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Extending a Chebyshev Subspace to a Weak Chebyshev Subspace of Higher Dimension and Related Results

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Abstract

Let G={g1,…,gn} be an n-dimensional Chebyshev sub-space of C[a, b] such that $1 \notin G$ and $U=(u_0, u_1,\dots,u_n)$ be an (n+1)-dimensional subspace of C[a, b] where $u_0=1$, $u_i=g_i$, i=1….. n. Under certain restriction on *G*, we proved that *U* is a Chebyshev subspace if and only if it is a Weak Chebyshev subspace. In addition, some other related results are established.

Keywords: Chebyshev system; Weak Chebyshev system

Introduction

The finite set of functions $\{g_1,\dots,g_n\}$ and C[a, b] is called a Chebyshev system on [a, b] if it is linearly independent and $D\begin{pmatrix} g_1,\dots, & g_n \\ x_1,\dots, & x_n \end{pmatrix}=Det\left[g_i(x_j)\right]\geq 0, i,j=1,\dots,n$ for all $\{x_j\}_{j=1}^n$ such that $a \le x_1<x_2<\dots<x_n \le b$, and the n-dimensional subspace $G=[g_1,\dots,g_n]$ of C[a, b] will be called a Chebyshev subspace [1-4]. Using the continuity of the determinant, it can be shown that the sign of the determinant is constant [5], so we will assume that the sign of the determinant is always positive through this paper (replace g_1 by $-g_1$ if necessary). And the finite set of functions $\{g_1,\dots,g_n\}$ and C[a, b] is called a Weak Chebyshev system on [a, b] if it is linearly independent and $D\begin{pmatrix} g_1,\dots, & g_n \\ x_1,\dots, & x_n \end{pmatrix}=Det\left[g_i(x_j)\right]\geq 0, i,j=1,\dots,n$ for all $\{x_j\}_{j=1}^n$ such that $a \le x_1 < x_2 <\dots< x_n \le b$ and the n-dimensional subspace $G=\{g_1,\dots,g_n\}$ of C[a, b] will be called a weak Chebyshev subspace, C[a, b] is the space of all real-valued continuous functions. Extending an n-dimensional Chebyshev subspace which does not contain a constant function to an (n+1)-dimensional Chebyshev subspace containing a constant function was investigated [4]. In what follows is the statement of the problem considered in this paper: Let $G=[g_1,\dots,g_n]$ be a Chebyshev subspace of C[a, b] such that $1\notin G$ and $U=\{u_0,u_1,\dots,u_n\}$ be an (n+1)-dimensional subspace of C[a, b] where $u_0=1$, $u_i=g_i$, i=1,…,n [6-8]. Our main purpose is to prove that, under certain restriction on G, U is a Chebyshev subspace of C[a, b] if and only if it is a Weak Chebyshev subspace of C[a, b]. An example illustrating that the preceding assertion is not true in general is presented and some related results are given at the end of the last section.

Preliminary

We start this section by the following well known theorem [3,5].

Theorem

For an n-dimensional subspace G of C[a, b], the following statements are equivalent.

(i) G is a Chebyshev subspace.

(ii) Every nontrivial function $g \in G$ has at most n-1 distinct zeros in [a, b].

(iii) For all points $a=t_0 \le t_1 < \dots < t_{n-1} \le t_n=b$, there exists a function $g \in G$ such that

$g(t)=0, t \in \{t_1,\dots,t_{n-1}\}$

$g(t)\neq 0, t \notin \{t_1,\dots,t_{n-1}\}$

$(-1)^I g(t)>0, t \in(t_{i-1}, t_i)$, i=1,…..,n

We need the following definitions:

Definition 1: Let *U* be a subspace of C[a, b], $x \in [a, b]$ and $f \in U$ such that f(x)=0. We call x an essential zero of *f* with respect to *U*, if and only if there is a $g \in U$ with $g(x) \neq 0$.

If no confusion arises, the term "with respect to *U* "will be omitted.

Definition 2: Let $f \in C[a, b]$ and $a \le t_1<,\dots,< t_n \le b$ be zeros of *f*. we say that these zeros are separated if and only if there are $s_1,\dots,s_{n-1}$ in [a, b] with

$t_i < s_i < t_{i+1}$

Such that

$f(s_i) \neq 0$, i=1,……n-1.

The following theorem is a version of theorem 1 of Stockenberg [6].

Theorem

Let **G** be an n-dimensional Weak Chebyshev subspace of C[a, b]. Then the following statements hold.

