(\(\alpha, \beta\))-fuzzy Lie algebras over an \((\alpha, \beta)\)-fuzzy field

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Abstract

The concept of \((\alpha, \beta)\)-fuzzy Lie algebras over an \((\alpha, \beta)\)-fuzzy field is introduced. We provide characterizations of an \((\in, \in \lor q)\)-fuzzy Lie algebra over an \((\in, \in \lor q)\)-fuzzy field.

2000 MSC: 17B99, 08A72

1 Introduction

Zadeh [12] formulated the notion of fuzzy sets and after that many scholars developed fuzzy system of different algebraic structures. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], has played a vital role in generating some different types of fuzzy subgroups. Using the belong-to relation \((\in)\) and quasi-coincidence with relation \((q)\) between fuzzy points and fuzzy sets, the concept of \((\alpha, \beta)\)-fuzzy subgroup was introduced by Bhakat and Das [4]. Akram [1] introduced \((\alpha, \beta)\)-fuzzy Lie subalgebras and investigated some of its properties. Nanda [9] introduced fuzzy algebra over fuzzy field. It is natural to investigate similar types of generalization of the existing fuzzy subsystem. In [3], we introduced fuzzy Lie algebra over a fuzzy field and some properties were discussed.

In this paper, we introduce the concept of \((\alpha, \beta)\)-fuzzy Lie algebra over an \((\alpha, \beta)\)-fuzzy field and investigate some of its properties.

2 Preliminaries

In this section, we present some definitions needed for our study. We denote a complete distributive lattice with the smallest element \(0\) and the largest element \(1\) by \(I\). By a fuzzy subset of a nonempty set \(X\), we mean a function from \(X\) to \(I\).

**Definition 2.1** (see [5]). Let \(X\) be a field and let \(F\) be a fuzzy subset of \(X\). Then \(F\) is called a fuzzy field of \(X\) if

\[(i) \text{ for all } \lambda, \gamma \in X, F(\lambda - \gamma) \geq F(\lambda) \land F(\gamma),
(ii) \text{ for all } \lambda, \gamma \neq 0 \text{ in } X, F(\lambda\gamma^{-1}) \geq F(\lambda) \land F(\gamma).\]

**Remark 2.2.** It is seen that if \(F\) is a fuzzy field of \(X\), then

\[F(0) \geq F(1) \geq F(\lambda) = F(-\lambda) = F(\lambda^{-1}) \text{ for all } \lambda \neq 0 \text{ in } X.\]

**Definition 2.3.** Let \(A\) be a fuzzy subset of a Lie algebra \(L\). Then \(A\) is called a fuzzy Lie algebra of \(L\) over a fuzzy field \(F\), if for all \(x, y \in L, \lambda \in X\),
A \textcolor{red}{{(i)}} A(x - y) \geq A(x) \land A(y), \\
(ii) \ A(\lambda x) \geq F(\lambda) \land A(x), \\
(iii) \ A([x, y]) \geq A(x) \land A(y).

3 The relations \textit{belong to} and \textit{quasi-coincidence with}

Let \( L \) be a Lie algebra over a field \( X \), let \( A : L \rightarrow [0, 1] \) be a fuzzy set on \( L \), and let \( F : X \rightarrow [0, 1] \) be a fuzzy set on \( X \). The support of fuzzy set \( A \) is the crisp set that contains all elements of \( L \) that have nonzero membership grades in \( A \).

\textbf{Definition 3.1} (see [10]). A fuzzy set \( A : L \rightarrow [0, 1] \) of the form

\[ A(y) = \begin{cases} 
t \in (0, 1], & \text{if } y = x, \\
0, & \text{if } y \neq x 
\end{cases} \]

is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \).

For a fuzzy point \( x_t \) and a fuzzy set \( A \) in a set \( L \), Pu and Liu [10] gave meaning to the symbol \( x_t \in A_{\alpha} \) where \( \alpha \in \{\in, \in q, \in \lor q\} \).

A fuzzy point \( x_t \) is said to \textit{belong to} a fuzzy set \( A \), written as \( x_t \in A \), if \( A(x) \geq t \). A fuzzy point \( x_t \) is said to be \textit{quasi-coincident with} a fuzzy set \( A \), denoted by \( x_t \in q A \), if \( A(x) + t > 1 \).

The following notations are used in this paper.

1. \( \in \lor q \) means that either \textit{belong to} or \textit{quasi-coincident with},
2. \( \in \) means that \( \alpha \) does not hold.

