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Fuzzy Implementation of Qubits Operators

Alexander Rybalov[1], Eugene Kagan[2], Alon Rapoport[3] and Irad Ben-Gal[4]*

[1]*Jerusalem College of Technology, Jerusalem, 91160, Israel*
[2]*Department of Industrial Engineering, Ariel University, Ariel, 40700, Israel*
[3]*Department of Mechanical Engineering, Tel-Aviv University, Ramat-Aviv, 69978, Israel*
[4]*Department of Industrial Engineering, Tel-Aviv University, Ramat-Aviv, 69978, Israel*

Abstract

In the paper, a complete minimal system of fuzzy logic operators that implements the operations over qubits is suggested. The model is based on a parametric system of the fuzzy not and phase operators and represents operators of quantum information theory acting on real amplitudes. An example of the system application is provided by its implementation to a mobile robot control, and its correspondence with the previously suggested models is illustrated by numerical simulations.

Keywords: Quantum information; Fuzzy logic; Mobile robots.

Introduction

Following the publication of the classical work by Birkhoff and von Neumann [1], the correspondence between the logic of quantum mechanics and non-Boolean logics has attracted significant theoretical and practical interest [2]. During the last decades, such an interest was enhanced with the development of quantum information and computation theory [3] and several approaches for the implementation of quantum logic by the use of fuzzy methods were presented[4-8]. The main goal of these studies is to construct a fuzzy model of quantum mechanical operators such that it can be implemented by fuzzy controllers [9]. In some cases, such models are complementary to the implementation of fuzzy control models by the use of quantum computations [10].

Recently, Hannachi et al. [5,6] suggested a direct fuzzy model of quantum computations over the qubits with real amplitudes. In their model, termed here as the H³-model, qubits are represented by pairs of membership functions, and main quantum gates are implemented by corresponding fuzzy operators on the unit square. Rybalov et al. [11] supplement the H³-model with reverse fuzzy Hadamard operator and implemented the resulting model in a mobile robot control system. The resulting model is indicated henceforth as the extended H³-model.

Although the extended H³-model provides basic representations of quantum information concepts by fuzzy logic methods and, in the completed form, can be used in certain applications, no general and minimal two-dimensional fuzzy model have been reported so far.

The objective of the paper is to present a minimal fuzzy model on a unit square that generalises the extended H³-model and, similar to quantum-mechanical models, preserves smoothness of membership functions and operators. The suggested model applies a parametric system of the fuzzy not and phase operators, such that their compositions provide a complete system of fuzzy operators and implement operators of quantum information theory for real amplitudes. The fuzzy not and phase operators are based on the uninorms [12].

The actions of the suggested minimal fuzzy model are illustrated by the use of a real mobile robot control system. In addition, numerical simulations and comparison with the control method based on the extended H³-model [11] are provided.

The obtained results can support a development of fuzzy logic tools for the models of quantum information theory and their implementation.

The paper is organized as follows. The background includes a brief presentation of the main fuzzy operators, which are included in the H³-model and extended H³-model. In Phase_α and not_α operators for H³-model, we defined the suggested parametric fuzzy not and phase operators and consider their actions. Uninorm for Phase_α and Not_α Operators and Projector section deals with the uninorm and the metrics for the suggested operators. In implementation of the suggested model and numerical analysis, an example, which illustrates an implementation of the suggested model for the mobile robot's control, is presented. In addition, the section includes results of numerical simulations and statistical comparison of the suggested model with the extended H³-model applied for the mobile robot's control.

Background

The states s of quantum-mechanical system [3] are defined by the use of qubits that are two elements column-vectors

$$|s\rangle = (a, b)^T = a|0\rangle + b|1\rangle = a(1, 0)^T + b(0, 1)^T$$

where vectors $|0\rangle = (1, 0)^T$ and $|1\rangle = (0, 1)^T$ are the basic qubits and $|a|^2 + |b|^2 = 1$. The basic states $|0\rangle$ and $|1\rangle$ of the system are usually denoted by "↑" ("spin up") and "↓" ("spin down"):

$$"\uparrow" = |s_1\rangle = (1, 0)^T, \qquad "\downarrow" = |s_2\rangle = (0, 1)^T,$$

while the states $"\leftarrow"$ and $"\rightarrow"$ are defined as follows:

$$"\rightarrow" = |s_3\rangle = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T, "\leftarrow" = |s_4\rangle = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T.$$

The states $|s_1\rangle, |s_2\rangle, |s_3\rangle$ and $|s_4\rangle$ form a complete system in such a sense that each state can be obtained by the use of the other states [11], i.e.:

$$|s_1\rangle = \frac{1}{\sqrt{2}}|s_3\rangle + \frac{1}{\sqrt{2}}|s_4\rangle, \quad |s_2\rangle = \frac{1}{\sqrt{2}}|s_3\rangle - \frac{1}{\sqrt{2}}|s_4\rangle,$$

