From the Kadomtsev-Petviashvili equation halfway to Ward's chiral model [1]

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Abstract

The "pseudodual" of Ward's modified chiral model is a dispersionless limit of the matrix Kadomtsev-Petviashvili (KP) equation. This relation allows to carry solution techniques from KP over to the former model. In particular, lump solutions of the $su(m)$ model with rather complex interaction patterns are reached in this way. We present a new example.

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1 Relation between KP and Ward's chiral model

Ward's chiral model in $2+1$ dimensions [15] (see [4] for further references) is given by

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y + [J^{-1}J_x, J^{-1}J_t] = 0 \tag{1.1}$$

for an $SU(m)$ matrix J, where $J_t = \partial J/\partial t$, etc. In terms of the new variables

$$x_1 := (t-x)/2, \quad x_2 := y, \quad x_3 := (t+x)/2 \tag{1.2}$$

this simplifies to $(J^{-1}J_{x_3})_{x_1} - (J^{-1}J_{x_2})_{x_2} = 0$, which is a straight reduction of Yang's equation, one of the potential forms of the self-dual Yang-Mills equation [12]. It extends to the hierarchy

$$(J^{-1}J_{x_{n+1}})_{x_m} - (J^{-1}J_{x_{m+1}})_{x_n} = 0, \quad m,n = 1,2,\ldots \tag{1.3}$$

The Ward equation is completely integrable[2] and admits soliton-like solutions, often called "lumps". It was shown numerically [13] and later analytically [16, 6, 3] that such lumps can interact in a nontrivial way, unlike usual solitons. In particular, they can scatter at right angles, a phenomenon sometimes referred to as "anomalous scattering".[3] Also the integrable KP equation, more precisely KP-I ("positive dispersion"), possesses lump solutions with anomalous scattering [5, 14, 1] (besides those with trivial scattering [9]). Introducing a potential ϕ for the real scalar function u via $u = \phi_x$, in terms of independent variables t_1, t_2 (spatial coordinates) and t_3 (time), the (potential) KP equation is given by

$$(4\,\phi_{t_3} - \phi_{t_1t_1t_1} - 6\,\phi_{t_1}\,\phi_{t_1})_{t_1} - 3\,\sigma^2\,\phi_{t_2t_2} = 0$$

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[2]In the sense of the inverse scattering method, the existence of a hierarchy, and various other characterisations of complete integrability. In the following "integrable" loosely refers to any of them.

[3]See also the references cited above for related work. Anomalous scattering has also been found in some related non-integrable systems, like sigma models, Yang-Mills-Higgs equation (monopoles) and the Abelian Higgs model or Ginzburg-Landau equation (vortices), see [10] for instance.

with $\sigma = i$ ($= \sqrt{-1}$) in case of KP-I and $\sigma = 1$ for KP-II. Could it be that this equation has a closer relation with the Ward equation? We are trying to compare an equation for a scalar with a matrix equation, and in [16] the appearance of nontrivial lump interactions in the Ward model had been attributed to the presence of the "internal degrees of freedom" of the latter. At first sight this does not match at all. However, the resolution lies in the fact that the KP equation possesses an integrable extension to a (complex) matrix version,

$$\Big(4\,\Phi_{t_3} - \Phi_{t_1t_1t_1} - 6\,\Phi_{t_1}\,Q\,\Phi_{t_1}\Big)_{t_1} - 3\,\sigma^2\,\Phi_{t_2t_2} = -6\,\sigma\,[\Phi_{t_1},\Phi_{t_2}]_Q$$

where we modified the product by introducing a constant $N \times M$ matrix Q, and the commutator is modified accordingly, so that

$$[\Phi_{t_1},\Phi_{t_2}]_Q = \Phi_{t_1}Q\Phi_{t_2} - \Phi_{t_2}Q\Phi_{t_1}$$

Here Φ is an $M \times N$ matrix. If $\mathrm{rank}(Q) = 1$, and thus $Q = VU^\dagger$ with vectors U and V, then any solution of this (potential) matrix KP equation determines a solution $\phi := U^\dagger\Phi V$ of the scalar KP equation.[4] More generally, this extends to the corresponding (potential) KP hierarchies.