Let \( \min\{t, s\} \) be denoted by \( m(t, s) \) and let \( \max\{t, s\} \) be denoted by \( M(t, s) \). Take \( I = [0, 1] \) and \( \land = \min, \lor = \max \) with respect to the usual order in Definitions 2.1 and 2.3.

\textbf{Lemma 3.2.} A fuzzy subset \( F \) of a field \( X \) is a fuzzy field if and only if it satisfies the following conditions:

(i) for all \( \lambda, \gamma \) in \( X \), \( \lambda_t, \gamma_s \in F \Rightarrow (\lambda - \gamma)_{m(t, s)} \in F \), \\
(ii) for all \( \lambda, \gamma \neq 0 \) in \( X \), \( \lambda_t, \gamma_s \in F \Rightarrow (\lambda \gamma^{-1})_{m(t, s)} \in F \),

for all \( t, s \in (0, 1] \).

\textbf{Lemma 3.3.} Let \( L \) be a Lie algebra over a field \( X \). Then a fuzzy subset \( A \) of Lie algebra \( L \) is a fuzzy Lie algebra over a fuzzy field \( F \) of \( X \) if and only if it satisfies the following conditions:

(i) \( x_t, y_s \in A \Rightarrow (x - y)_{m(t, s)} \in A \), \\
(ii) \( x_t \in A, \ \lambda_r \in F \Rightarrow (\lambda x)_{m(t, r)} \in A \), \\
(iii) \( x_t, y_s \in A \Rightarrow ([x, y])_{m(t, s)} \in A \),

for all \( x, y \in L \), for all \( \lambda \in X \), for all \( t, s, r \in (0, 1] \).
4 \((\alpha, \beta)\)-fuzzy Lie algebras over an \((\alpha, \beta)\)-fuzzy field

Let \(\alpha\) and \(\beta\) denote any one of \(\varepsilon, q, \in \vee q\) unless otherwise specified.

**Definition 4.1.** Let \(X\) be a field and let \(F : X \rightarrow [0, 1]\) be a fuzzy subset of \(X\). Then \(F\) is called an \((\alpha, \beta)\)-fuzzy field of \(X\), if it satisfies the following conditions:

(i) for all \(\lambda, \gamma \in X\), \(\lambda \alpha F, \gamma \alpha F \Rightarrow (\lambda - \gamma)_{m(t,s)} \beta F\),

(ii) for all \(\lambda, \gamma \neq 0\) in \(X\), \(\lambda t \alpha F, \gamma s \alpha F \Rightarrow (\lambda \gamma - 1)_{m(t,s)} \beta F\),

for all \(t, s \in (0, 1]\).

**Definition 4.2.** Let \(L\) be a Lie algebra over a field \(X\), and let \(F : X \rightarrow [0, 1]\) be an \((\alpha, \beta)\)-fuzzy field of \(X\). Then a fuzzy subset \(A : L \rightarrow [0, 1]\) is called an \((\alpha, \beta)\)-fuzzy Lie algebra of \(L\) over an \((\alpha, \beta)\)-fuzzy field \(F\) of \(X\), if it satisfies the following conditions:

(i) \(x t \alpha A, y s \alpha A \Rightarrow (x - y)_{m(t,s)} \beta A\),

(ii) \(x t \alpha A, \lambda \alpha F \Rightarrow (\lambda x)_{m(t,s)} \beta A\),

(iii) \(x t \alpha A, y s \alpha A \Rightarrow ([x, y])_{m(t,s)} \beta A\),

for all \(x, y \in L\), for all \(\lambda \in X\), for all \(t, s, r \in (0, 1]\).

**Example 4.3.** In the real vector space \(\mathbb{R}^3\), define \([x, y] = x \times y\), where ‘\(\times\)’ is cross product of vectors for all \(x, y \in \mathbb{R}^3\). Then \(\mathbb{R}^3\) is a Lie algebra over the field \(\mathbb{R}\).

