Folding of Digraphs
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Abstract
In this paper we introduced the definition of dibipartite graphs, complete dibipartite graphs and digraph folding, and then we proved that any dibipartite graph can be folded but the complete dibipartite graph can be folded to an arc. By using adjacency matrices we described the digraph folding.

Keywords: Digraphs; Dibipartite graphs; Complete dibipartite graphs; Folding of dibipartite graphs; Adjacency matrices

Introduction
1) A digraph D consists of a set of elements, called vertices and a list of ordered pairs of these elements, called arcs. The set of vertices is called the vertex set of D, denoted by V(D), and the list of arcs is called the arc list of D, denoted by A(D). If v and w are vertices of D, then an arc of the form vw is said to be directed from v to w [1].

2) A dibipartite graph is a digraph whose vertex set can be split into sets A and B in such a way that each arc (directed edge) of the digraph runs from a vertex in A to a vertex in B (or a vertex of B to a vertex of A). We can distinguish the vertices in A from those in B by drawing the former in black and the latter in white, so that each arc is incident from a black (or white) vertex to a white (or black) vertex (Figure 1).

3) A complete dibipartite graph is a dibipartite graph in which each black (or white) vertex is joined to each white (or black) vertex by exactly one arc. The complete dibipartite graph with r black vertices and s white vertices is denoted by K_{rs}. We call a complete dibipartite graph of the form K_{1,s} star sink or star source digraphs (Figure 2).

4) Let D_1 and D_2 be digraphs and f : D → D is a continuous function. Then f is called a digraph map if,
   a) For each vertex v ∈ V(D), f(v) is a vertex in V(D).
   b) For each arc e ∈ A(D), dim(f(e)) ≤ dim(e).

Folding of Dibipartite Graphs
Definition
Let D_1 and D_2 be simple digraphs, we call a digraph map f : D → D a digraph folding if f maps vertices to vertices and arcs to arcs, i.e., for each v ∈ V(D_1), f(v) ∈ V(D_2) and for each e ∈ A(D_1), f(e) ∈ A(D_2).

If the digraph contains loops, then the digraph folding must send loops to loops but of the same direction. We denote the set of digraph foldings between digraphs D_1 and D_2 by D_2(D_1, D_2) and the set of diagraph foldings of D into itself by D(D).

Theorem
Any dibipartite graph D can be folded.
Proof:
Let D be the digraph shown in Figure 3. Then the graph map f : D → D defined by f(v_1,...,v_4)=(v_3,v_2,v_3,v_4) and f(e_1,e_2,e_3,e_4,e_5)=(e_4,e_3,e_3,e_4,e_5) is a digraph folding. The image f(D) is shown in Figure 3. From now on the omitted vertices or arcs will be mapped into themselves.

Example
Let D be the digraph shown in Figure 3. Then the graph map f : D → D defined by f(v_1,...,v_4)=(v_3,v_2,v_3,v_4) and f(e_1,e_2,e_3,e_4,e_5)=(e_4,e_3,e_3,e_4,e_5) is a digraph folding. The image f(D) is shown in Figure 3. From now on the omitted vertices or arcs will be mapped into themselves.

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Example

Let $D_1$ be the dibipartite graph shown in Figure 4. A digraph folding $f : D \rightarrow D_f$ can be defined as follows $f(v_i,v_j,v_k, v_l) = (v_i,v_j,v_k)$ and $f(e_i,e_j,e_k,e_l) = (e_i,e_j,e_k,e_l)$. The image $D_f = D_2$ is shown in Figure 4.

Theorem

Any complete dibipartite graph $D$ can be folded to an arc.

Proof:

Let $D$ be a complete dibipartite graph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $B = \{v_i, v_{i+1}, \ldots, v_r\}$, such that each vertex of $A$ is joined to each vertex of $B$ by exactly one arc.

Thus

$$A(D) = \{(v_i,v_{i+1}), \ldots, (v_r,v_1), (v_r,v_{i+1}), \ldots, (v_1,v_{i+1}), \ldots, (v_r,v_1), \ldots, (v_r,v_{i+1}), \ldots, (v_1,v_{i+1})\}$$

Now let $f : D \rightarrow D$ be a digraph map defined by

$$f(v_i) = \begin{cases} v_i, & \text{if } k = 1, \ldots, r \\ v_{i+1}, & \text{if } k = r + 1, \ldots, r \end{cases}$$

Thus the image of any arc of $A(D)$ will be the arc $(v_i,v_{i+1})$. Of course, this map is a digraph folding.

Example

Consider the complete dibipartite graph $K_{2,4}$ shown in Figure 5. A digraph folding $f$ of $K_{2,4}$ into itself may be defined as follows $f(v_i,v_j,v_k) = (v_i,v_j,v_k)$ and $f(e_i,e_j) = (e_i,e_j)$, $i = 1, \ldots, 8$. This may be done by the composition of the two digraph folding $f_1$ and $f_2$ shown in Figure 5.
Then we can fold first $D_1$ by folding the multiple arc into itself to get the digraph $D_2$. In this case $M(D_1)$ is nothing but $M(D_2)$ after replacing the number 2 by the number 1. Then a digraph folding $g \in D(D_2)$ can be defined by using $M(D_2)$ by mapping the vertex $v_i$ to the vertex $v_j$ since the first and the third row of $M(D_2)$ have the same entries. Thus the arcs $(v_j,v_i)$ and $(v_j,v_i')$ will be mapped to the arcs $(v_j,v_i)$ and $(v_j,v_i')$ respectively, since the first and the third row are the same.

(b) Let $D$ be the digraph shown in Figure 8.

The adjacency matrix $M(D)$ is given by

$$M(D) = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

Then a digraph folding $g: D \rightarrow D$ can be defined by using $M(D)$ by mapping the vertices $v_j$, $v_k$, and $v_l$ to $v_3$, $v_4$, and $v_5$ respectively. Also the arcs $(v_j,v_i)$ and $(v_j,v_i')$ will be mapped to the arcs $(v_j,v_i)$ and $(v_j,v_i')$ respectively since $g(v_j)=v_3$, $g(v_k)=v_4$, and $g(v_l)=v_6$. Also the image of the arc $(v_j,v_i)$ is $(v_j,v_i)$ since the first and third columns are the same. Finally the image of the arc $(v_j,v_i)$ is $(v_j,v_i)$ since the second and fourth rows are the same, and so on. See the adjacency matrix $M(D)$.

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