Next we look for a relation between the matrix KP and the Ward equation. Indeed, there is a dispersionless (multiscaling) limit of the above "noncommutative" (i.e. matrix) KP equation,

$$\Phi_{x_1x_3} - \sigma^2\,\Phi_{x_2x_2} = -\sigma\,[\Phi_{x_1},\Phi_{x_2}]_Q \tag{1.4}$$

obtained by introducing $x_n = n\,\epsilon\,t_n$ with a parameter ϵ, and letting $\epsilon \to 0$ (assuming an appropriate dependence of the KP variable Φ on ϵ) [4]. If $\mathrm{rank}(Q) = m$, and thus $Q = VU^\dagger$ with an $M \times m$ matrix U and an $N \times m$ matrix V, then the $m \times m$ matrix $\varphi := \sigma\,U^\dagger\Phi V$ solves

$$\varphi_{x_1x_3} - \sigma^2\,\varphi_{x_2x_2} = -[\varphi_{x_1},\varphi_{x_2}] \tag{1.5}$$

if Φ solves (1.4).[5] This is a straight reduction of another potential form of the self-dual Yang-Mills equation [12]. In terms of the variables x, y, t, it becomes

$$\varphi_{tt} - \varphi_{xx} - \sigma^2\,\varphi_{yy} + [\varphi_t - \varphi_x, \varphi_y] = 0 \tag{1.6}$$

Now we note that the cases $\sigma = i$ and $\sigma = 1$ are related by exchanging x and t, hence they are equivalent.[6] We choose $\sigma = 1$ in the following. Then (1.5) extends to the hierarchy

$$\varphi_{x_mx_{n+1}} - \varphi_{x_{m+1}x_n} = [\varphi_{x_n},\varphi_{x_m}], \quad m,n = 1,2,\dots \tag{1.7}$$

The circle closes by observing that this is "pseudodual" to the hierarchy (1.3) of Ward's chiral model in the following sense. (1.7) is solved by

$$\varphi_{x_n} = -J^{-1}\,J_{x_{n+1}}, \quad n = 1,2,\dots \tag{1.8}$$

and the integrability condition of the latter system is the hierarchy (1.3). Rewriting (1.8) as $J_{x_{n+1}} = -J\,\varphi_{x_n}$, the integrability condition is the hierarchy (1.7). All this indeed connects the Ward model with the KP equation, but more closely with its matrix version, and not quite on a level which would allow a closer comparison of solutions. Note that the only nonlinearity that survives in the dispersionless limit is the commutator term, but this drops out in the "projection" to scalar KP. On the other hand, we established relations between *hierarchies*, which somewhat ties their solution structure together.[7]

In the Ward model, J has values in $SU(m)$, thus φ must have values in the Lie algebra $su(m)$, so has to be traceless and anti-Hermitian. Corresponding conditions have to be imposed on Φ to achieve this. The KP-I counterpart is the reality condition for the scalar ϕ.

[4] See e.g [11, 2] for related ideas.

[5] (1.5) and its four-dimensional extension are sometimes called "Leznov equation", see [7].

[6] Note also that this transformation leaves the conserved density (2.2) invariant.

[7] We note, however, that e.g the singular shock wave solutions of the dispersionless limit of the *scalar KdV* equation have little in common with KdV solitons.

2 Exact solutions of the pdCM hierarchy

Via the dispersionless limit, methods of constructing exact solutions can be transfered from the (matrix) KP hierarchy to the *pseudodual chiral model* (pdCM) hierarchy (1.7). From [4] we recall the following result. It determines in particular various classes of (multi-) lump solutions of the $su(m)$ pdCM hierarchy.

Theorem 2.1. *Let P, T be constant $N \times N$ matrices such that $T^\dagger = -T$ and $P^\dagger = TPT^{-1}$, and V a constant $N \times m$ matrix. Suppose there is a constant solution K of $[P, K] = -VV^\dagger T$ $(= Q)$ such that $K^\dagger = TKT^{-1}$. Let X be an $N \times N$ matrix solving $[X, P] = 0$, $X^\dagger = TXT^{-1}$ and $X_{x_{n+1}} = X_{x_1} P^n$, $n = 1, 2, \ldots$. Then $\varphi := -V^\dagger T(X - K)^{-1} V$ solves the $su(m)$ pdCM hierarchy.*