1. If there is a $g \in G$ with n separated, essential zeros $a \le t_1 <\dots< t_n \le b$, then g(t)=0 for all t with $t \le t_1 \ge t_n$.

2. No $g \in G$ has more than n separated, essential zeros.

The Main Result

We start this section with the following lemma.

Lemma

Let $G=\{g_1,\dots,g_n\}$ be an n –dimensional Chebyshev subspace of

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C[a, b] such that 1 ∉ and $U=\{u_0,u_1,\ldots,u_n\}$ be an (n+1)-dimensional subspace of C[a,b] where $u_0=1$, $u_i=g_i$, I=1,…..,n. If there are two non-trivial functions h, k∈ U and a set of n points $\{x_j\}_{j=1}^{n}$ with

$a \le x_1 < x_2 < \ldots < x_n \le b$

such that

$h(x_i)=k(x_i)=0, \quad i=1,\ldots,n,$

then there is a nonzero constant λ such that h(x) – λk(x) for every x∈ [a, b].

Proof: Write $h=a_0+\sum_{i=1}^{n} aigi$, if $a_0=0$ then h ∈ G and from theorem (1) h(x)=0 for every x ∈[a, b], so $a_0 \neq 0$ then h (x_i)=0,i=1,….,n, where

$$\bar{h}=\frac{I}{a_0}h=1+\sum_{i=1}^{n}\frac{a_i}{a_0}gi,$$

Similarly, if $\bar{k}=b_o+\sum_{b_0}^{n} b_i g_i$, then $b_0 \neq 0$ and $\bar{k}$ (x_i)=0, I=1, …,n, where

$$\bar{k}=\frac{1}{b_o}k=1+\sum_{b_0}^{n}\frac{b_i}{b_0}g_i$$

Now let $f=\bar{h}-\bar{k}$, then f is an element of the n-dimensional Chebyshev subspace G with $f(x_i)$=0, i=1,….,n, so $f_{\equiv 0}$ and $\bar{h}=\bar{k}$, taking $\lambda=\frac{a_0}{b_0}$, we have λ ≠ 0 and h(x)=λ k(x) for every x ∈ [a, b].

***Assumption A*:** We say that the subspace G of C[a, b] satisfies assumption A if for each f∈ G such that f(x)=f(y) for some x, y ∈[a, b] with x< y there is appoint z, x<z<y such v that f (z) ≠ f (x).

Lemma

Let $G=(g_1,\ldots,g_n)$ be an n-dimensional Chebyshev subspace of C[a, b] such that 1∈G and $U=(u_0,u_1,\ldots,u_n)$ be an (**n**+1)-dimensional subspace of C[a, b] where $u_0=1$, $u_i=g_i$, i=1,…, n. If **G** satisfies Assumption A, then the zeros of each nontrivial function h ∈ U are separated and essential.

Proof: Let h be a nontrivial element of U such that h(x)=h(y)=0 for some x, y with a ≤ x < y ≤ b. If h ∈ G, then n ≤ 3, for otherwise h ≡ 0, and since G is an n-dimensional Chebyshev subspace of C[a, b], there is a point z ∈ (x, y) such that h(z) ≠ 0. If h∉ G, then h=α+g, where α ≠ 0 and g∈G, hence g(x)=g(y)=-α, but G satisfies Assumption A, so there is a point z∈ (x, y) such that g(z) ≠-α that is h (z)≠ 0, this shows that the zeros of h are separated. For the second part of the assertion of the lemma, it is clear that each zero of any nontrivial element of u is an essential zero, that is because 1∈ U.

***Remark 1*:** Note that if 1 ∉ U and 0 ∉ f ∈U, f(x)=0, then since G is a Chebyshev space, x is an essential zero for f. Indeed, there is an element g ∈G such that g(x) ≠ 0.

Theorem

Let $G=(g_1,\ldots g_n)$ be an n-dimensional Chebyshev subspace of C[a, b] such that 1 ∈ G and $U=(u_0, u_1,\ldots, u_n)$ be an (n+1)-dimensional subspace of C[a, b] where $u_0=1$, $u_i=g_i$, i=1,…., n. If G satisfies Assumption A, then U is a Chebyshev subspace of C[a, b] if and only if it is a Weak Chebyshev subspace of C[a, b].

Proof: One direction is trivial.