Define \(A : \mathbb{R}^3 \rightarrow [0, 1]\) for all \(x = (a, b, c) \in \mathbb{R}^3\) by

\[
A(a, b, c) = \begin{cases} 
1 & \text{if } a = b = c = 0, \\
0.5 & \text{if } a \neq 0, b = 0, c = 0, \\
0 & \text{otherwise},
\end{cases}
\]

and define \(F : \mathbb{R} \rightarrow [0, 1]\) for all \(\lambda \in \mathbb{R}\), by

\[
F(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \mathbb{Q}, \\
0.5 & \text{if } \lambda \in \mathbb{Q}(\sqrt{2}) - \mathbb{Q}, \\
0 & \text{if } \lambda \in \mathbb{R} - \mathbb{Q}(\sqrt{2}).
\end{cases}
\]

(i) Then by actual computation, it follows that \(F\) is an \((\varepsilon, \varepsilon)\)-fuzzy field of \(\mathbb{R}\) and \(A\) is an \((\varepsilon, \varepsilon)\)-fuzzy Lie algebra of \(\mathbb{R}^3\) over the \((\varepsilon, \varepsilon)\)-fuzzy field \(F\) of \(\mathbb{R}\). Also it can be verified that \(A\) is an \((\varepsilon, \varepsilon \vee q)\)-fuzzy Lie algebra of \(\mathbb{R}^3\) over an \((\varepsilon, \varepsilon \vee q)\)-fuzzy field \(F\) of \(\mathbb{R}\).

(ii) Let \(x = (1, 0, 0), y = (2, 0, 0), t = 0.4, s = 0.3\). Then \(A(x - y) = 0.5\) and \(m(t, s) = 0.3\). \(A(x - y) + m(t, s) < 1\). So \((x - y)_{m(t,s)} \gamma A\). Hence \(A\) is not an \((\varepsilon, q)\)-fuzzy Lie algebra.

(iii) Let \(x = (0, 0, 0), y = (2, 0, 0)\) be elements in \(\mathbb{R}^3\) and \(t = 0.4, s = 0.6\). Then \(x t q A\) and \(y s q A\). But \(A(x - y) + m(t, s) = 0.5 + 0.4 < 1\). This shows that \((x - y)_{m(t,s)} \gamma A\). Hence \(A\) is not a \((q, q)\)-fuzzy Lie algebra.

**Theorem 4.4.** Let \(X\) be a field. Then a fuzzy subset \(F : X \rightarrow [0, 1]\) is a fuzzy field if and only if \(F\) is an \((\varepsilon, \varepsilon)\)-fuzzy field of \(X\).

**Proof.** The result follows immediately from Lemma 3.2. \(\square\)

**Theorem 4.5.** Let \(L\) be a Lie algebra over a field \(X\). Then a fuzzy subset \(A\) of \(L\) is a fuzzy Lie algebra over a fuzzy field \(F\) of \(X\) if and only if \(A\) is an \((\varepsilon, \varepsilon)\)-fuzzy Lie algebra of \(L\) over an \((\varepsilon, \varepsilon)\)-fuzzy field \(F\) of \(X\).
Proof. The result follows immediately from Lemmas 3.2 and 3.3.\qed

**Theorem 4.6.** Let $X$ be a field and let $F : X \rightarrow [0, 1]$ be a fuzzy subset of $X$. Then $F$ is an $(\varepsilon, \in \vee q)$-fuzzy field of $X$ if and only if

1. for all $\lambda, \gamma$ in $X$, $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$,
2. for all $\lambda, \gamma \neq 0$ in $X$, $F(\lambda \gamma^{-1}) \geq m(F(\lambda), F(\gamma), 0.5)$.

**Proof.** Suppose that $F$ is an $(\varepsilon, \in \vee q)$-fuzzy field of $X$. It is clear that

$$m(F(\lambda), F(\gamma), 0.5) = m(m(F(\lambda), F(\gamma), 0.5)).$$

We consider two possibilities.

**Case 1.** Let $m(F(\lambda), F(\gamma)) < 0.5$. Then, $m(F(\lambda), F(\gamma), 0.5) = m(F(\lambda), F(\gamma))$. If possible, let $F(\lambda - \gamma) < m(F(\lambda), F(\gamma), 0.5) = m(F(\lambda), F(\gamma))$. Let $r, s \in (0, 1)$ be such that $F(\lambda - \gamma) < r < s < m(F(\lambda), F(\gamma))$. Then $F(\lambda) > r$, $F(\gamma) > s$ and so $\lambda r, \gamma s \in F$ and $F(\lambda - \gamma) < m(r, s)$ shows that $(\lambda - \gamma)m(r, s) \in F$ and $F(\lambda - \gamma) + m(r, s) < m(r, s) + m(r, s) < 1$ shows that $m(\lambda - \gamma)m(r, s) \notin qF$. Therefore, $(\lambda - \gamma)m(r, s) \notin qF$, a contradiction.