Example 2.1. Let $m = 2$, $N = 2$, and

$$P = \begin{pmatrix} p & 0 \\ 0 & p^* \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} f & 0 \\ 0 & f^* \end{pmatrix}, \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with complex parameters a, b, c, d, p and a function f (with complex conjugate f^*). Then $X_{x_{n+1}} = X_{x_1} P^n$ is satisfied if f is an arbitrary holomorphic function of

$$\omega := \sum_{n \geq 1} x_n \, p^{n-1} \tag{2.1}$$

Furthermore, $[P, K] = -VV^\dagger T$ has a solution iff $ac^* + bd^* = 0$ and $\beta := 2\Im(p) \neq 0$ (where $\Im(p)$ denotes the imaginary part of p). Without restriction of generality we can set the diagonal part of K to zero, since it can be absorbed by redefinition of f in the formula for φ. We obtain the following components of φ,

$$\varphi_{11} = -\varphi_{22} = \frac{i\,\beta}{D}\Big(|bc|^2 - |ad|^2 + 2\beta\,\Im(a^* c f)\Big)$$
$$\varphi_{12} = -\varphi_{21}^* = \frac{\beta}{D}\Big(-2i\,(|c|^2 + |d|^2)\,a^* b + \beta\,(a^* d\, f - bc^* f^*)\Big)$$

where

$$D := (|a|^2 + |b|^2)(|c|^2 + |d|^2) + \beta^2\,|f(\omega)|^2 > 0 \quad \text{if } \det(V) \neq 0$$

If f is a non-constant polynomial in ω, the solution is regular, rational and localized. It describes a simple lump if f is linear in ω. Otherwise it attains a more complicated shape (see [4] for some examples). □

Fixing the values of $x_4, x_5, \ldots$, we concentrate on the first pdCM hierarchy equation. In terms of the variables x, y, t given by (1.2), we then have

$$\omega = \frac{1}{2}(t - x + 2py + p^2(t + x))$$

subtracting a constant that can be absorbed by redefinition of the function f in the solution in Example 2.1. This solution becomes *stationary*, i.e. t-independent, if $p = \pm i$. The conserved density

$$\mathcal{E} := -\mathrm{tr}[(\varphi_t - \varphi_x)^2 + {\varphi_y}^2]/2 \tag{2.2}$$

of (1.6) is non-negative and will be used below to display the behaviour of some solutions.

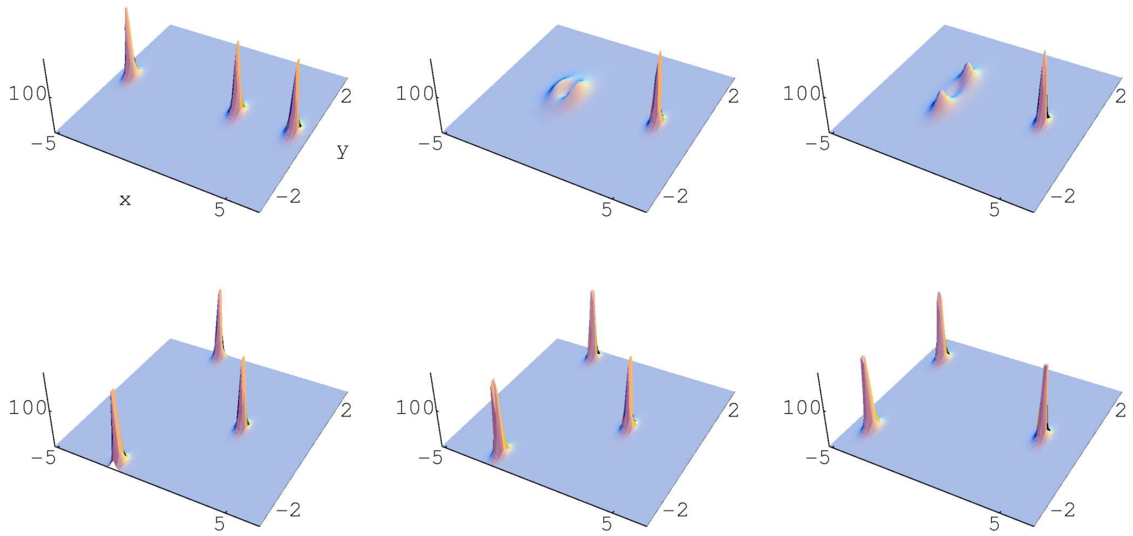

Figure 1. Plots of $\mathcal{E}$ at times $t = -90, -55, -53, 0, 30, 80$ for the solution in Example 2.2 with $p_1 = -i(1-\epsilon)$ and $p_2 = i(1+\epsilon)$ where $\epsilon = 1/20$, $f_1(\omega_1) = 4i\,\omega_1$, $f_2(\omega_2) = i\,\omega_2^2$.