For the other direction, suppose $U=(u_0, u_1, \ldots u_n)$ is an (n+1) – dimensional Weak Chesbyshev subspace of C [a, b] where $u_0=1$, $u_i=g_i$, I=1, n and $G=(g_1,\ldots g_n)$ is an n-dimensional Chesbyshev subspace of C[a, b] satisfying Assumption A. Let ū be a nontrivial element of U such that we

$\bar{u}(x_i)=0,\ldots i=1,\ldots d,$

$a \le x_1 <. x_{n+1} \le b,$

If d>n+1, then by lemma (2) together with theorem (2) we must have $\bar{u}$ ≡ 0, so d ≥ n+1, if d≤ n, then is nothing to prove, so to this end, we will assume that d=n+1 and

$\bar{u}(x_i)=0$, i=1,….n+1,

$a \le x_1 < \ldots < x_{n+1} \le b,$

again from lemma (2) and theorem (2) we must have $a=x_1$ or $x_{n+1}=b$ and ū(x) ≠ 0 for all $x\in[a,b]\setminus\{x_j\}_{j=1}^{n}$ Writing $\hat{u}=\alpha_0+\sum_{i=1}^{n}\bar{\alpha}_i g_i$, then $\alpha_0 \neq 0$ that is because G is an n-dimensional Chebyshev subspace of C[a, b].

Taking $u=\frac{1}{\alpha_0}\bar{u}$, then

$$u=1+\sum_{i=1}^{n}\alpha_i g_i, \text{where}\, \alpha_i=\frac{\bar{\alpha}}{\alpha_0} i=1,\ldots.n$$

$u(x_i)$=0, I=1,…, n+1

and

u (x) ≠ 0 for all $x\in[a,b]\setminus\{x_j\}_{j=1}^{n}$

The rest of the proof is divided into several cases.

***Case A*:** $a=x_1$ and $x_{1+1}=b$.

Since G is an n-dimensional Chebyshev subspace of C [a, b], then for any point q ∈ (x_n, b) there is a function $g=\sum_{i=1}^{n}\beta_i g_i$. ∈ G such that

g (y_i)=1, I=1, ….., n, where $y_i=x_{i,\, i=1,\ldots\ldots n-1}$ and $y_n=q$.

Taking

$$v=1-g=1-\sum_{i=1}^{n}\beta_i g_i,$$

Then v is a nontrivial element of U with

v (y_i)=0, i=1,…..n,

$a=y_1 < \ldots, < y_n < b,$

and if there is a point $t\in[a,b]\setminus\{y_i\}_{i=1}^{n}$ such that v(t)=0, then by theorem (2) we must have t=b, hence u and v are two dimensional elements of U such that

u (x_{n+1})=v(x_{n+1})=0,

u (x_i)=v(x_i)=0, i=1, ….,n-1

And by lemma (1) there is a non-zero constant λ such that u=λ v, this implies that

u (t_i)=0, I=1, ….n+2

Where $t_i=x_i$, i=1,….n, $t_{n+1}=y_n$

And $t_{n+2}=x_{n+1}=b$

This means that u has at least n+2 separated zeros in [a, b] which implies that u=v ≡ 0 contradicting the fact u and v are nontrivial elements of U, hence $v(x)^{1}0$, $x\in[a,b]\setminus\{y_i\}_{i=1}^{n}$. It is clear that

$u(x) \neq 0$, $x \in (x_n, b)$, $u(x_n)=u(b)=0$

and

$v(y_n)=0$, $y_n \in (x_n, b)$, $v(t) \neq 0$ for all $t \in [x_n, b] \setminus \{y_n\}$,

and if $x \in [x_n, y_n]$, $y \in (y_n, b)$ then sign $v(x)$=-sign $v(y)$, subsequently,

We treat four different subcases.

Case A1: $u(x) < 0$ for all $x \in (x_n, b)$ and $v(x) > 0$ for all $x \in [x_n, y_n]$,

then $v(x) < 0$ for all $x \in (y_n, b]$, taking $w=u-v$, we have

$w(x_n)=-v(x_n) < 0$ and $w(y_n)=u(y_n) > 0$

by the continuity of w, there is a point $s \in (x_n, y_n)$ such that $w(s)=0$,

hence we have:

$w(z_i)=0$, $i=1,\ldots.n$, where $z_i=x_i$, $I=1,\ldots.n-1$ and $z_n=s$.

But w belongs to the n-dimensional Chebyshev subspace G of C [a,b].

Hence $w \equiv 0$ and it follows that u=v and

$u(t_i)=0$, $I=1, \ldots n+2$

Where

$t_i=x_i$, $I=1, \ldots n$,

$t_{n+1}=y_n$ and $t_{n+2}=x_{n+1}=b$

So u must be identically zero.

