

From Monge-Ampere-Boltzman to Euler Equations

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Abstract

This paper concerns with the convergence of the Monge-Ampere-Boltzman system to the incompressible Euler Equations in the quasi-neutral regime.

Keywords: Vlasov-Monge-Ampère-Boltzman system; Euler equations of the incompressible fluid

Introduction and Main Results

In this paper, we are interested in the hydrodynamical limit of the Boltzman-Monge-Ampere system (BMA)

$$f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_{f^\varepsilon} f^\varepsilon = Q(f^\varepsilon, f^\varepsilon) \quad (1.1)$$

$$\det(\mathbb{I}_d + \varepsilon^2 D^2 \phi^\varepsilon) = \rho^\varepsilon \quad (1.2)$$

where $f^\varepsilon(t, x, \xi) \geq 0$ the electronic density at time $t \geq 0$ point $x \in [0, 1]^d = \mathbb{T}^d$, and with a velocity $\xi \in \mathbb{R}^d$, and Id is the identity matrix defined by

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

The spatially periodic electric potential is coupled with ϕ^ε through the nonlinear Monge-Ampere equation (1.2). The quantities $\varepsilon > 0$ and

$$\rho^\varepsilon(t, x) \geq 0 \text{ denote respectively the vacuum electric permittivity and } \rho^\varepsilon(t, x) = \int_{\mathbb{R}^d} f^\varepsilon(t, x, \xi) d\xi. \quad (1.3)$$

$Q(f^\varepsilon, f^\varepsilon)$ is the Boltzman collision integral. This integral operates only on the ξ -argument of the distribution f^ε and is given by

$$Q(f^\varepsilon, f^\varepsilon)(t, x, \xi) = \iint_{S_+^{d-1} \times \mathbb{R}^d} \left((f_1^\varepsilon)' (f_1^\varepsilon) - f_1^\varepsilon f_1^\varepsilon \right) b(\xi - \xi_1, \sigma) d\sigma d\xi_1,$$

where the terms f_1^ε , $(f_1^\varepsilon)'$ and $(f_1^\varepsilon)'$ defines, respectively the values

$$f^\varepsilon(t, x, \xi_1), f^\varepsilon(t, x, \xi') \text{ and } f^\varepsilon(t, x, \xi_1')$$

with ξ' and ξ_1' given in terms of $\xi, \xi_1 \in \mathbb{R}^d$, and $\sigma \in S_+^{d-1} = \{ \sigma \in S^{d-1} / \sigma \cdot \xi \geq \sigma \cdot \xi_1 \}$ by

$$\xi' = \frac{\xi + \xi_1}{2} + \frac{\xi - \xi_1}{2} \sigma, \xi_1' = \frac{\xi + \xi_1}{2} - \frac{\xi - \xi_1}{2} \sigma.$$

The aim of this work is to investigate the hydrodynamic limit of the (BMA) system with optimal transport techniques.

Note that

$$\int_{\mathbb{R}^d} Q(f^\varepsilon, f^\varepsilon) d\xi = \int_{\mathbb{R}^d} \xi_i Q(f^\varepsilon, f^\varepsilon) d\xi = \int_{\mathbb{R}^d} |\xi|^2 Q(f^\varepsilon, f^\varepsilon) d\xi = 0, i = 1, 2, \dots, d.$$

The linearization of the determinant about the identity matrix gives

$$\det(\mathbb{I}_d + \varepsilon^2 D^2 \phi^\varepsilon) = 1 + \varepsilon^2 \Delta \phi^\varepsilon + O(\varepsilon^4).$$

Where \mathbb{I}_d represents the identity matrix.

So, one can see that the BMA system is considered as a fully non-linear version of the Vlasov Poisson-Boltzman (VPB) system defined by

$$\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_{f^\varepsilon} f^\varepsilon = Q(f^\varepsilon, f^\varepsilon) \quad (1.4)$$

$$\varepsilon^2 \Delta \phi^\varepsilon = \rho^\varepsilon - 1 \quad (1.5)$$

The analysis of the VPB system has been considered by many authors and many results can be found in a vast literature [1-10].