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*Corresponding author: Irad Ben-Gal, Department of Industrial Engineering, Tel-Aviv University, Ramat-Aviv, 69978, Israel, Tel: 13126136968; E-mail: bengal@tau.ac.il

$$|s_3\rangle = \frac{1}{\sqrt{2}}|s_1\rangle + \frac{1}{\sqrt{2}}|s_2\rangle, \quad |s_4\rangle = \frac{1}{\sqrt{2}}|s_1\rangle - \frac{1}{\sqrt{2}}|s_2\rangle,$$

and, as it is usual in quantum mechanics, the states $|s\rangle$ and $(-1)\cdot|s\rangle$ are not distinguished.

According to the Hannachi et al. [5,6], in the direct fuzzy model of quantum computations to each quantum state $|s\rangle = (a,b)^T$ with real amplitudes fuzzy state $\tilde{s} = (\mu_a, \mu_b)$, in which the membership functions $\mu_a : [0,1] \to [0,1]$ and $\mu_b : [0,1] \to [0,1]$ are defined as follows:

$$\mu_a = \frac{2}{\pi}\arcsin\sqrt{\frac{1}{2}\left(sign(a)a^2 - sign(b)b^2 + 1\right)}, \mu_b = \frac{2}{\pi}\arcsin\sqrt{\frac{1}{2}\left(sign(a)a^2 + sign(b)b^2 + 1\right)}.$$

Inversely, given values of membership functions μ_a and μ_b, the amplitudes $a : [0,1] \to [0,1]$ and $b : [0,1] \to [0,1]$ are obtained by the following equations [5,6]:

$$a = sign(\mu_a + \mu_b - 1)\sqrt{\left|\sin^2 \frac{\pi}{2}\mu_a + \sin^2 \frac{\pi}{2}\mu_b - 1\right|},$$

$$b = sign(\mu_b - \mu_a)\sqrt{\left|\sin^2 \frac{\pi}{2}\mu_b - \sin^2 \frac{\pi}{2}\mu_a\right|}.$$

The Hannachi and the H³-model [5,6] include fuzzy representations of the main qubit operators (Pauli operators I, X, Z, Hadamard H operator and negative unitary operator $\bar{\text{I}}$) that are used in quantum information theory [8]. The correspondence between quantum and fuzzy operators is the following. Let us denote fuzzy operators by $\tilde{\text{I}}$, $-\tilde{\text{I}}$, $\tilde{\text{X}}$, $\tilde{\text{Z}}$ and $\tilde{\text{H}}$ in correspondence to the quantum operators. Then an application of the fuzzy operators to a fuzzy state $\tilde{s} = (\mu_a, \mu_b)$ results in the following states [5,6]:

$$\tilde{\text{I}}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}, \; -\tilde{\text{I}}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}1-\mu_a \\ 1-\mu_b\end{pmatrix}, \; \tilde{\text{X}}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}1-\mu_a \\ \mu_b\end{pmatrix},$$

$$\tilde{\text{Z}}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}\mu_b \\ \mu_a\end{pmatrix},$$

$$\tilde{\text{H}}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}\max\left\{0, \min\left\{1, \frac{1}{2} - \mu_a + \mu_b\right\}\right\} \\ \min\left\{1, \max\left\{0, \mu_a + \mu_b - \frac{1}{2}\right\}\right\}\end{pmatrix}.$$

Since the fuzzy state $\tilde{s}$ is not a vector, its row and column representation are used here interchangeably.

To complete the correspondence between quantum and fuzzy models, Rybalov et al. [11] suggested the reverse fuzzy Hadamard operator $\tilde{\text{H}}^R$ that corresponds to the quantum reverse Hadamard operator. For a fuzzy state $\tilde{s} = (\mu_a, \mu_b)$ this operator is defined as follows:

$$\tilde{\text{H}}^R\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}\min\left\{1, \max\left\{0, \frac{3}{2} - \mu_a - \mu_b\right\}\right\} \\ \max\left\{0, \min\left\{1, \frac{1}{2} - \mu_a + \mu_b\right\}\right\}\end{pmatrix}.$$

The H³-model with the reverse Hadamard operator $\tilde{\text{H}}^R$ is called extended H³-model.

Phase_α and not_α operators for H³-model

Both the original and the extended H³-models are rather complex. Below we suggest a simpler fuzzy model that includes two operators, to which the transformations that are provided by the extended H³-model can be reduced.

Let $\tilde{s} = (\mu_a, \mu_b)$ be a fuzzy state and assume that the values of membership functions μ_a and μ_b are obtained according to the H³-model. However, the fuzzy states $\tilde{s} = (\mu_a, \mu_b)$ are not restricted and the suggested operators can be applied for any pair of membership functions (μ_a, μ_b).