**Case 2.** Let $m(F(\lambda), F(\gamma)) \geq 0.5$. Then, $m(F(\lambda), F(\gamma), 0.5) = 0.5$. If possible, let $F(\lambda - \gamma) < 0.5$. Then $\lambda 0.5 \in F$, $\gamma 0.5 \in F$, but $(\lambda - \gamma)0.5 \notin qF$, a contradiction. Therefore, it follows that $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$. Similarly, (ii) is proved.

Conversely, suppose that conditions (i) and (ii) are satisfied by a fuzzy set $F$ of $X$. Let $\lambda \in F$, $\gamma \in F$, for $\lambda, \gamma \in X$ and $r, s \in (0, 1]$. Then $F(\lambda) \geq r$, $F(\gamma) \geq s$ and so $m(F(\lambda), F(\gamma)) \geq m(r, s)$. Since $F$ satisfies condition (i),

$$F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5) \geq m(r, s, 0.5).$$

Now consider the possibilities $m(r, s) \leq 0.5$ or $m(r, s) > 0.5$. If $m(r, s) \leq 0.5$, then $m(r, s, 0.5) = m(r, s)$ and $F(\lambda - \gamma) \geq m(r, s)$ and so $(\lambda - \gamma)m(r, s) \in F$. If $m(r, s) > 0.5$, then $m(r, s, 0.5) = 0.5$ and $F(\lambda - \gamma) \geq 0.5$. So, $F(\lambda - \gamma) + m(r, s) \geq 0.5 + m(r, s) > 0.5 + 0.5 = 1$ and hence $(\lambda - \gamma)m(r, s) \notin qF$. Therefore, it follows that if $\lambda r \in F$, $\gamma s \in F$, then $(\lambda - \gamma)m(r, s) \notin qF$. Similarly, if $\lambda r \in F$, $\gamma s \in F$ for all $\lambda, \gamma \neq 0$ in $X$, then $(\lambda - \gamma)m(r, s) \notin qF$. Hence $F$ is an $(\varepsilon, \in \vee q)$-fuzzy field of $X$. \qed

**Theorem 4.7.** Let $L$ be a Lie algebra over a field $X$. Then a fuzzy subset $A$ of $L$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \in \vee q)$-fuzzy field $F$ of $X$ if and only if

1. for all $x, y \in L$, $A(x - y) \geq m(A(x), A(y), 0.5)$,
2. for all $x \in L$, $A(\lambda x) \geq m(F(\lambda), A(x), 0.5)$,
3. for all $x, y \in L$, $A([x, y]) \geq m(A(x), A(y), 0.5)$.

**Proof.** Suppose that $A$ is an $(\varepsilon, \in \vee q)$-fuzzy Lie algebra over an $(\varepsilon, \in \vee q)$-fuzzy field $F$ of $X$. It is clear that $m(F(\lambda), A(x), 0.5) = m(F(\lambda), A(x), 0.5)$. We consider two possibilities.

**Case 1.** Let $m(F(\lambda), A(x)) < 0.5$. Then, $m(F(\lambda), A(x), 0.5) = m(F(\lambda), A(x))$. If possible, let $A(\lambda x) < m(F(\lambda), A(x), 0.5) = m(F(\lambda), A(x))$. Let $t \in (0, 1]$ be such that $A(\lambda x) < t < m(F(\lambda), A(x))$. Then, $F(\lambda) > t$ and $A(x) > t$. So, $\lambda t \in F$ and $x t \in A$. But $A(\lambda x) < t$ and $A(\lambda x) + t < t + t < 2m(F(\lambda), A(x)) < 1$. This shows that $(\lambda x)t \notin qA$, a contradiction.

**Case 2.** Let $m(F(\lambda), A(x)) \geq 0.5$. If possible, let $A(\lambda x) < m(F(\lambda), A(x), 0.5) = 0.5$. Then we have $\lambda 0.5 \in F$ and $x 0.5 \in A$, but $(\lambda x)0.5 \notin qA$, a contradiction. Therefore, it follows that $A(\lambda x) \geq m(F(\lambda), A(x), 0.5)$. Thus, (ii) is proved. Similarly, (i) and (iii) are proved.
Conversely, suppose that the conditions (i), (ii), and (iii) are satisfied by a fuzzy set \( A \) of \( L \). Let \( x_t \in A, y_s \in A \), for \( x, y \in L \) and \( t, s \in (0, 1] \). Then, \( A(x) \geq t, A(y) \geq s \) and so \( m(A(x), A(y)) \geq m(t, s) \). Since \( A \) satisfies condition (iii),

\[
A([x, y]) \geq m(A(x), A(y), 0.5) \geq m(t, s, 0.5).
\]

