

## Fuzzy $n$ -Lie Algebras

Davvaz B<sup>1</sup> and Dudek WA<sup>2\*</sup>

<sup>1</sup>Department of Mathematics, Yazd University, Yazd, Iran

<sup>2</sup>Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspińskiego 27, 50-370 Wrocław, Poland

### Abstract

Properties of fuzzy subalgebras and ideals of  $n$ -ary Lie algebras are described. Methods of construction fuzzy ideals are presented. Connections with various fuzzy quotient  $n$ -Lie algebras are proved.

**Keywords:** Fuzzy set;  $n$ -ary Lie algebra; Subalgebra; Ideal; Fuzzy ideal

### Introduction

In 1985 Filippov [1] proposed a generalization of the concept of a Lie algebra by replacing the binary operation by  $n$ -ary one. He defined an  $n$ -ary Lie algebra structure on a vector space  $L$  as an operation which associates with each  $n$ -tuple  $(x_1, \dots, x_n)$  of elements in  $L$  another element  $[x_1, \dots, x_n]$  which is  $n$ -linear, skew-symmetric:

$$[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = \text{sign}(\sigma)[x_1, \dots, x_n]$$

and satisfies the generalized Jacobi identity (called also the Filippov identity):

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n],$$

where  $\sigma \in S_n$ .

Now, such structures are also called  $n$ -Lie algebras or Filippov algebras. For  $n=2$  we obtain a classical Lie algebras.

Note that such an  $n$ -ary operation, realized on the smooth function algebra of a manifold and additionally assumed to be an  $n$ -derivation, is an  $n$ -Poisson structure. This general concept, however, was not introduced neither by Filippov, nor by other mathematicians that time. It was done much later in 1994 by Takhtajan [2] in order to formalize mathematically the  $n$ -ary generalization of Hamiltonian mechanics proposed by Nambu [3]. Apparently Nambu was motivated by some problems of quark dynamics and the  $n$ -bracket operation he considered was:

$$[f_1, \dots, f_n] := \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

where  $L = R[x_1, \dots, x_n]$  is the vector space of polynomials in  $n$ -variables.

Nambu does not mention that the  $n$ -bracket operation satisfies the generalized Jacobi identity but Filippov reports this operation in his paper [1] among other examples of  $n$ -Lie algebras. The formal proof is given in [4].

Ternary Lie algebras were studied [5,6]. For other generalizations and applications see ref. [7].

The study of fuzzy Lie algebras was initiated in refs. [8,9], and continued in various directions by many authors (for example [10-12]). The study of fuzzy  $n$ -ary algebras was initiated by Dudek [13]. Davvaz and Dudek described fuzzy  $n$ -ary groups as a generalization of

Rosenleld's fuzzy groups [14].

In this paper we describe fuzzy  $n$ -ary Lie algebras.

### Preliminaries

Let  $X$  be a non-empty set. A fuzzy subset  $\mu$  of  $X$  is a function  $\mu: X \rightarrow [0,1]$ . Let  $\mu$  and  $\lambda$  be two fuzzy subsets of  $X$ , we say that  $\mu$  is contained in  $\lambda$ , if  $\mu(x) \leq \lambda(x)$  for all  $x \in X$ . The set  $\bar{\mu}_t = \{x \in X | \mu(x) \geq t\}$ ,  $t \in [0,1]$  is called a level subset of  $\mu$ .

#### Definition 2.1

Let  $V$  be a vector space over a field  $F$ . A fuzzy subset  $\mu$  of  $V$  is called a fuzzy subspace of  $V$  if for all  $x, y \in V$  and  $\alpha \in F$ , the following conditions are satisfied:

- $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$  for all  $x, y \in V$ ,
- $\mu(\alpha x) \geq \mu(x)$  for all  $x \in V, \alpha \in F$ .