Let α be a real number such that $0 \le \alpha \le 1$. For fuzzy state $\tilde{s} = (\mu_a, \mu_b)$ and the number α, we define two operators $phase_\alpha$ and not_α as follows:

$$phase_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}(1-\alpha)\cdot\mu_a + \alpha\cdot(1-\mu_b) \\ \alpha\cdot(1-\mu_a) + (1-\alpha)\cdot(1-\mu_b)\end{pmatrix},$$

$$not_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}(1-\alpha)\cdot(1-\mu_a) + \alpha\cdot\mu_b \\ \alpha\cdot\mu_a + (1-\alpha)\cdot\mu_b\end{pmatrix}.$$

It is easy to verify that the operators $phase_\alpha$ and not_α have the following properties:

$$phase_{\alpha=0}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}\mu_a \\ 1-\mu_b\end{pmatrix}, \qquad not_{\alpha=0}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}1-\mu_a \\ \mu_b\end{pmatrix},$$

$$phase_{\alpha=1}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}1-\mu_b \\ 1-\mu_a\end{pmatrix}, \qquad not_{\alpha=1}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \begin{pmatrix}\mu_b \\ \mu_a\end{pmatrix}.$$

In addition, for the compositions of the operators $phase_\alpha$ and not_α with any $\alpha \in [0,1]$, one obtains:

$$phase_\alpha\left(phase_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = not_\alpha\left(not_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = \begin{pmatrix}(2\alpha^2 - 2\alpha + 1)\cdot\mu_a + \alpha\cdot(1-\alpha) \\ (2\alpha^2 - 2\alpha + 1)\cdot\mu_b + \alpha\cdot(1-\alpha)\end{pmatrix},$$

$$phase_\alpha\left(not_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = not_\alpha\left(phase_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = \begin{pmatrix}1 - (2\alpha^2 - 2\alpha + 1)\cdot\mu_a - \alpha\cdot(1-\alpha) \\ 1 - (2\alpha^2 - 2\alpha + 1)\cdot\mu_b - \alpha\cdot(1-\alpha)\end{pmatrix} =$$

$$= \begin{pmatrix}1 \\ 1\end{pmatrix} - phase_\alpha\left(phase_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = \begin{pmatrix}1 \\ 1\end{pmatrix} - not_\alpha\left(not_\alpha\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right).$$

In particular, for $\alpha = 0$ and $\alpha = 1$, the compositions of the operators result in the following states:

$$phase_{\alpha=0}\left(phase_{\alpha=0}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = not_{\alpha=0}\left(not_{\alpha=0}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) =$$

$$= phase_{\alpha=1}\left(phase_{\alpha=1}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = not_{\alpha=1}\left(not_{\alpha=1}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = \begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix},$$

$$phase_{\alpha=0}\left(not_{\alpha=0}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = not_{\alpha=0}\left(phase_{\alpha=0}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) =$$

$$= phase_{\alpha=1}\left(not_{\alpha=1}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = not_{\alpha=1}\left(phase_{\alpha=1}\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix}\right) = \begin{pmatrix}1-\mu_a \\ 1-\mu_b\end{pmatrix}.$$

Let us apply the defined operators $phase_\alpha$ and not_α and their compositions with $\alpha = 0$ and $\alpha = 1$ to the fuzzy states $\tilde{s}_1 = (1,1)$, $-\tilde{s}_1 = (0,0)$, $\tilde{s}_2 = (0,1)$ and $-\tilde{s}_2 = (1,0)$, in which the values of membership

functions are integer (including zero). The implementation of the presented formulas results in the states with integer values of membership functions, as it is shown in Figure 1.

Now, let us present the actions of the operators $phase_\alpha$ and not_α over the states $\widetilde{s}_3 = (1/2, 1)$, $-\widetilde{s}_3 = (1/2, 0)$, $\widetilde{s}_4 = (1, 1/2)$ and $-\widetilde{s}_4 = (0, 1/2)$. For $\alpha = 0$ and $\alpha = 1$, these operators act as it is shown in Figure 2.

Finally, let us consider the actions of the operators $phase_\alpha$ and not_α with $\alpha = 1/2$. Direct calculations show that for the fuzzy states $\widetilde{s}_1 = (1, 1)$, $-\widetilde{s}_1 = (0, 0)$, $\widetilde{s}_2 = (0, 1)$ and $-\widetilde{s}_2 = (1, 0)$ with integer values of membership functions the operators $phase_{\alpha=1/2}$ and $not_{\alpha=1/2}$ result in the states that lie on the perimemter of the unit square.