In Hsiao et al. [11] study the convergence of the VPB system to the weak solution of Vlasov-Monge-Ampère converge to a solution of the incompressible Euler equations when the parameter goes to 0, Brenier [12] and Loeper [13] for details. So, is a legitim question to look for the convergence of a weak solution of BMA (of course if such solution exists) to a solution of the incompressible Euler equations when the parameter goes to 0.

The study of the existence and uniqueness of solution to the BMA system seems a difficult matter. Here we assume the existence and uniqueness of smooth solution to the BMA and we just look to the asymptotic analysis of this system.

Definitions and Recalls

Definition 1

For a fixed bounded convex open set W of \mathbb{R}^d and a positive measure on \mathbb{R}^d of total mass $|W|$, we note by $F[\rho]$ the unique up to a constant convex function on W satisfying

$$\forall g \in C^0(\mathbb{R}^d) \cap L^1(d\rho), \int_{\mathbb{R}^d} g(x) d\rho(x) = \int_{\Omega} g(\Phi[\Omega, \rho])(y) dy.$$

Its Legendre-Fenchel transform denoted $\Psi[\Omega, \rho]$ the function satisfying (1.6) $\forall g \in C^0(\mathbb{R}^d) \cap L^1(\Omega, dy) \int_{\mathbb{R}^d} g(\nabla \Psi[\Omega, \rho]) d\rho(x) = \int_{\Omega} g(y) dy$. we may write Φ (resp. Ψ) instead of $\Phi[\Omega, \rho]$ (resp. $\Psi[\Omega, \rho]$) if no confusion is possible.

Remark 2

- Existence and uniqueness of Φ is due to the polar factorization theorem.
- By setting the change of variables $y = \nabla \Psi(x)$, we get $dy = \det D^2 \Psi(x) dx$. So (1.6) can be transformed to:

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$\forall g \in C^0(\mathbb{R}^d) \cap L^1(\Omega, dy)$,

$$\int_{\mathbb{R}^d} g(\nabla \Psi[\Omega, \rho]) \rho(x) dx = \int_{\Omega} g(\nabla \Psi(x)) \det D^2 \Psi(x) dx \tag{1.6}$$

Which is a weak version of the Monge–Ampere equation

$$\det D^2 \Psi(x) = \rho(x)$$

∇ maps $\text{supp}(\rho)$ in Ω

We assume that BMA system has a renormalized solution in the sense of DiePerna and Lions [3].

For simplicity, we set

$$\tilde{\varphi}[\rho] = \frac{|x|^2/2 - \Psi[\rho]}{\varepsilon}$$

and

$$\bar{\vartheta}[\rho] = \frac{\Phi[\rho] - |x|^2/2}{\varepsilon}$$

So that, $\nabla \phi[\rho] = \nabla \tilde{\varphi}[\rho] \circ \nabla \Phi[\rho]$, and the (BMA) $_{\rho}$ (ρ stands for periodic) system takes the following form

$$\begin{cases} \partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon} \nabla \tilde{\varphi}[\rho] \cdot \nabla_{\xi} f^\varepsilon = Q(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon(0, \cdot) = f_0^\varepsilon \end{cases} \tag{1.7}$$

The energy is given by

$$\begin{aligned} E(t) &= \frac{1}{2} \int f(t, x, \xi) |\xi|^2 dx d\xi + \frac{1}{2} \int |\nabla \phi|^2 dx \\ &= \frac{1}{2} \int f(t, x, \xi) |\xi|^2 dx d\xi + \frac{1}{2} \int \rho |\nabla \tilde{\varphi}|^2 dx \end{aligned}$$

It has been shown [2] that the energy is conserved.

The Euler equation for incompressible fluids reads

$$\begin{cases} \partial_t \bar{v} + \bar{v} \cdot \nabla \bar{v} = -\nabla p, \\ \nabla \cdot \bar{v} = 0. \end{cases} \tag{1.8}$$

One can find in Loeper [13] more details for this kind of equations.