Note that the second condition implies,  $\mu(-x) \geq \mu(x)$  for all  $x \in V$ ,

#### Lemma 2.2

If  $\mu$  is a fuzzy subspace of a vector space  $V$ , then  $\mu(x) \leq \mu(0)$  for all  $x \in V$ , and

- $\mu(x) = \mu(-x)$ ,
- $\mu(x-y) = \mu(0) \Rightarrow \mu(x) = \mu(y)$ ,
- $\mu(x) < \mu(y) \Rightarrow \mu(x-y) = \mu(x) = \mu(y-x)$

for all  $x, y \in V$ .

**Proof.** Directly from the definition we obtain  $\mu(x) \leq \mu(0)$  and  $\mu(x) = \mu(-x)$ . Moreover, for all  $x, y \in V$  we have

$$\begin{aligned} \min\{\mu(x-y), \mu(y)\} &\geq \min\{\min\{\mu(x), \mu(-y)\}, \mu(y)\} = \min\{\mu(x), \mu(y)\} \\ &= \min\{\mu((x-y)+y), \mu(y)\} \geq \min\{\min\{\mu(x-y), \mu(y)\}, \mu(y)\} \\ &= \min\{\mu(x-y), \mu(y)\}, \end{aligned}$$

which implies

**\*Corresponding author:** Dudek WA, Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyspińskiego 27, 50-370 Wrocław, Poland, Tel: 71 320-31-62; Fax: (+48)-(71)-328-07-51; E-mail: [wieslaw.dudek@pwr.edu.pl](mailto:wieslaw.dudek@pwr.edu.pl)

Received April 28, 2017; Accepted June 05, 2017; Published June 12, 2017

**Citation:** Davvaz B, Dudek WA (2017) Fuzzy  $n$ -Lie Algebras. J Generalized Lie Theory Appl 11: 268. doi: 10.4172/1736-4337.1000268

**Copyright:** © 2017 Davvaz B, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

$$\min\{\mu(x-y), \mu(y)\} = \min\{\mu(x), \mu(y)\}.$$

Similarly

$$\min\{\mu(x-y), \mu(x)\} = \min\{\mu(x), \mu(y)\}.$$

Hence

$$\min\{\mu(x-y), \mu(y)\} = \min\{\mu(x-y), \mu(x)\} = \min\{\mu(x), \mu(y)\}.$$

This for  $\mu(x-y)=\mu(0)$  gives  $\mu(x)=\mu(y)$ , and  $\mu(x-y)=\mu(x)$  for  $\mu(x)<\mu(y)$ .

**Theorem 2.3**

For a fuzzy subset  $\mu$  of a vector space  $V$ , the following statements are equivalent.

- $\mu$  is a fuzzy subspace of  $V$ .
- Each non-empty  $\overline{\mu}_t$  is a subspace of  $V$ .

This theorem firstly proved in ref. [15] is a consequence of the Transfer Principle for fuzzy sets described in ref. [16].

Let  $\{\mu_i\}_{i \in I}$  be a collection of fuzzy subsets of  $X$ . Then, we define the fuzzy subsets  $\bigcap_{i \in I} \mu_i$  and  $\bigcup_{i \in I} \mu_i$  by:

$$\begin{aligned} \left(\bigcap_{i \in I} \mu_i\right)(x) &= \inf_{i \in I} \{\mu_i(x)\} \text{ for all } x \in X, \\ \left(\bigcup_{i \in I} \mu_i\right)(x) &= \sup_{i \in I} \{\mu_i(x)\} \text{ for all } x \in X. \end{aligned}$$

**Fuzzy Subalgebras and Ideals**

Recall that a non-empty subset  $S$  of an  $n$ -Lie algebra  $L$  is its subalgebra if it is a subspace of a vector space  $L$  and  $[x_1, \dots, x_n] \in S$  for all  $x_1, \dots, x_n \in S$ .

A subspace  $S$  of an  $i$ -ideal of  $L$  if for all  $x_1, \dots, x_{n \in L}$  and  $y \in S$  we have  $[x_1, \dots, x_{[i-1]}, y, x_{[i+1]}, \dots, x_n] \in L$ .

Two  $n$ -Lie algebras  $L_1, L_2$  over the same field  $F$  are isomorphic if there exists a vector space isomorphism  $\varphi: L_1 \rightarrow L_2$  such that for all  $\varphi([x_1, \dots, x_n]) = [\varphi(x_1), \dots, \varphi(x_n)]$  for all  $x_1, \dots, x_n \in L$ .