Case A2: $u(x) > 0$ for all $x \in (x_n, b)$ and $v(x) < b0$ for all $x \in [x_n, y_n]$, then $v(x) < 0$ for all $x \in [x_n, y_n]$,

then $v(x) > 0$ for all $x \in (y_n, b]$, again taking w=u-v, we have

$w(y_n)=u(y_n) > 0$ $w(b)=-v(b) < 0$,

and there is a point $s \in (y_n, b]$, such that $w(s)=0$, so w has at least n distinct zeros in [a,b]. A similar argument as in case A1 shows that u must be identically zero.

Case A3: $u(x) < 0$ for all $x \in (x_n, b)$ and $v(x) < 0$ for all $x \in [x_n, y_n)$,

then $v(x) > 0$ for all $x \in (y_n, b]$, taking w=u-v, we have

$w(y_n)=-v(x_n) > 0$ and $w(y_n)=u(y_n) < 0$,

and continuing exactly as in case A1, we conclude that u must be identically zero.

Case A4: $u(x) < 0$ for all $x \in (x_n, b)$ and $v(x) > 0$ for all $x \in [x_n, y_n)$,

then $v(x) < 0$ for all $x \in (y_n, b]$, taking w=u-v, we have

$w(y_n)=u(y_n) < 0$ and $w(b)=-v(b) > 0$ and an argument similar to that of case A2 shows be identically zero.

Case B: $a < x_1$ and $x_{n+1}=b$

As in case A, for any $q \in (x_n, b)$ there is a function $g=\sum_{i=1}^{n} \beta_i g_i . \in G$

Such that $g(y_i)=1$, $I=1,\ldots\ldots,n$, where

$y_i=x_i$, $i=1,\ldots n-1$ and $y_n=q$

Taking $v=1-g=1-\sum_{i=1}^{n} \beta_i g_i$,

Then v is a nontrivial element of U with

$v(y_i)=0$, $I=1, \ldots.n$,

$a < y_1 < \ldots\ldots < y_n < b$

and if there is a point $t \in [a,b]\{y_i\}_{i=1}^{n}$ such that $v(t)=0$,then by theorem (2) we must have t=b or t=a.

If t=b, then u and v are two nontrivial elements of U such that

$U(x_{n+1})=v(x_{n+1})=0$,

$u(x_i)=v(x_i)=0$, $I=1, \ldots n-1$

and by lemma 1 there is a nonzero constant λ such that u=λ v, this implies that

$u(t_i)=0$, $i=1,\ldots.n+2$

where $t_i=x_i$, $i=1, \ldots..n$,

$t_{n+1}=y_n$

and $t_{n+2}=x_{n+1}=b$

so u has at least n+2 separated zeros in[a, b] which implies that u=v $\equiv 0$ and this is a contradiction.

so $t \neq b$ and the situation becomes exactly as in case A, proceedings as in case A we conclude that u must be identically zero.

Case C: $a=x_1$ and $x_{n+1} < b$

The proof of this case requires that $n \geq 2$ and the proof for n=1 will be given in remark (2).

Now, for any point $p \in (a,x_1)$ there is a function $g=\sum_{i=1}^{n} \beta_i g_i \in G$ such that

$G(y_i)=1$, $i=1,\ldots., n$,

Where $y_1=p$ and $y_{i-1}=x_i$, $i=3,\ldots n+1$.

Taking

$v=1-g=1-\sum_{i=1}^{n} \beta_i g_i$,

Then v is a nontrivial element of U with

$v(y_i)=0$, $i=1,\ldots.,n$,

$a < y_1 < \ldots.. < y_n < b$,

and if there is a point $t \in [a,b]/\{y_i\}_{i=1}^{n}$ such that $v(t)=0$, then by therome (2)we must have t=a or t=b.

If t=a, then u and v are two nontrivial elements of U such that

$U(x_1)=v(x_1)=0$, $u(x_i)=0$, $i=3,\ldots.,n+1$.

A similar argument to that of the other cases leads to a contradiction.

So $t \neq a$ and on the interval $[a, x_2]$ we have

$u(a)=u(x_2)=0$, $u(x) \neq 0$, $x \in (a,x_2)$

and

$v(y_1)=0$, $y_1 \in (a,x_2)$,$v(t) \neq 0$ for every $t \in [a,x_2] \setminus \{y_1\}$.

If $x \in [a,y_1)$, $y \in (y_1,x_2]$ then sign $v(x)$=-sign $v(y)$, and as in the other cases we are presented with four different subcases. In each case, a similar argument to that of the cases in A can be used to show that the function u-v in G has at least n zeros which leads to the conclusion that u must be identically zero. Hence U is a Chebyshev subspace of C [a,b].