Theorem 3

Let f^ε be a weak solution of (1.7) with finite energy, let $(t, x) \mapsto \bar{v}(t, x)$ be a smooth $C^2([0, T] \times T^d)$ solution of (1.8) for $t \in [0, T]$, and $p(t, x)$ the corresponding pressure, let

$$H\varepsilon(t) = \frac{1}{2} \int f(t, x, \xi) |\xi - \bar{v}(t, x)|^2 dx d\xi + \frac{1}{2} \int |\nabla \phi(t, x)|^2 dx$$

Then

$$H\varepsilon(t) \leq C \exp(Ct) (H\varepsilon(0) + \varepsilon^2), \forall t \in [0, T]$$

The constant C depends only on

$$\sup_{0 \leq s \leq T} \|\bar{v}(s, \cdot), p(s, \cdot), \partial_t p(s, \cdot), \nabla p(s, \cdot)\|_{W^{1, \infty}(T^d)}$$

Proof of the Theorem 3

Later, in the section, we need the following Lemma

Lemma 4:

Let $G: T^d \rightarrow \mathbb{R}$ be Lipschitz continuous such that $\int_{T^d} G = 0$ then for all $R > 0$ has one

$$\left| \int_{\rho} G = 0 \right| \leq \frac{1}{2} \|\nabla G\|_{L^\infty} \left(\frac{1}{R} \varepsilon^2 + RH\varepsilon \right)$$

We have

$$\begin{aligned} \frac{d}{dt} H\varepsilon &= \frac{d}{dt} \left[\frac{1}{2} \int f(t, x, \xi) (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi \right] \\ &= \int f(t, x, \xi) (\partial_t \bar{v} \cdot \bar{v} - \partial_t \bar{v} \cdot \xi) + \frac{1}{2} \int \partial_t f (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi \end{aligned}$$

From the BMA we have

$$\begin{aligned} \int \partial_t f (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi &= - \int \nabla_x \cdot (f(t, x, \xi) \xi) (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi \\ &\quad + \int \nabla_{\xi} \cdot \left(\frac{1}{\varepsilon} \nabla \tilde{\varphi} f(t, x, \xi) \xi \right) (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi \\ &\quad + \int Q(f^\varepsilon, f^\varepsilon) (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi \end{aligned}$$

The last term is equal to zero from the property of Boltzman Operator [1,3,5,7-9,11].

It follows by integrating by party that

$$\int \partial_t f (|\bar{v}|^2 - 2\xi \cdot \bar{v}) dx d\xi = 2 \int f(t, x, \xi) \xi \nabla \bar{v} \cdot (\bar{v} - \xi) dx d\xi + 2 \int f(t, x, \xi) \frac{1}{\varepsilon} \nabla \tilde{\varphi} \cdot \bar{v} \xi dx d\xi.$$

Thus

$$\begin{aligned} \frac{d}{dt} H\varepsilon &= \int f(t, x, \xi) (\partial_t \bar{v} \cdot \bar{v} - \partial_t \bar{v} \cdot \xi) dx d\xi + \int f(t, x, \xi) \xi \nabla \bar{v} \cdot (\bar{v} - \xi) dx d\xi + \int f(t, x, \xi) \frac{1}{\varepsilon} \nabla \tilde{\varphi} \cdot \bar{v} \xi dx d\xi \\ &= - \int f(t, x, \xi) (\bar{v} - \xi) \nabla \bar{v} \cdot (\bar{v} - \xi) dx d\xi + \int f(t, x, \xi) (\partial_t \bar{v} \cdot (\bar{v} - \xi)) dx d\xi + \int f(t, x, \xi) \bar{v} \nabla \bar{v} \cdot (\bar{v} - \xi) dx d\xi + \int f(t, x, \xi) \frac{1}{\varepsilon} \nabla \tilde{\varphi} \cdot \bar{v} \xi dx d\xi \\ &= - \int f(t, x, \xi) (\bar{v} - \xi) \nabla \bar{v} \cdot (\bar{v} - \xi) dx d\xi + \int f(t, x, \xi) (\bar{v} - \xi) (\partial_t \bar{v} + \bar{v} \nabla \bar{v}) dx d\xi + \int f(t, x, \xi) \frac{1}{\varepsilon} \nabla \tilde{\varphi} \cdot \bar{v} \xi dx d\xi \\ &= A + B + D \end{aligned}$$