Let  $L$  be an  $n$ -Lie algebra. Fixing in  $[x_1, x_2, \dots, x_n]$  elements  $x_2, \dots, x_{[n-1]}$  we obtain a new binary operation  $\langle x, y \rangle = [x, x_2, \dots, x_{[n-1]}, y]$  with the property  $\langle x_k, y \rangle = \langle y, x_k \rangle = 0$  for all  $k=2, \dots, n-1$  and all  $y \in L$ . It is easily to see that  $L$  with respect to this new operation is an classical Lie algebra. It is called a binary retract. Fixing various  $x_2, \dots, x_{[n-1]}$  we obtain various (generally non-isomorphic) retracts. Obviously, any subalgebra (ideal) of an  $n$ -Lie algebra is a subalgebra (ideal) of each binary retract of  $L$ . The converse is not true. Hence results obtained for  $n$ -Lie algebras are essential generalizations of results proved for Lie algebras.

Basing on the idea of fuzzyfications of algebras with one  $n$ -ary operation proposed in ref. [13] we present a fuzzyfication of  $n$ -Lie algebras.

**Definition 3.1**

Let  $L$  be an  $n$ -Lie algebra. A fuzzy subalgebra of  $L$  is a fuzzy subspace  $\mu$  such that

$$\mu([x_1, \dots, x_n]) \geq \min\{\mu(x_1), \dots, \mu(x_n)\} \text{ for all } x_1, \dots, x_n \in L.$$

**Definition 3.2**

Let  $L$  be an  $n$ -Lie algebra. A fuzzy ideal of  $L$  is a fuzzy subspace  $\mu$  such that

$$\mu([x_1, \dots, x_n]) \geq \mu(x_i) \text{ for all } x_1, \dots, x_n \in L \text{ and } 1 \leq i \leq n.$$

The following facts are obvious. Their proofs are very similar to the proofs of analogous results for fuzzy  $n$ -ary systems [13] and fuzzy Lie algebras [9].

**Proposition 3.3:**

A fuzzy subspace  $\mu$  of an  $n$ -Lie algebra  $L$  is its fuzzy ideal if and only if

$$\mu([x_1, \dots, x_n]) \geq \max\{\mu(x_1), \dots, \mu(x_n)\} \tag{1}$$

for all  $x_1, \dots, x_n \in L$ .

**Proposition 3.4:** If  $\mu$  is a fuzzy ideal of an  $n$ -Lie algebra  $L$ , then

$$L_\mu = \{x \in L | \mu(x) = \mu(0)\}$$

is an ideal of  $L$  contained in every non-empty level subset of  $\mu$ .

**Proposition 3.5:** Let  $\mu$  and  $\lambda$  be two fuzzy ideals of an  $n$ -Lie algebra  $L$  such that  $\mu(0)=\lambda(0)$ . Then  $L_{\mu \cap \lambda} = L_\mu \cap L_\lambda$ .

**Theorem 3.6**

Let  $\varphi: L \rightarrow L'$  be an  $n$ -Lie algebra homomorphism of an  $n$ -Lie algebra  $L$  onto an  $n$ -Lie algebra  $L'$ . Then the following conditions hold:

- if  $\mu$  is a fuzzy ideal of  $L$ , then  $\varphi(\mu)$  is a fuzzy ideal of  $L'$ ,
- if  $\nu$  is a fuzzy ideal of  $L'$  then  $\varphi^{-1}(\nu)$  is a fuzzy ideal of  $L$ ,
- $\overline{\varphi^{-1}(\nu)}_t = \varphi^{-1}(\overline{\nu}_t)$  for every  $t \in [0, 1]$  and every fuzzy ideal  $\nu$  of  $L'$ .

**Proposition 3.7:** Let  $L$  be an  $n$ -Lie algebra. Then the intersection of any family of fuzzy subalgebras (ideals) of  $L$  is again a fuzzy subalgebra (ideal) of  $L$ .