Now consider the possibilities \( m(t, s, 0.5) \leq 0.5 \) or \( m(t, s) > 0.5 \). If \( m(t, s) \leq 0.5 \), then \( m(t, s, 0.5) = m(t, s) \) and \( A([x, y]) \geq m(t, s) \), and so \( ([x, y])_{m(t, s)} \in A \). If \( m(t, s) > 0.5 \), then \( m(t, s, 0.5) = 0.5 \) and \( A([x, y]) \geq 0.5 \). So \( A([x, y]) + m(t, s) \geq 0.5 + m(t, s) > 0.5 + 0.5 = 1 \) and hence \( ([x, y])_{m(t, s)} \in \mathcal{Q}A \). Therefore, it follows that if \( x_t \in A, y_s \in A \), then \( ([x, y])_{m(t, s)} \in \mathcal{Q}A \). Similarly, if \( x_t \in A, y_s \in A \), then \( (x - y)_{m(t, s)} \in \mathcal{Q}A \) and if \( \lambda_r \in F, x_t \in A \), then \( (\lambda x)_{m(r, t)} \in \mathcal{Q}A \). Hence, \( A \) is an \((\varepsilon, \in \mathcal{Q})\)-fuzzy Lie algebra of \( L \) over an \((\varepsilon, \in \mathcal{Q})\)-fuzzy field \( F \) of \( X \).

**Proposition 4.8.** Let \( L \) be a Lie algebra over a field \( X \). Then every \((\varepsilon, \in)\)-fuzzy Lie algebra of \( L \) over an \((\varepsilon, \in \mathcal{Q})\)-fuzzy field of \( X \) is an \((\varepsilon, \in \mathcal{Q})\)-fuzzy Lie algebra of \( L \) over an \((\varepsilon, \in \mathcal{Q})\)-fuzzy field of \( X \).

**Proof.** Suppose \( A \) is an \((\varepsilon, \in)\)-fuzzy Lie algebra of \( L \) over an \((\varepsilon, \in)\)-fuzzy field \( F \) of \( X \). Let \( \lambda, \gamma \in X, r, s \in (0, 1] \). Since \( F \) is an \((\varepsilon, \in)\)-fuzzy field of \( X \), \( \gamma_s \in F \Rightarrow (\lambda - \gamma)_{m(r, s)} \in F \), then \( F(\lambda - \gamma) \geq m(r, s) \) shows that \( (\lambda - \gamma)_{m(r, s)} \in \mathcal{Q}F \). Similarly, \((\varepsilon, \in \mathcal{Q})\)-fuzzy Lie algebra, for \( x, y \in L, t, s \in (0, 1] \), \( x_t \in A, y_s \in A \Rightarrow ([x, y])_{m(t, s)} \in A \). Thus, \( A([x, y]) \geq m(t, s) \). Then by definition \( ([x, y])_{m(t, s)} \in \mathcal{Q}A \). Similarly, \( x_t \in A, y_s \in A \Rightarrow (x - y)_{m(t, s)} \in \mathcal{Q}A \) and \( x_t \in A, \lambda_s \in F \Rightarrow (\lambda x)_{m(t, s)} \in \mathcal{Q}A \). Hence \( A \) is an \((\varepsilon, \in \mathcal{Q})\)-fuzzy Lie algebra of \( L \) over an \((\varepsilon, \in \mathcal{Q})\)-fuzzy field \( F \) of \( X \).

**Remark 4.9.** The converse of this proposition may not be true as seen in the following example.

**Example 4.10.** Let \( L = \mathbb{R}^3 \) and \([x, y] = x \times y\), where ‘\( \times \)’ is cross product for all \( x, y \in L \). Then \( L \) is a Lie algebra over the field \( \mathbb{R} \). Define \( A : \mathbb{R}^3 \rightarrow [0, 1] \) for all \( x = (a, b, c) \in \mathbb{R}^3 \) by

\[
A(a, b, c) = \begin{cases} 
0.6 & \text{if } a = b = c = 0, \\
0.8 & \text{if } a \neq 0, b = 0, c = 0, \\
0.5 & \text{otherwise}, 
\end{cases}
\]

and define \( F : \mathbb{R} \rightarrow [0, 1] \) for all \( \lambda \in \mathbb{R} \) by

\[
F(\lambda) = \begin{cases} 
0.6 & \text{if } \lambda \in \mathbb{Q}, \\
0.8 & \text{if } \lambda \in \mathbb{Q}(\sqrt{2}) - \mathbb{Q}, \\
0.5 & \text{if } \lambda \in \mathbb{R} - \mathbb{Q}(\sqrt{2}). 
\end{cases}
\]