However, an application of the operators $phase_{\alpha=1/2}$ and $not_{\alpha=1/2}$ to the resulting states $\widetilde{s}_3 = (1/2, 1)$, $-\widetilde{s}_3 = (1/2, 0)$, $\widetilde{s}_4 = (1, 1/2)$ and $-\widetilde{s}_4 = (0, 1/2)$ with non-integer values of membership functions does not give the original states. The states that correspond to these values are located in the internal part of the unit square instead of its perimeter. To project such states to the perimeter let us define the unique projector $\widetilde{P}$ such that for any fuzzy state (μ'_a, μ'_b) it results in the state $(\mu_a, \mu_b) = \widetilde{P}(\mu'_a, \mu'_b)$, which lies on the closest intersection point of the perimeter, and a line that follows via the point (μ'_a, μ'_b)
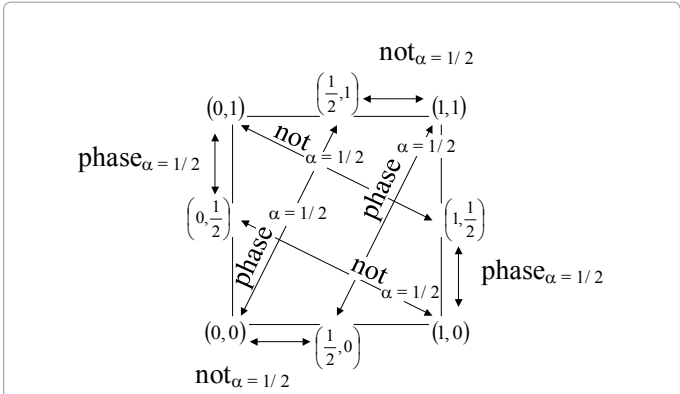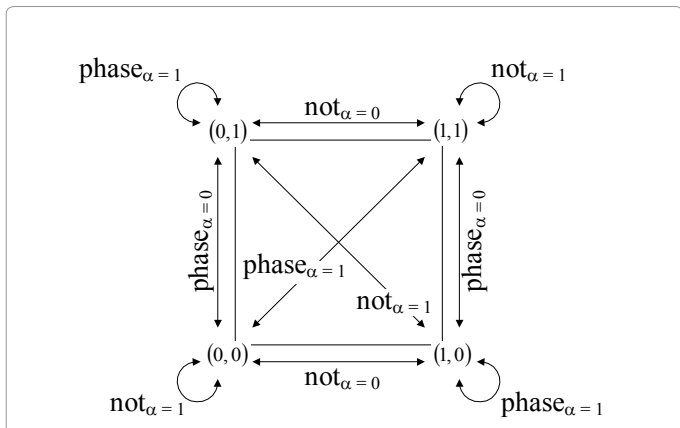

Figure 1: Actions of $phase_\alpha$ and not_α operators over the fuzzy states with integer values of membership functions.

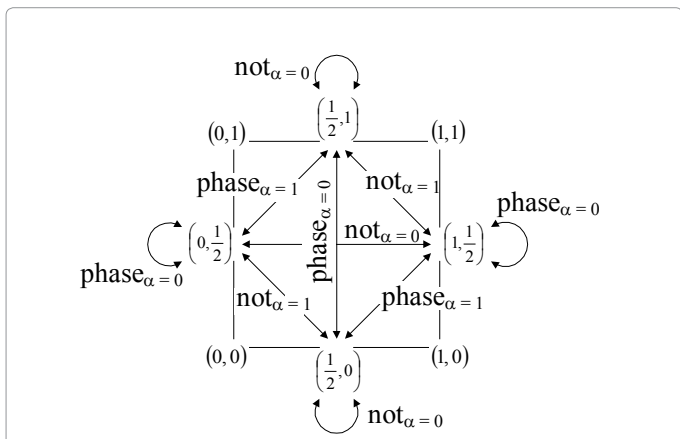**Figure 2:** Actions of $phase_\alpha$ and not_α operators over the fuzzy states with non-integer values of membership functions.

Figure 3: Actions of $phase_\alpha$ and not_α operators with $\alpha = 1/2$.

Quantum operator	Extended H³ model	$phase_\alpha$ - not_α model
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mu_a \mapsto \mu_a$ $\mu_b \mapsto \mu_b$	$phase_\alpha^2$ or not_α^2, $\alpha = 0$ or $\alpha = 1$
$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mu_a \mapsto 1 - \mu_a$ $\mu_b \mapsto 1 - \mu_b$	$phase_\alpha \circ not_\alpha$, $\alpha = 0$ or $\alpha = 1$
$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\mu_a \mapsto 1 - \mu_a$ $\mu_b \mapsto \mu_b$	not_0
$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mu_a \mapsto \mu_b$ $\mu_b \mapsto \mu_a$	not_1
$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} \mapsto \widetilde{H} \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$	$not_{1/2}$
$H^R = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} \mapsto \widetilde{H}^R \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$	$phase_{1/2} \circ not_0 \circ not_1$

In the table, $phase_\alpha^2$ and not_α^2 stand for a composition of the operators $phase_\alpha^2 = phase_\alpha \circ phase_\alpha$ and $not_\alpha^2 = not_\alpha \circ not_\alpha$, correspondingly.