It is easy to see that the union of fuzzy subalgebras (ideals) of an  $n$ -Lie algebra  $L$  is not a fuzzy subalgebra (ideal) of  $L$ , in general. But we have the following proposition on the union of fuzzy subalgebras (ideals) of  $L$ .

**Proposition 3.8:** Let  $\{\mu_n\}$  be a chain of fuzzy subalgebras (ideals) of an  $n$ -Lie algebra  $L$ . Then  $\bigcup_n \mu_n$  is a fuzzy subalgebra (ideal) of  $L$ .

**Theorem 3.9**

For a fuzzy subset  $\mu$  of an  $n$ -Lie algebra  $L$ , the following statements are equivalent.

- $\mu$  is a fuzzy subalgebra (ideal) of  $L$ .
- Each non-empty  $\overline{\mu}_t$ , is a subalgebra (ideal) of  $L$ .

**Proof.** Let  $\mu$  be a fuzzy ideal of  $L$ . Since  $\mu$  is a fuzzy subspace of  $L$ , by Theorem 2.3, each non-empty  $\overline{\mu}_t$  is a subspace of  $L$ . Therefore, it is enough to prove that  $[\overline{\mu}_t, \dots, \overline{\mu}_t, \overline{\mu}_t, \dots, \overline{\mu}_t] \subseteq \overline{\mu}_t$ . For every  $y \in \overline{\mu}_t$  and  $x_1, \dots, x_n \in L$  we show that  $[x_1, \dots, x_{[i-1]}, y, x_{[i+1]}, \dots, x_n] \in \overline{\mu}_t$ . Since  $\mu$  is a fuzzy ideal, we have

$$t \leq \mu(y) \leq \mu([x_1, \dots, x_{[i-1]}, y, x_{[i+1]}, \dots, x_n])$$

$$\text{and so } [x_1, \dots, x_{[i-1]}, y, x_{[i+1]}, \dots, x_n] \in \overline{\mu}_t.$$

Conversely, assume that every non-empty  $\overline{\mu}_t$  is an ideal of  $L$ . Therefore,  $\overline{\mu}_t$  is a subspace of  $L$  and so by Theorem 2.3,  $\mu$  is a fuzzy subspace of  $L$ . Now, for every  $y \in L$ , we put  $t_0 = \mu(y)$ . Then,  $y \in \overline{\mu}_{t_0}$ . Therefore,

for every  $x_1, \dots, x_n \in L$  we have  $[x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n] \in \overline{\mu_{t_0}}$  which implies that  $\mu([x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n]) \geq t_0 = \mu(y)$ . So,  $\mu$  is a fuzzy ideal.

For subalgebras the proof is analogous.

**Proposition 3.10:** Let  $L$  be an  $n$ -Lie algebra and  $\mu$  be a fuzzy subalgebra of  $L$ . Let  $\overline{\mu_{t_1}}$  and  $\overline{\mu_{t_2}}$  (with  $t_1 < t_2$ ) be any two level subalgebras of  $\mu$ . Then  $\overline{\mu_{t_1}} = \overline{\mu_{t_2}}$  if and only if there is no  $x$  in  $L$  such that  $t_1 \leq \mu(x) < t_2$ .

**Theorem 3.11**

Let  $\{S | \lambda \in \lambda\}$ , where  $\emptyset \neq \lambda \subseteq [0,1]$ , be a collection of ideals of an  $n$ -Lie algebra  $L$  such that

- (i)  $L = \bigcup_{\lambda \in \Lambda} S_\lambda$
- (ii)  $\alpha > \beta \Leftrightarrow S_\alpha \subset S_\beta$  for all  $\alpha, \beta \in \Lambda$

Then  $\mu$  defined by

$$\mu(x) = \sup\{\lambda \in \Lambda \mid x \in S_\lambda\}$$

is a fuzzy ideal of  $L$ .

**Proof.** By Theorem 3.9, it is sufficient to show that every non-empty level  $\overline{\mu_\alpha}$  is an ideal of  $L$ .

