2i\rho^{\{\gamma}_{ab}\big(\varepsilon^{\beta\}\gamma}V^a - \varepsilon^aB^{\beta\}\gamma}\big) + i\rho^\delta_{\beta[a}\big(\varepsilon^\delta_{b]}V^a - \varepsilon^a\eta^\delta_{b]}\big) + i\rho^\gamma_{b\{\alpha}\big(\varepsilon_{\beta\}\gamma}\psi^\alpha - \varepsilon^\alpha B_{\beta\}\gamma}\big)\end{aligned} \tag{4.8}$$

having used $\mathcal{D}\varepsilon^{ab} = \mathcal{D}\varepsilon^{\alpha\beta} = \mathcal{D}\varepsilon^{b\beta} = 0$ in (4.5). The variations of V^a, ω^{ab}, ψ are unchanged and given in (2.11)–(2.13).

In summary: the new formulation of $D = 11$ supergravity proposed here contains the fields

$$V^a,\ \omega^{ab},\ \psi,\ B^{ab},\ B^{\alpha\beta},\ \eta^{a\beta}. \tag{4.9}$$

The transformation rules of these fields, given in (2.11)–(2.13) and (4.6)–(4.8) close under the same conditions necessary for the closure of the algebra (3.5)–(3.7), since it is just a reformulation of this algebra. The only “spurious” element in this reformulation is the fact that some of the parameters (i.e. ε^{ab}, $\varepsilon^{a\beta}$, $\varepsilon^{\alpha\beta}$) must be taken to be covariantly constant. Even in this case the algebra (3.5)–(3.7) closes only provided that the Bianchi identities (2.2) hold, which implies the $D = 11$ field equations (2.7)–(2.9). Thus, the closure of the transformation rules on the fields (4.9) requires the $D = 11$ field equations, a situation analogous to the one in type IIB supergravity [12, 13].

Finally, we can relate the covariant curl $\mathcal{D}_{[\mu} B^{ab}_{\nu]}$ to the curl of the 3-form $F_{\mu\nu ab}$: indeed the curvature of B^{ab}, according to the definition (3.1), reads

$$T^{ab} = \mathcal{D}B^{ab} - \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi \tag{4.10}$$

(see also [2]) where we have used the structure constants deduced by recasting the diffeomorphism algebra (3.5)–(3.7) in the form (3.2). Comparing the $V^c V^d$ components of the definition of T^{ab} (4.10) and its solution (4.2) yields

$$\left(\mathcal{D}B^{ab}\right)_{cd} = 24F^{ab}{}_{cd},$$

$(\mathcal{D}B^{ab})_{cd}$ being the $V^c V^d$ components of the two-form $\mathcal{D}B^{ab}$. The other fields $B^{\alpha\beta}$, $\eta^{a\beta}$ are auxiliary: their curvature solutions, given respectively in (4.3) and (4.4), have no spacetime (VV)-components, and the external components only contain the gravitino curvature.

In conclusion, we have found a set of transformation rules on the dynamical fields V^a, ω^{ab}, ψ, B^{ab} and auxiliary fields $B^{\alpha\beta}$, $\eta^{a\beta}$ that close on the (usual) field equations of $D = 11$ supergravity, F^{abcd} being now related to the curl of B^{ab}.

Acknowledgments This work is partially supported by the European Community’s Human Potential Program under contract MRTN-CT-2004-005104 and by the Italian MIUR under contract PRIN-2003023852 on the project “Superstrings, branes and fundamental interactions”.

References

[1] E. Bergshoeff and E. Sezgin, *New spacetime superalgebras and their Kac-Moody extension*, Phys. Lett. B, 232 (1989), 96–103.

[2] L. Castellani, *Lie derivatives along antisymmetric tensors, and the M-theory superalgebra*. Preprint, hep-th/0508213.

[3] L. Castellani, *Group-geometric methods in supergravity and superstring theories*, Internat. J. Modern Phys. A, 7 (1992), 1583–1625.

[4] L. Castellani, R. D’Auria, and P. Fré, *Supergravity and Superstrings: A Geometric Perspective*, World Scientific, Teaneck, NJ, 1991.

[5] L. Castellani, P. Fré, F. Giani, K. Pilch, and P. van Nieuwenhuizen, *Gauging of* $d = 11$ *supergravity?*, Ann. Physics, 146 (1983), 35–77.

[6] L. Castellani and A. Perotto, *Free differential algebras: their use in field theory and dual formulation*, Lett. Math. Phys., 38 (1996), 321–330.

[7] E. Cremmer and B. Julia, *The* $\mathrm{SO}(8)$ *supergravity*, Nuclear Phys. B, 159 (1979), 141–212.

[8] E. Cremmer, B. Julia, and J. Scherk, *Supergravity in theory in* 11 *dimensions*, Phys. Lett. B, 76 (1978), 409–412.

[9] A. D’Adda, R. D’Auria, P. Fré, and T. Regge, *Geometrical formulation of supergravity theories on orthosymplectic supergroup manifolds*, Riv. Nuovo Cimento (3), 3 (1980), 81.

[10] R. D’Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, Nuclear Phys. B, 201 (1982), 101–140.

[11] R. D’Auria, P. Fré, and T. Regge, *Graded-Lie-algebra cohomology and supergravity*, Riv. Nuovo Cimento (3), 3 (1980), 1–37.

[12] M. B. Green and J. H. Schwarz, *Supersymmetrical string theories*, Phys. Lett. B, 109 (1982), 444–448.

[13] M. B. Green and J. H. Schwarz, *Extended supergravity in ten dimensions*, Phys. Lett. B, 122 (1983), 143–147.

[14] Y. Ne’eman and T. Regge, *Gauge theory of gravity and supergravity on a group manifold*, Riv. Nuovo Cimento (3), 1 (1978), 1–43.

[15] Y. Ne’eman and T. Regge, *Gravity and supergravity as gauge theories on a group manifold*, Phys. Lett. B, 74 (1978), 54–56.

[16] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math., (1977), 269–331.

[17] P. K. Townsend, *Four lectures on M-theory*, in Proceedings of “High Energy Physics and Cosmology”, Trieste, Italy, 1996, 385–438. hep-th/9612121.

[18] P. K. Townsend, *M-theory from its superalgebra*, in Proceedings of “Strings, Branes and Dualities”, Cargese, France, 1997, 141–177. hep-th/9712004.

[19] P. van Nieuwenhuizen, *Free graded differential superalgebras*, in Group Theoretical Methods in Physics, M. Serdaroğlu and E. Ínönü, eds., vol. 180 of Lecture Notes in Physics, Springer-Verlag, Berlin, 1983, 228–247.