Table 1: Correspondence between quantum, H³and $phase_\alpha$-not_α models.

and a center of the square $(1/2, 1/2)$. The strict meaning of the projector is considered in the next section. The actions of the operators $phase_\alpha$ and not_α with $\alpha = 1/2$ (composed with the projector $\widetilde{p}$, i.e. under the assumption that $phase_\alpha \equiv phase_\alpha \circ \widetilde{P}$ and $not_\alpha \equiv not_\alpha \circ \widetilde{P}$) are shown in Figure 3.

For convenience, the correspondence between quantum and H³-model operators and their representation by the use of $phase_\alpha$ and not_α operators is summarized in Table 1.

The above-presented consideration demonstrates that the suggested $phase_\alpha$ and not_α operators completely implement the transformations, which are provided by the extended H³-model that includes fuzzy reverse Hadamard operator $\widetilde{H}^R$.

Uninorm for Phase$_\alpha$ and not$_\alpha$ Operators and Projector $\widetilde{P}$

Let us clarify the meaning of the above-introduced projecting operator $\widetilde{P}$. Below we show that this operator provides a monotony

property of the uninorms that are defined by compositions of the operators $phase_\alpha$ and not_α.

Let us start with the uninorm that is defined over the membership functions. The uninorm R is defined as a mapping $R : [0,1] \times [0,1] \to [0,1]$ such that for any values of membership functions μ_a, μ_b, μ_c, $\mu_d \in [0,1]$, the following properties hold [12]:

commutativity: $R(\mu_a, \mu_b) = R(\mu_b, \mu_a)$;

monotonicity: $R(\mu_a, \mu_b) \geq R(\mu_c, \mu_d)$ if $\mu_a \geq \mu_c$ and $\mu_b \geq \mu_d$;

associativity: $R(\mu_a, R(\mu_b, \mu_c)) = R(R(\mu_a, \mu_b), \mu_c)$;

existence of identity: for any $\mu \in [0,1]$ there exists $\varepsilon \in [0,1]$ such that $R(\mu, \varepsilon) = \mu$.

Let us define a similar uninorm U that is applicable to $phase_\alpha$ and not_α operators, which act on the pairs (μ_a, μ_b) and result in the pairs (μ'_a, μ'_b), $\mu_a, \mu_b, \mu'_a, \mu'_b \in [0,1]$. Below we denote these operators by A, B, C and D.

Assume that the uninorm $U(A, B)$ is an operator over the fuzzy states, which is defined as a composition of the operators A, B, i.e. $U(A, B) = B \circ A$. Since, as indicated above, it holds true that $phase_\alpha \circ not_\alpha = not_\alpha \circ phase_\alpha$, and $phase_\alpha^2(\mu_a, \mu_b) = not_\alpha^2(\mu_a, \mu_b) = (\mu_a, \mu_b)$, and acting of such uninorm U on the $phase_\alpha$ and not_α operators meets the commutativity and associativity requirements, and an identity $E = phase_\alpha \circ phase_\alpha$ (or $E = not_\alpha \circ not_\alpha$) exists.

The consideration of the monotonicity of the uninorm U requires definition of the relations "<" and ">" between the operators. At first, notice that the uninorm U for the operators acts on the pairs of pairs of membership functions and results in the pair of membership functions, i.e.,

$$U((\mu_a, \mu_b), (\mu_c, \mu_d)) \mapsto (\mu_r, \mu_s), \quad \mu_r, \mu_s \in [0,1].$$

In addition, notice that the pair $(1/2, 1/2)$ represents a fixed point for the operators $phase_\alpha$ and not_α with any value of parameter α. Hence, this point can be considered as a rapper or *null-point*.

By the use of the point $\tilde{\theta} = (1/2, 1/2)$, the relations "<" and ">" between the operators can be defined as follows. Let $d : (\tilde{s}', \tilde{s}'') \to m \in [0, \infty)$ be a metric defined on the unit square, where $\tilde{s}' = (\mu'_a, \mu'_b)$, $\tilde{s}'' = (\mu''_a, \mu''_b)$ and $\mu'_a, \mu'_b, \mu''_a, \mu''_b \in [0,1]$. Then, for any fuzzy state $\tilde{s} = (\mu_a, \mu_b)$ the distance from this state to the null-point $\tilde{\theta} = (1/2, 1/2)$ is given by the formula:

$$m(\mu_a, \mu_b) = d(\tilde{s}, \tilde{\theta}) = d\left((\mu_a, \mu_b), \left(\frac{1}{2}, \frac{1}{2}\right)\right),$$