Let  $\overline{\mu_\alpha} \neq \emptyset$  for some fixed  $\alpha \in [0,1]$ . Then

$$\alpha = \sup\{\lambda \in \Lambda \mid \lambda < \alpha\} = \sup\{\lambda \in \Lambda \mid S_\alpha \subset S_\lambda\}$$

or

$$\alpha \neq \sup\{\lambda \in \Lambda \mid \lambda < \alpha\} = \sup\{\lambda \in \Lambda \mid S_\alpha \subset S_\lambda\}.$$

In the first case we have  $\overline{\mu_\alpha} = \bigcup_{\lambda < \alpha} S_\lambda$ , because

$$x \in \overline{\mu_\alpha} \Leftrightarrow (x \in S_\lambda \text{ for all } \lambda < \alpha) \Leftrightarrow x \in \bigcap_{\lambda < \alpha} S_\lambda.$$

In the second, there exists  $\varepsilon > 0$  such that  $(\alpha - \varepsilon, \lambda) \cap \lambda = \emptyset$ . In this case

$\overline{\mu_\alpha} = \bigcup_{\lambda \geq \alpha} S_\lambda$ . Indeed, if  $x \in \bigcup_{\lambda \geq \alpha} S_\lambda$ , then  $x \in S_\lambda$  for some  $\lambda \geq \alpha$ , which gives

$$\mu(x) \geq \lambda \geq \alpha. \text{ Thus } x \in \overline{\mu_\alpha}, \text{ i.e., } \bigcup_{\lambda \geq \alpha} S_\lambda \subseteq \overline{\mu_\alpha}.$$

Conversely, if  $x \notin S_\lambda$ , then  $x \notin S_\lambda$  for all  $\lambda \geq \alpha$ , which implies  $x \notin S_\lambda$  for all  $\lambda > \alpha - \varepsilon$ , i.e., if  $x \in S_\lambda$  then  $\lambda \leq \alpha - \varepsilon$ . Thus  $\mu(x) \leq \alpha - \varepsilon$ . Therefore  $x \notin \overline{\mu_\alpha}$ . Hence  $\overline{\mu_\alpha} \subseteq \bigcup_{\lambda \geq \alpha} S_\lambda$ , and consequently  $\overline{\mu_\alpha} = \bigcup_{\lambda \geq \alpha} S_\lambda$ . This completes our proof.

**Theorem 3.12**

Let  $\mu$  be a fuzzy subset defined on an  $n$ -Lie algebra  $L$  and let  $Im(\mu) = \{t_0, t_1, t_2, \dots\}$ , where  $1 \geq t_0 > t_1 > t_2 \dots \geq 0$ . If  $S_0 \subset S_1 \subset S_2 \dots$  are subalgebras (ideals) of  $L$  such that  $\mu(S_k \setminus S_{k-1}) = t_k$  for  $k=0,1,2,\dots$ , where  $S_{-1} = \emptyset$ , then  $\mu$  is a fuzzy subalgebra (ideal) of  $L$ .

**Proof.** First consider the case when all  $S_i$  are subalgebras. If  $[x_1, \dots, x_n] \in L \setminus \bigcup_k S_k$  then also at least one of  $x_1, \dots, x_n$  is in  $L \setminus \bigcup_k S_k$  because in the opposite case  $x_1, \dots, x_n$  and  $[x_1, \dots, x_n]$  will be in some  $S_k$ . So, in this case

$$\mu([x_1, \dots, x_n]) = 0 = \min\{\mu(x_1), \dots, \mu(x_n)\}.$$

It is clear that for arbitrary elements  $x_1, \dots, x_n \in L$  there exists only one  $k$  such that  $[x_1, \dots, x_n] \in S_k \setminus S_{k-1}$  and only one  $k_i$  such that  $x_i \in S_{k_i} \setminus S_{k_i-1}$ .

Thus  $\mu([x_1, \dots, x_n]) = t_k, \mu(x_i) = t_{k_i}$ .