Remark 2: The following is the proof for theorem (3) when n=1 which is somehow more direct:

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Suppose g is a non constant continuous function on [a,b] such that G=[g] is a Chebyshev subspace of C[a,b] of dimension 1 satisfying Assumption A and U=[1,g] is a subspace of C[a,b] of dimension 2. If *U* is not a Chebyshev subspace, then there is a nontrivial element u=α +βg of *U* such that $u(x_1)=u(x_2)=0$ where $a \le x_1 < x_2 \le b$, clearly α ≠ 0 and β ≠ 0, so $g(x_1) = g(x_2) = c = -\frac{\alpha}{\beta} \neq 0$.

By lemma (2) there is a point y_1, $x_1 < y_1 < x_2$ such that $g(y_1)=d \neq c$. Taking $x_1=z_1$, $y_1=z_2$ and $x_2=z_3$, then

$$a \le z_1 < z_2 < z_3 \le b$$

$$D\begin{pmatrix} 1 & g \\ z_1 & z_2 \end{pmatrix} = \mathrm{Det}\begin{pmatrix} 1 & 1 \\ c & d \end{pmatrix} = d - c \neq 0$$

And

$$D\begin{pmatrix} 1 & g \\ z_2 & z_3 \end{pmatrix} = c - d \neq 0,$$

Hence

$$sign\, D\begin{pmatrix} 1 & g \\ z_1 & z_2 \end{pmatrix} = -sign\, D\begin{pmatrix} 1 & g \\ z_2 & z_3 \end{pmatrix}$$

This shows that *U* is not a weak Chebyshev subspace and the theorem is proved.

The followings example illustrates that theorem (3) is not true in general is proved.

Example 1

Let

$$g(\mathrm{x}) = \begin{pmatrix} 1 & 0 \le x \le 1 \\ x & 1 < x \le 2 \end{pmatrix}$$

G={g} is a Chebyshev Subspace of C[0,2] of dimension 1, if *U*=(1,g) and $\le x_1 < x_2 \le 2$, then

$$D\begin{pmatrix} 1 & g \\ x_1 & x_2 \end{pmatrix} = 0 \text{ if } x_2 \in [0,1]$$

And

$$D\begin{pmatrix} 1 & g \\ x_1 & x_2 \end{pmatrix} > 0 \text{ if } x_2 \in (1,2]$$

That is *U* is a 2-dimensional weak Chebyshev Subspace of C[0,2] but not a Chebyshev Subspace.

If H is n-dimensional subspace of C[a, b], then it is possible that H is a Chebyshev subspace on one of the intervals (a, b] or [a, b) but not on the closed interval [a,b] as illustrated in the following example.

Example 2

Let H=(sin x, cos x), it can be easily checked that H is a Chebyshev subspace of dimension 2 on each of the intervals (0,π] of dimension 2.

In next result we give a necessary and sufficient condition for an n-dimensional Chebyshev H on (a,b] or [a,b) to be a chebyshev subspace on the closed interval [a,b].

Theorem

Let H be an n-dimensional subspace of C [a, b] such that H is a Chebyshev subspace on (a,b] or on [a,b), then H is a Chebyshev subspace on [a, b] if and only if each function h_i, I=1,…n can have at most n-1 distinct zeros on [a,b] whenever $H=[h_1,\ldots h_n]$.

Proof: If $(h_1,\ldots h_n)$ is a basis of H such that for some $s \in \{1,\ldots n\}$,h_s has at least n zeros on [a, b],then clearly H is not a Chebyshev Subspace on [a,b].For the other direction, suppose H is Chebyshev subspace on.

I=(a, b] but not on [a, b], there is a non-trivial element u ∈ *U* such that

$u(z_1) \neq 0$, $u(z_i)=0$, i=2, …n.

Since H is a Chebyshev subspace on (a, b], there is a subset $E=\{h_1,\ldots h_n\}$ and H such that

$$h_i(z_j) = \delta_{ij} = \begin{cases} 1 & if\ i=j \\ 0 & \text{otherwise} \end{cases}$$

The elements of E are linearly independent and $H=\{h_1,\ldots h_2\}$,write

$$u = \sum_{i=1}^{n} a_i h_i, a_i \in \mathbb{R}\, i = 1,\ldots n,$$

Then

$0 \neq u(z_1)=a_1h_1(z_1)=a_1$,

$0=u(z_i)=a_ih_i(z_i)=a_i$, I=2, ……n,

Hence $h_1 = \frac{1}{a_1 u}$ which has n zeros on [a, b] and this is a contradiction.

Using a similar argument when *I*=[a, b) leads to a contradiction and the theorem is proved.

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