The relations "<" and ">" between the states $\tilde{s}' = (\mu'_a, \mu'_b)$ and $\tilde{s}'' = (\mu''_a, \mu''_b)$ are specified by the use of the distances as follows. We say that the state $\tilde{s}' = (\mu'_a, \mu'_b)$ is greater than the state $\tilde{s}'' = (\mu''_a, \mu''_b)$ and write $\tilde{s}' > \tilde{s}''$ if in the defined metric d it holds true that

$$m(\mu'_a, \mu'_b) = d(\tilde{s}', \tilde{\theta}) > m(\mu''_a, \mu''_b) = d(\tilde{s}'', \tilde{\theta}),$$

and, similarly, we say that $\tilde{s}'$ is less the state $\tilde{s}''$ and write $\tilde{s}' < \tilde{s}''$ if it holds true that

$$m(\mu'_a, \mu'_b) = d(\tilde{s}', \tilde{\theta}) < m(\mu''_a, \mu''_b) = d(\tilde{s}'', \tilde{\theta}).$$

If both $\tilde{s}' > \tilde{s}''$ and $\tilde{s}' < \tilde{s}''$ are satisfied, then we say that the states are equal in the sense of distance from the null-point θ and write $\tilde{s}' = \tilde{s}''$.

The relations between the operators A and B that act over the fuzzy states, and, correspondingly, between the results of the applications of uninorm U are defined by the relations between the resulting states as follows. Let $A(\mu_a, \mu_b) = (\mu'_a, \mu'_b)$ and $B(\mu_a, \mu_b) = (\mu''_a, \mu''_b)$. Then, we say that $A > B$ if $(\mu'_a, \mu'_b) > (\mu''_a, \mu''_b)$ and $A < B$ if $(\mu'_a, \mu'_b) < (\mu''_a, \mu''_b)$.

Finally, let A, B, C and D be fuzzy operators, and let $U(A, B) = B \circ A = (\mu_a, \mu_b)$ and $U(C, D) = D \circ C = (\mu_c, \mu_d)$. Then, we say that

$$U(A, B) > U(C, D) \text{ if } (\mu_a, \mu_b) > (\mu_c, \mu_d), \text{ and}$$
$$U(A, B) < U(C, D) \text{ if } (\mu_a, \mu_b) < (\mu_c, \mu_d).$$

Now, let us consider the role of the projecting operator $\tilde{P}$. Assume that the metric d is a Manhattan metric, i.e.

$$m(\mu_a, \mu_b) = d(\tilde{s}, \tilde{\theta}) = \left| \mu_a - \frac{1}{2} \right| + \left| \mu_b - \frac{1}{2} \right|.$$

Let (μ_a, μ_b) be a state, and assume that fuzzy operators A, B, C and D without projector $\tilde{P}$ act as follows:

$$A(\mu_a, \mu_b) = \left(\frac{1}{8}, \frac{7}{8}\right), \quad B(\mu_a, \mu_b) = \left(\frac{7}{8}, \frac{7}{8}\right), \quad C(\mu_a, \mu_b) = \left(0, \frac{1}{2}\right),$$

$$D(\mu_a, \mu_b) = \left(1, \frac{1}{2}\right).$$

Then, by the use of the projector $\tilde{P}$, i.e., under an assumption that $A \equiv A \circ \tilde{P}$, $B \equiv B \circ \tilde{P}$, $C \equiv C \circ \tilde{P}$ and $D \equiv D \circ \tilde{P}$, one obtains:

$$A(\mu_a, \mu_b) = (0, 1), \qquad B(\mu_a, \mu_b) = (1, 1),$$

while the results of the operators C and D are still the same. Hence the inequalities $A > C$ and $B > D$ hold both for original results and for the states, which are obtained by the use of the projector $\tilde{P}$.

Direct calculations show that for the operators A, B, C and D without projector $\tilde{P}$ it follows that

$$m(U(A, B)) = m\left(A\left(\frac{7}{8}, \frac{7}{8}\right)\right) \leq \frac{3}{4} \text{ and } m(U(C, D)) = m\left(C\left(1, \frac{1}{2}\right)\right) \leq 1.$$

Thus, for the operators without projector $\tilde{P}$ in spite of the inequalities $A > C$ and $B > D$, the relation $U(A, B) > U(C, D)$ that is required by monotonicity, in general, is not satisfied.

However, similar considerations that include the projector $\tilde{P}$ result in the following inequalities $m(U(A, B)) \leq 1$ and $m(U(C, D)) \leq 1$. Notice that both $A(\mu_a, \mu_b) = (0, 1)$ and $B(\mu_a, \mu_b) = (1, 1)$ lie on the perimeter of the unit square, i.e. they are boundary states. Moreover, the results of the compositions $U(A, B) = B \circ A$ and $U(C, D) = D \circ C$ are also boundary states. Then, according to definitions of the operators $phase_\alpha$ and not_α, the transformations of the states over the boundary of the unit square are continuous. Hence, from the inequalities $A > C$ and $B > D$ it follows that $U(A, B) > U(C, D)$, as it is required by the

monotonicity property of the uninorm. This property of the suggested operators allows their consideration in the framework of general fuzzy operators.