Then by Theorem 4.7, it follows that \( A \) is an \((\varepsilon, \in \mathcal{Q})\)-fuzzy Lie algebra of \( \mathbb{R}^3 \) over an \((\varepsilon, \in \mathcal{Q})\)-fuzzy field \( F \) of \( \mathbb{R} \).

But this is not an \((\varepsilon, \in)\)-fuzzy Lie algebra of \( \mathbb{R}^3 \) over an \((\varepsilon, \in)\)-fuzzy field of \( \mathbb{R} \). Let \( x = (1, 0, 0) \). Then \( A(1, 0, 0) = 0.8 > 0.65 > 0.62 \). So \( x_{0.65} \in A \) and \( x_{0.62} \in A \). But \( (x - x)_{m(0.65, 0.62)} = (0)_{0.62}^+ \mathcal{A} \). It is clear that \( A(0) + 0.62 = 0.6 + 0.62 > 1 \) and so \( (0)_{0.62} \in \mathcal{Q}A \). Therefore \( A \) is not an \((\varepsilon, \in)\)-fuzzy Lie algebra of \( \mathbb{R}^3 \) over an \((\varepsilon, \in)\)-fuzzy field \( F \) of \( \mathbb{R} \).
\textbf{Theorem 4.11.} Let $A$ be an $(\varepsilon, \varepsilon \lor q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon \lor q)$-fuzzy field $F$ of $X$ such that $M(A(x), F(\lambda)) < 0.5$ for all $x \in L$ and for all $\lambda \in X$. Then $A$ is an $(\varepsilon, \varepsilon)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon)$-fuzzy field $F$ of $X$.

\textbf{Proof.} Suppose that $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon \lor q)$-fuzzy field $F$ of $X$. Let $\lambda, \gamma \in X$ and $t, s \in [0, 1]$ be such that $\lambda t \in F$, $\gamma s \in F$. Then, $F(\lambda) \geq t, F(\gamma) \geq s$ and so $m(F(\lambda), F(\gamma)) \geq m(t, s)$. It follows from Theorem 4.6 that $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$.

Given that $M(A(x), F(\lambda)) < 0.5$ for all $x \in L$, for all $\lambda \in X$,

then, we have $m(F(\lambda), F(\gamma)) < 0.5$.

\textbf{Proof.} Suppose that $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon \lor q)$-fuzzy field $F$ of $X$. Let $\lambda, \gamma \in X$ and $t, s \in [0, 1]$ be such that $\lambda t \in F$, $\gamma s \in F$. Then, $F(\lambda) \geq t, F(\gamma) \geq s$ and so $m(F(\lambda), F(\gamma)) \geq m(t, s)$. It follows from Theorem 4.6 that $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$.

Given that $M(A(x), F(\lambda)) < 0.5$ for all $x \in L$, for all $\lambda \in X$,

\textbf{Proposition 4.12.} If $A$ is an $(\varepsilon, \varepsilon \lor q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon \lor q)$-fuzzy field $F$, then

\begin{enumerate}
  \item $A(0) \geq m(A(x), 0.5)$,
  \item $A(-x) \geq m(A(x), 0.5)$,
  \item $A(x + y) \geq m(A(x), A(y), 0.5)$.
\end{enumerate}

\textbf{Proof.} Let $x \in L$, $y \in L$. Then, from Theorem 4.7, the following hold.

\begin{enumerate}
  \item $A(0) = A([x, x]) \geq m(A(x), 0.5)$. So, $A(0) \geq m(A(x), 0.5)$.
  \item $A(-x) = A(0 - x) \geq m(A(0), A(x), 0.5) = m(m(A(x), 0.5), A(0)) = m(A(x), 0.5)$.
  \item $A(x + y) = A(x - (-y)) \geq m(A(x), A(-y), 0.5) \geq m(A(x), m(A(y), 0.5), 0.5) = m(A(x), A(y), 0.5)$.
\end{enumerate}

\textbf{Theorem 4.13.} Let $A$ be an $(\varepsilon, \varepsilon \lor q)$-fuzzy Lie algebra of $L$ over an $(\varepsilon, \varepsilon \lor q)$-fuzzy field $F$ of $X$. Then, for $t \in (0, 0.5]$, $A_t$ is a Lie subalgebra over $F_t$ when $F_t$ contains at least two elements.