More complicated solutions are obtained by superposition in the following sense. Given data (X_1, P_1, T_1, V_1) and (X_2, P_2, T_2, V_2) that determine solutions according to theorem 2.1, we build

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

The diagonal blocks of the new big matrix K will be K_1 and K_2. It only remains to solve

$$P_1 K_{12} - K_{12} P_2 = -V_1 V_2^\dagger T_2$$

for the upper off-diagonal block of K and set $K_{21} = T_2^{-1} K_{12}^\dagger T_1$. In particular, one can superpose lump solutions as given in the preceding example.

Example 2.2. Superposition of two single lumps with $V_1 = V_2 = I_2$, the 2×2 unit matrix, yields

$$\varphi_{11} = -\varphi_{22} = -\frac{i}{\mathcal{D}}\Big(\beta_2|ah_1|^2 + \beta_1|ah_2|^2 + 2\beta_1\beta_2\,\Im(a^* h_1 h_2^*) + (\beta_1+\beta_2)|b|^4\Big)$$

$$\varphi_{12} = -\varphi_{21}^* = \frac{1}{\mathcal{D}}\Big(a|h_1|^2\beta_2 h_2 + a^*\beta_1 h_1|h_2|^2 + (b^*)^2(a\beta_1 h_1 + a^*\beta_2 h_2)\Big)$$

where $\beta_i := 2\,\Im(p_i)$, $a := p_1 - p_2^*$, $b := p_1 - p_2$, $h_1 := a\beta_1 f_1$, $h_2 := a^*\beta_2 f_2$ with arbitrary holomorphic functions $f_1(\omega_1)$ (where ω_1 is (2.1) built with p_1), respectively $f_2(\omega_2)$, and

$$\mathcal{D} := (|b|^2 + |h_1|^2)(|b|^2 + |h_2|^2) + \beta_1\beta_2|h_1 - h_2|^2$$

This solution is again regular if $p_1 \neq p_2$ [4]. For $|f_1| \to \infty$ (resp. $|f_2| \to \infty$) we recover the single lump solution (2.1) with $V = I_2$ and f replaced by f_2 (resp. f_1).

Choosing $p_1 = i(1-\epsilon)$ and $p_2 = i(1+\epsilon)$ (or correspondingly with i replaced by $-i$) with $0 < \epsilon \ll 1$, and f_1, f_2 linear in ω_1, respectively ω_2, one observes scattering at right angle (cf. [16] for the analogous case in the Ward model).

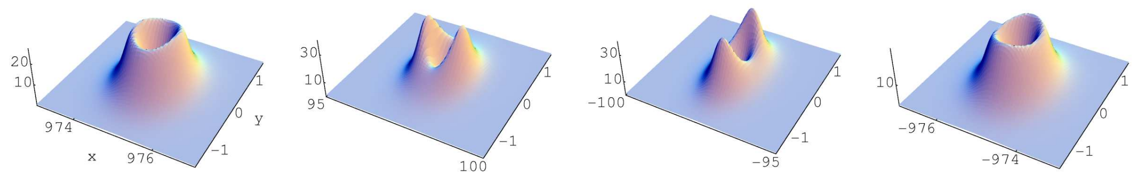

Figure 2. Origin and fate of the lump pair parts appearing in Fig. 1 (to the right in the first plot and to the left in the last). Plots of $\mathcal{E}$ at $t = -20000, -2000, 2000, 20000$.