Implementation of the Suggested Model and Numerical Analysis

The suggested model that is based on $phase_\alpha$ and not_α operators, as well as extended H³-model, were implemented in a mobile robot controller, and for both models comparative numerical simulations and field experiments were carried out. As indicated above, the detailed consideration of the experiments and simulation results is presented in our work [11].

The correspondence between the robot's states and its orientation has been specified according to the quantum "spin orientations" as follows:

north, $0^0 \equiv 360^0$: $\tilde{s}_1 = (1,1)$, $-\tilde{s}_1 = (0,0)$, south, 180^0: $\tilde{s}_2 = (0,1)$, $-\tilde{s}_2 = (1,0)$,

east, 270^0: $\tilde{s}_3 = \left(\frac{1}{2},1\right)$, $-\tilde{s}_3 = \left(\frac{1}{2},0\right)$, west, 90^0: $\tilde{s}_4 = \left(1,\frac{1}{2}\right)$,

$-\tilde{s}_4 = \left(0,\frac{1}{2}\right)$.

The implemented sequence of actions, in the terms of extended H³-model, has the following form:

$$\text{Initial state} \rightarrow \tilde{H} \rightarrow \tilde{H} \rightarrow \tilde{H}^R \rightarrow \tilde{H}^R \rightarrow \tilde{H}^R \rightarrow \tilde{H} \rightarrow \tilde{H} \rightarrow \tilde{H} \rightarrow \tilde{H}^R \rightarrow \tilde{H}^R \rightarrow \tilde{H}^R \rightarrow \tilde{H}.$$

According to this sequence, the robot conducts the following actions:

- moving one step forward;
- turning to the right (direct Hadamard operator);
- moving one step forward;
- turning to the left (application of the direct Hadamard operator over eastern orientation of the robot turns the robot to the left returning it to the initial orientation;
- moving one step forward;
- turning to the right (in contrast to the direct Hadamard operator, reverse Hadamard operator turns the robot left from its initial orientation);
- moving one step forward,

and so on up to the last step when the robot returns to its initial state. The obtained trajectory is simple enough for analysis, and, at the same time, represents all possible turns of the robot, so that all transitions between the states can be tested.

In the field experiments, the mobile robot was programmed so that it implemented the presented sequence of actions according to either the extended H³-model or the suggested model based on $phase_\alpha$ and not_α operators. An example of the obtained trajectories of the robot, while at the initial state it was oriented to the north, are shown in Figure 4.

The Figure demonstrates the difference between turns that are governed by the extended H³-model and the suggested model based on $phase_\alpha$ and not_α operators. Notice that the turns of the robot controlled by $phase_\alpha$ and not_α model differ from the turns that were

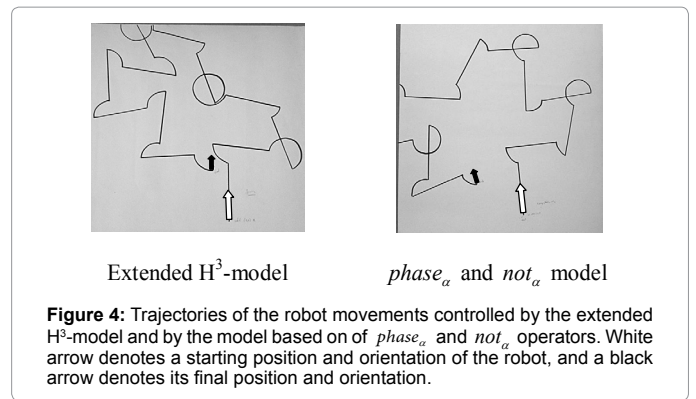Extended H³-model $phase_\alpha$ and not_α model

Figure 4: Trajectories of the robot movements controlled by the extended H³-model and by the model based on of $phase_\alpha$ and not_α operators. White arrow denotes a starting position and orientation of the robot, and a black arrow denotes its final position and orientation.

Extended H³-model

Mean = -0.07, st. dev. = 6.95

$phase$ and not model

Mean = 0.22, st. dev. = 10.41

(a): 1% turns' errors.

Extended H³-model

Mean = -2.34, st. dev. = 39.15

$phase$ and not model

Mean= -3.20, st. dev. = 46.09

(b): 5% turns' errors.