\textbf{Proof.} For $t \in (0, 0.5]$, suppose $F_t$ contains at least two elements.

Let $\lambda, \gamma \in F_t$. Then $\lambda t \in F$, $\gamma s \in F$ and so $F(\lambda) \geq t, F(\gamma) \geq t$. This shows that $m(F(\lambda), F(\gamma)) \geq t$ and so $m(F(\lambda), F(\gamma), 0.5) \geq m(t, 0.5)$. Therefore,

$F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5) \geq m(t, 0.5) = t$
and hence \((\lambda - \gamma)I_1 = F\). Thus, \(\lambda - \gamma \in F_1\). Similarly, \(\lambda\gamma^{-1} \in F_1\) for all \(\lambda, \gamma \neq 0\) in \(F_1\). Therefore, \(F_1\) is a subfield of \(X\).

Suppose \(x, y \in A_t\). Then \(A(x) \geq t, A(y) \geq t\) and \(m(A(x), A(y), 0.5) \geq m(t, 0.5) = t\). So \(A(x + y) \geq m(A(x), A(y), 0.5) \geq t\) and hence \((x + y) \in A_t\). Let \(\lambda \in F_1, x \in A_t\). Then \(F(\lambda) \geq t, A(x) \geq t\) and \(m(F(\lambda), A(x)) \geq t\). Thus, \(m(F(\lambda), A(x), 0.5) \geq t\) and so \(A(\lambda x) \geq m(F(\lambda), A(x), 0.5) \geq t\). Hence, \(\lambda x \in A_t\).

Similarly, for \(x, y \in A_t\), \([x, y] \in A_t\). Therefore, \(A_t\) is a Lie subalgebra over the field \(F_t\). \(\square\)

Let \(f : L \rightarrow L'\) be a function. If \(A\) and \(B\) are fuzzy subsets of \(L\) and \(L'\), respectively, then \(f(A)\) and \(f^{-1}(B)\) are defined using Zadeh’s extension principle [6]. If \(\alpha\) is one of \(\in, \in\dot{\varepsilon}\), \(\in\cap\varepsilon\) or \(\forall\varepsilon\), it follows that \(x_\alpha f^{-1}(B)\) if and only if \((f(x))_\alpha B\) for all \(x \in L\) and for all \(t \in [0, 1]\).

**Theorem 4.14.** Let \(L\) and \(L'\) be Lie algebras over a field \(X\) and let \(f : L \rightarrow L'\) be a homomorphism. If \(B\) is an \((\in, \in\dot{\varepsilon})\)-fuzzy Lie algebra of \(L'\) over an \((\in, \in\dot{\varepsilon})\)-fuzzy field \(F\) of \(X\), then \(f^{-1}(B)\) is an \((\in, \in\dot{\varepsilon})\)-fuzzy Lie algebra of \(L\) over the \((\in, \in\dot{\varepsilon})\)-fuzzy field \(F\) of \(X\).

**Proof.** Let \(x, y \in L\) and \(t, s \in [0, 1]\) be such that \(x_t \in f^{-1}(B)\) and \(y_s \in f^{-1}(B)\). Then \((f(x))_t \in B\), \((f(y))_s \in B\). Since \(B\) is an \((\in, \in\dot{\varepsilon})\)-fuzzy Lie algebra of \(L'\) over an \((\in, \in\dot{\varepsilon})\)-fuzzy field \(F\) of \(X\),

\[
(f(x - y))_{m(t, s)} = (f(x) - f(y))_{m(t, s)} \in \in\dot{\varepsilon}B.
\]

So we have \((x - y)_{m(t, s)} \in \in\dot{\varepsilon}f^{-1}(B)\). Similarly, \([x, y]_{m(t, s)} \in \in\dot{\varepsilon}f^{-1}(B)\).