Suppose  $t_{k_i} > t_k$  for all  $i=1,2,\dots,n$ . Then, by the assumption,  $k_i < k$  and  $S_{k_i} \subseteq S_s \subseteq S_{k-1} \subset S_k$ , where  $s = \max\{k_1, \dots, k_n\}$ . Hence  $x_1, \dots, x_n \in S_{k-1}$  and, in the consequence,  $[x_1, \dots, x_n] \in S_{k-1}$  because  $S_{k-1}$  is a subalgebra. This is a contradiction. Therefore there is at least one  $t_{k_i} \leq t_k$ . In this case  $\mu([x_1, \dots, x_n]) = t_k \geq t_{k_i} \geq \min\{\mu(x_1), \dots, \mu(x_n)\}$ . Since  $\mu$  also is a fuzzy subspace of a vector space  $L$ , it is a fuzzy subalgebra of  $L$ .

Now, let all  $S_i$  be ideals and let  $[x_1, \dots, x_n] \in S_k \setminus S_{k-1}$  for some  $x_1, \dots, x_n \in L$ . Then these  $x_1, \dots, x_n$  are in  $L \setminus S_{k-1}$ . If not, then there exists  $x_i \in S_{k-1}$ . But in this case  $[x_1, \dots, x_n] \in S_{k-1}$  because  $S_{k-1}$  is an ideal. This is a contradiction. So, all  $x_i \in L \setminus S_{k-1}$ . Hence  $\max\{\mu(x_1), \dots, \mu(x_n)\} \leq t_k = \mu([x_1, \dots, x_n])$ . Now, if  $[x_1, \dots, x_n] \in L \setminus \bigcup_k S_k$ , then also all  $x_1, \dots, x_n$  are in  $L \setminus \bigcup_k S_k$ . Thus  $\max\{\mu(x_1), \dots, \mu(x_n)\} = \mu([x_1, \dots, x_n])$ . This completes the proof that  $\mu$  is a fuzzy ideal.

**Corollary 3.13**

For any chain  $S_0 \subset S_1 \subset S_2 \dots$  of subalgebras (ideals) of an  $n$ -Lie algebra  $L$  and any chain of reals  $1 \geq t_0 > t_1 > \dots \geq 0$  there exists a fuzzy subalgebra (ideal)  $\mu$  of  $L$  such that  $\overline{\mu_{t_k}} = S_k$ .

**Theorem 3.14**

Let  $Im(\mu) = \{t_i \mid i \in I\}$  be the image of a fuzzy subalgebra (ideal)  $\mu$  of an  $n$ -Lie algebra  $L$ . Then

- (a) There exists a unique  $t_0 \in Im(\mu)$  such that  $t_0 \geq t_i$  for all  $t_i \in Im(\mu)$ ,
- (b)  $L$  is the set-theoretic union of all  $\overline{\mu_{t_i}}, t_i \in Im(\mu)$ ,
- (c)  $\Omega = \{\overline{\mu_{t_i}} \mid t_i \in Im(\mu)\}$  is linearly ordered by inclusion,
- (d)  $\Omega$  contains all level subalgebras (ideals) of  $\mu$  if and only if  $\mu$  attains its infimum on all subalgebras (ideals) of  $L$ .

**Proof.** (a) Follows from the fact that  $t_0 = \mu(0) \geq \mu(x)$  for all  $x \in L$ .

(b) If  $x \in L$ , then  $\mu(x) = t_x \in Im(\mu)$ . Thus  $x \in \bigcup \overline{\mu_{t_i}} \subseteq L$ , where  $t_i \in Im(\mu)$ , which proves (b).

(c) Since  $\overline{\mu_{t_i}} \subseteq \overline{\mu_{t_j}} \Leftrightarrow t_i \geq t_j$  for  $i, j \in I$ , then  $\Omega$  linearly ordered by inclusion.