Let us begin with the first term A. Use Holder inequality and that $|\nabla \bar{v}| \leq C$ to decompose

$$\begin{aligned} A &= \int f^{\frac{1}{2}}(t, x, \xi) (\bar{v} - \xi) \cdot f^{\frac{1}{2}}(t, x, \xi) \nabla \bar{v} \cdot (\xi - \bar{v}) dx d\xi \\ &\leq \left[\int f(t, x, \xi) |\xi - \bar{v}|^2 dx d\xi \right]^{\frac{1}{2}} \cdot \left[C \int f(t, x, \xi) |\xi - \bar{v}|^2 dx d\xi \right]^{\frac{1}{2}} \\ &\leq C H\varepsilon^{\frac{1}{2}} \cdot H\varepsilon^{\frac{1}{2}} = C H\varepsilon \end{aligned}$$

From the second term D, one has

$$D = \int \int f(t, x, \xi) \frac{1}{\varepsilon} \nabla \tilde{\varphi} \cdot \bar{v} \xi dx d\xi = \frac{1}{\varepsilon} \int \rho \bar{v} \nabla \tilde{\varphi} = \frac{1}{\varepsilon} \int \bar{v} \nabla \tilde{\varphi} d\rho$$

From the definition of Φ , we have

$$\begin{aligned} D &= \frac{1}{\varepsilon} \int \bar{v} \cdot (\nabla \Phi) \cdot \nabla \tilde{\varphi} (\nabla \Phi) dx = \frac{1}{\varepsilon} \int \bar{v} \cdot (x + \varepsilon \nabla \phi) \cdot \nabla \phi \\ &= \frac{1}{\varepsilon} \int \bar{v} \nabla \Phi + (\bar{v} \cdot (x + \varepsilon \nabla \phi) - \bar{v} \cdot x) \cdot \nabla \phi. \end{aligned}$$

Since \bar{v} is divergence free, once gets

$$B \leq 0 + C \cdot \int |\nabla \phi|^2 \leq C H\varepsilon$$

Consider now the last term D.

$$\begin{aligned} D &= - \int f(t, x, \xi) \bar{v} \cdot \nabla p dx d\xi + \frac{1}{2} \int f(t, x, \xi) \xi \cdot \nabla p dx d\xi \\ &= - \int f(t, x) \bar{v} \cdot \nabla p dx d\xi + \int f(t, x, \xi) \xi \cdot \nabla p dx d\xi \end{aligned}$$

But since \bar{v} is divergence free we have $\int \bar{v} \cdot \nabla p = 0$. Thus form Lemma 4

$(G = \bar{v} \cdot \nabla p)$, once has

$$- \int \rho \bar{v} \cdot \nabla p \leq C \cdot (\varepsilon^2 + H\varepsilon) \cdot$$

Since it costs no generality to suppose that for all $t \in [0, T], \int P(t, x) dx = 0$,

we get from the equation of conservation of mass

$$\int f(t, x, \xi) \xi \cdot \nabla p = - \int \nabla (f(t, x, \xi) \xi) p = \int \partial_i \rho_i p = \frac{d}{dt} \int \rho p - \int \rho \frac{\partial \rho}{\partial t}$$

By Lemma 4 and setting $Q(t) = - \int \rho p$ we can deduct that

$$\int f(t, x, \xi) \xi \cdot \nabla p \leq C(\varepsilon^2 + H_\varepsilon) - \frac{dQ}{dt}$$

Thus

$$D \leq C(H_\varepsilon + \varepsilon^2) - \frac{dQ}{dt}$$

We deduce then the following inequality

$$\frac{d}{dt}(H_\varepsilon + Q) \leq CH_\varepsilon + \theta(\varepsilon^2) \quad (2.1)$$

Still using 4,

$$|Q(t)| = \left| \int \rho p \right| \leq C\varepsilon^2 + \frac{1}{2}H_\varepsilon(t)$$

Thus

$$H_\varepsilon + Q \geq \frac{1}{2}H_\varepsilon(t) - C\varepsilon^2$$

So once can transform (2.1) as

$$\frac{d}{dt}(H_\varepsilon + Q) \leq C(H_\varepsilon + Q) + C\varepsilon^2$$

And by Gronwall's inequality [11] yields

$$H_\varepsilon(t) + Q(t) \leq (H_\varepsilon(0) + Q(0)) \exp(Ct)$$

Finally we conclude that

$$H_\varepsilon(t) \leq C(H_\varepsilon(0) + C\varepsilon^2) \exp(Ct)$$

Which achieves the proof of the theorem.

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