Let \(\lambda \in X\), \(x, y \in L\) and \(r, t \in [0, 1]\) be such that \(\lambda x_t \in F\) and \(x_t \in f^{-1}(B)\). Then \((f(x))_t \in B\) and so

\[
(f(\lambda x))_{m(r, t)} = (\lambda f(x))_{m(r, t)} \in \in\dot{\varepsilon}B
\]

and hence \((\lambda x)_{m(r, t)} \in \in\dot{\varepsilon}f^{-1}(B)\).

Therefore, \(f^{-1}(B)\) is an \((\in, \in\dot{\varepsilon})\)-fuzzy Lie algebra of \(L\) over the \((\in, \in\dot{\varepsilon})\)-fuzzy field \(F\) of \(X\). \(\square\)

**Definition 4.15.** A fuzzy set \(\mu\) of a set \(Y\) is said to possess sup property if for every nonempty set \(S\) of \(Y\), there exists \(x_0 \in S\) such that

\[
\mu(x_0) = \text{Sup} \{\mu(x) \mid x \in S\}.
\]

**Theorem 4.16.** Let \(L\) and \(L'\) be Lie algebras over a field \(X\) and let \(f : L \rightarrow L'\) be an onto homomorphism. Let \(A\) be an \((\in, \in\dot{\varepsilon})\)-fuzzy Lie algebra of \(L\) over an \((\in, \in\dot{\varepsilon})\)-fuzzy field \(F\) of \(X\), which satisfies the sup property. Then \(f(A)\) is an \((\in, \in\dot{\varepsilon})\)-fuzzy Lie algebra of \(L'\) over the \((\in, \in\dot{\varepsilon})\)-fuzzy field \(F\) of \(X\).

**Proof.** Let \(a, b \in L'\) and \(t, s \in [0, 1]\) be such that \(a_t \in f(A)\) and \(b_s \in f(A)\). Then \(f(A)(a) \geq t\) and \(f(A)(b) \geq s\) and so

\[
\text{Sup} \{A(z) \mid z \in f^{-1}(a)\} \geq t \quad \text{and} \quad \text{Sup} \{A(w) \mid w \in f^{-1}(b)\} \geq s.
\]

Since \(f\) is onto, \(f^{-1}(a)\) and \(f^{-1}(b)\) are nonempty subsets of \(L\) and by the sup property of \(A\), there exists \(x \in f^{-1}(a)\) and \(y \in f^{-1}(b)\) such that

\[
A(x) = \text{Sup} \{A(z) \mid z \in f^{-1}(a)\} \quad \text{and} \quad A(y) = \text{Sup} \{A(w) \mid w \in f^{-1}(b)\},
\]
then \( x_t \in A \) and \( y_s \in A \). Since \( A \) is an \((\in, \in \lor)\)-fuzzy Lie algebra of \( L \) over an \((\in, \in \lor)\)-fuzzy field \( F \) of \( X \), we have \( ([x, y])_{m(t,s)} \in \lor A \) and so \( A([x, y]) \geq m(t, s) \) or \( A([x, y]) + m(t, s) > 1 \). Now \( f(x) = a, f(y) = b \) and so \([x, y] \in f^{-1}([a, b])\). Therefore,

\[
f(A)([a, b]) = \text{Sup} \left\{ A(z) \mid z \in f^{-1}([a, b]) \right\} \geq A([x, y])
\]

and so \( f(A)([a, b]) \geq m(t, s) \) or \( f(A)([a, b]) + m(t, s) > 1 \). Thus, \(([a, b])_{m(t,s)} \in \lor q f(A)\). Also \((x - y)_{m(t,s)} \in \lor A \) shows that \((a - b)_{m(t,s)} \in \lor q f(A)\).

Let \( \lambda \in X, b \in L' \) and \( r, s \in (0, 1] \) be such that \( \lambda_r \in F \) and \( b_s \in f(A) \). Then it follows that \( \lambda_r \in F \) and \( y_s \in A \). So \((\lambda y)_{m(r,s)} \in \lor A \). Thus, \( A(\lambda y) \geq m(r, s) \) or \( A(\lambda y) + m(r, s) > 1 \). But \( f(A)(\lambda b) = \text{Sup} \{ A(w) \mid w \in f^{-1}(\lambda b) \} \geq A(\lambda y) \). This shows that \((\lambda b)_{m(r,s)} \in \lor q f(A)\).

Therefore, \( f(A) \) is an \((\in, \in \lor)\)-fuzzy Lie algebra of \( L' \) over the \((\in, \in \lor)\)-fuzzy field \( F \) of \( X \).

\[ \square \]

**References**


Received July 14, 2009
Revised September 15, 2009