If $p_1 = -i(1-\epsilon)$ and $p_2 = +i(1+\epsilon)$, one observes the following phenomenon: two lumps approach one another, meet, then separate in the orthogonal direction up to some maximal distance, reproach, merge again, and then separate again while moving in the original direction [4].[8] In the limit $\epsilon \to 0$, a vanishes and φ becomes constant (assuming f_1, f_2 independent of ϵ), so that $\mathcal{E}$ vanishes. For other choices of f_1 and f_2 more complex phenomena occur, including a kind of "exchange process" described in the following. Fig. 1 shows plots of $\mathcal{E}$ at successive times t for the above solution with f_1 linear in ω_1 and f_2 quadratic in ω_2. The latter function then corresponds to a bowl-shaped lump (see the left of the plots in Fig. 2) which, at early times, moves to the left along the x-axis, deforming into the lump *pair*, shown on the right hand side of the first plot in Fig. 1, under the increasing influence of the simple lump (corresponding to the linear function f_1) that moves to the right. When the latter meets the first partner of the lump pair, they merge, separate in y-direction to a maximal distance, move back toward each other and then continue moving as a lump pair (shown on the left hand side of the last plot in Fig. 1) into the negative x-direction. Meanwhile the remaining partner of the lump pair, that evolved from the original bowl-lump, retreats into the (positive) x-direction, with diminishing influence on the new lump pair, which then finally evolves into a bowl-shaped lump (see the right of the plots in Fig. 2). The smaller the value of ϵ, the larger the range of the interaction. □

Other classes of solutions are obtained by taking for P matrices of Jordan normal form, generalizing T appropriately, and building superpositions in the aforementioned sense. Some examples in the $su(2)$ case have been worked out in [4]. This includes examples exhibiting (asymptotic) π/n scattering of n-lump configurations. The pdCM (and also the Ward model) thus exhibits surprisingly complex lump interaction patterns, which are comparatively well accessible via the above theorem, though a kind of systematic classification is by far out of reach.

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References

[1] M. J. Ablowitz, S. Chakravarty, A. D. Trubatch, and J. Villarroel. A novel class of solutions of the non-stationary Schrödinger and the Kadomtsev-Petviashvili I equations. Phys. Lett. A, **267** (2000), 132–146.

[2] B. Carl and C. Schiebold. Nonlinear equations in soliton physics and operator ideals. Nonlinearity, **12** (1999), 333–364.

[8] See also [8] for an analogous phenomenon in case of KP-I lumps.

[3] B. Dai and C.-L. Terng. Bäcklund transformations, Ward solitons, and unitons. J. Diff. Geom. **75** (2007), 57–108.

[4] A. Dimakis and F. Müller-Hoissen. Dispersionless limit of the noncommutative potential KP hierarchy and solutions of the pseudodual chiral model in $2+1$ dimensions. Preprint `arXiv:0706.1373`.

[5] K. A. Gorshkov, D. E. Pelinovskii, and Yu. A. Stepanyants. Normal and anomalous scattering, formation and decay of bound states of two-dimensional solitons described by the Kadomtsev-Petviashvili equation. JETP, **77** (1993), 237–245.

[6] T. Ioannidou. Soliton solutions and nontrivial scattering in an integrable chiral model in $(2+1)$ dimensions. J. Math. Phys. **37** (1996), 3422–3441.

[7] A. N. Leznov. Equivalence of four-dimensional self-duality equations and the continuum analog of the principal chiral field problem. Theor. Math. Phys, **73** (1987), 1233–1237.

[8] Z. Lu, E. M. Tian, and R. Grimshaw. Interaction of two lump solitons described by the Kadomtsev-Petviashvili I equation. Wave Motion, **40** (2004), 123–135.

[9] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev. Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction. Phys. Lett. A, **63** (1977), 205–206.

[10] N. Manton and P. Sutcliffe. *Topological Solitons.* Cambridge University Press, Cambridge, 2004.

[11] V. A. Marchenko. *Nonlinear Equations and Operator Algebras.* Reidel, Dordrecht, 1988.

[12] L. J. Mason and N. M. J. Woodhouse. *Integrability, Self-Duality, and Twistor Theory.* Clarendon Press, Oxford, 1996.

[13] P. M. Sutcliffe. Non-trivial soliton scattering in an integrable chiral model in $(2+1)$-dimensions. J. Math. Phys., **33** (1992), 2269-2278.

[14] J. Villarroel and M. J. Ablowitz. On the discrete spectrum of the nonstationary Schrödinger equation and multipole lumps of the Kadomtsev-Petviashvili I equation. Comm. Math. Phys. **207** (1999), 1–47.

[15] R. S. Ward. Soliton solutions in an integrable chiral model in $2+1$ dimensions. J. Math. Phys., **29** (1988), 386–389.

[16] R. S. Ward. Nontrivial scattering of localized solitons in a $(2+1)$-dimensional integrable system. Phys. Lett. A, **208** (1995), 203–208.

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