Extended H³-model

Mean = -34.64, st. dev. = 88.03

$phase$ and not model

Mean = -39.65, st. dev. = 92.93
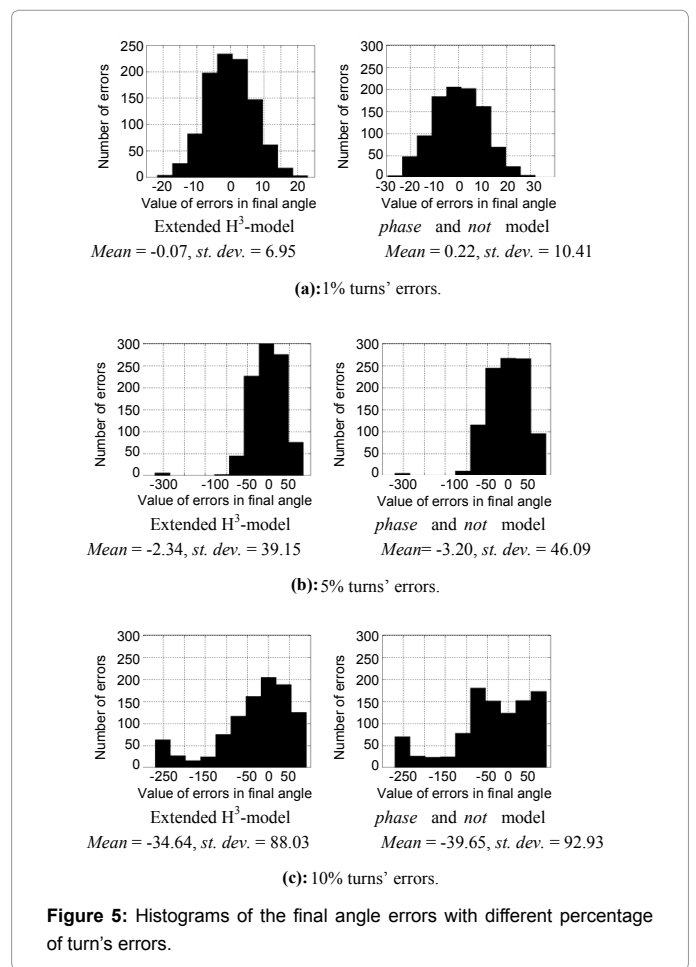
(c): 10% turns' errors.

Figure 5: Histograms of the final angle errors with different percentage of turn's errors.

previously obtained for the model of quantum control [11], so the topology of the states space for the $phase_\alpha$ and not_α model differs from both the extended H³ and the quantum models.

Numerical analysis was focused on the errors of the final angle that reflect the turns' errors. The above-indicated sequence of direct and reverse Hadamard operators was implemented numerically, while after each application of the operator a random error to the resulting state was added. There were implemented three values of errors: 1% error, 5% error and 10% error. The 1% error corresponds to the adding to the robot's orientation a randomly chosen angle from the interval $\left(-3.6^0, 3.6^0\right)$, the 5% error means adding a randomly chosen

angle form the interval $\left(-18^0, 18^0\right)$, and 10% error corresponds to the addition of the angle randomly chosen from the interval $\left(-36^0, 36^0\right)$. For each interval, the simulation was trialed 1000 times and the differences between the obtained and expected final orientations, i.e., the errors in final orientation angles were logged.

Histograms of the orientation errors in the final angle of the robots are shown in Figure 5. The histograms in Figure 5a correspond to the 1% errors in the robot's turns; the histograms in Figure 5b–to 5% turns' errors; and the histograms in Figure 5c–to a 10% turns' errors.

By the use of the Jarque-Bera test with significance level 0.01, it was found that in the case of 1% errors, the distributions of errors for both models of control are approximately normal, while in cases of 5% and 10% errors the distributions of errors can not be considered as normal. A series of t-tests with a significance level of 0.01 showed that the difference between the resulting errors is insignificant. Following the Kolmogorov-Smirnov test with a significance level 0.01, in all three cases of errors the obtained samples correspond to different continuous distributions.

The obtained results confirm our expectation that the model that is based on $phase_\alpha$ and not_α operators acts differently in comparison to the extended H^3-model, while the results of application of the suggested operators are equivalent. As indicated above, the difference between the actions of the extended H^3-model and the model based on $phase_\alpha$ and not_α operators is provided by different topologies of the states space.

Conclusion

In the paper, we suggested a system of fuzzy operators that implements the operations over qubits. The system includes two parameterized $phase_\alpha$ and not_α operators that act on the states obtained by the extended H^3-model. The transformations of the states that correspond to the operators of the extended H^3-model are determined by compositions of the defined $phase_\alpha$ and not_α operators. For the suggested operators, the operational uninorm was defined and analyzed, what introduces the suggested operators into general fuzzy logic approach to quantum computations.

The suggested system was illustrated by implementation for a mobile robot control and was analyzed by numerical simulations. Comparison of the control based on the suggested control demonstrated that the results of the actions of the suggested minimal fuzzy system (that includes two operators) are equivalent to the corresponding results obtained by the extended H^3-model (that includes six operators).

The obtained results can support a development of fuzzy logic tools for the models of quantum information theory.

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