(d) Suppose that  $\Omega$  contains all levels of  $\mu$ . Let  $S$  be a subalgebra (ideal) of  $L$ . If  $\mu$  is constant on  $S$ , then we are done. Assume that  $\mu$  is not constant on  $S$ . We have two cases: (1)  $S=L$  and (2)  $S \neq L$ . For  $S=L$  let  $\beta = \inf Im(\mu)$ . Then  $\beta \leq t \in Im(\mu)$ , i.e.,  $\overline{\mu_\beta} \supseteq \overline{\mu_t}$  for all  $t \in Im(\mu)$ . But  $\overline{\mu_0} = L \in \Omega$  because  $\Omega$  contains all levels of  $\mu$ . Hence there exists  $t' \in Im(\mu)$  such that  $\overline{\mu_{t'}} = L$ . It follows that  $\overline{\mu_\beta} \supset \overline{\mu_{t'}} = L$  so that  $\overline{\mu_\beta} = \overline{\mu_{t'}} = L$  because every level of  $\mu$  is a subalgebra (resp. ideal) of  $L$ .

Now it sufficient to show that  $\beta = t'$ . If  $\beta < t'$ , then there exists  $t'' \in Im(\mu)$  such that  $\beta \leq t'' < t'$ . This implies  $\overline{\mu_{t''}} \supset \overline{\mu_{t'}} = L$ , which is a contradiction. Therefore  $\beta = t' \in Im(\mu)$ .

In the case  $S \neq L$  we consider the fuzzy set  $\mu_s$  defined by

$$\mu_s(x) = \begin{cases} \alpha & \text{for } x \in S, \\ 0 & \text{for } x \in L \setminus S. \end{cases}$$

Clearly  $\mu_s$  is a fuzzy subalgebra (ideal) of  $L$  if  $S$  is a subalgebra (ideal).

Let

$$J = \{i \in I \mid \mu(x) = t_i \text{ for some } x \in S\}.$$

Then  $\Omega_S = \{\overline{\mu_{t_i}} \mid i \in J\}$  contains (by the assumption) all levels of  $\mu_S$ . This means that there exists  $x_0 \in S$  such that  $\mu(x_0) = \inf\{\mu_S(x) \mid x \in S\}$ , i.e.,  $\mu(x_0) = \mu_S(x)$  for some  $x \in S$ . Hence  $\mu$  attains its infimum on all subalgebras (ideals) of  $L$ .

To prove the converse let  $\overline{\mu_\alpha}$  be a level subalgebra of  $\mu$ . If  $\alpha = t$  for some  $t \in \text{Im}(\mu)$ , then  $\overline{\mu_\alpha} \in \Omega$ . If  $\alpha \neq t$  for all  $t \in \text{Im}(\mu)$ , then there does not exist  $x \in L$  such that  $\mu(x) = \alpha$ .

Let  $S = \{x \in L \mid \mu(x) > \alpha\}$ . Obviously  $0 \in S$  and  $\mu(x_i) > \alpha$  for all  $x_i \in S$ . From the fact that  $\mu$  is a fuzzy subalgebra we obtain

$$\mu([x_1, \dots, x_n]) \geq \min\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\} > \alpha,$$

which proves  $[x_1, \dots, x_n] \in S$ . Hence  $S$  is a subalgebra. By hypothesis, there exists  $y \in S$  such that  $\mu(y) = \inf\{\mu(x) \mid x \in S\}$ . But  $\mu(y) \in \text{Im}(\mu)$  implies  $\mu(y) = t'$  for some  $t' \in \text{Im}(\mu)$ . Hence  $\inf\{\mu(x) \mid x \in S\} = t' > \alpha$ .

Note that there does not exist  $z \in L$  such that  $\alpha \leq \mu(z) < t'$ . This gives  $\overline{\mu_\alpha} = \overline{\mu_{t'}}$ . Hence  $\overline{\mu_\alpha} \in \Omega$ . Thus  $\Omega$  contains all level subalgebras of  $\mu$ .

**Theorem 3.15**

If every fuzzy subalgebra (ideal)  $\mu$  defined on an  $n$ -Lie algebra  $L$  has a finite number of values, then every descending chain of subalgebras (ideals) of  $L$  terminates at finite step.

**Proof.** Suppose there exists a strictly descending chain

$$S_0 \supset S_1 \supset S_2 \supset \dots$$

of ideals of  $L$  which does not terminate at finite step. We prove that  $\mu$  defined by

$$\mu(x) = \begin{cases} \frac{k}{k+1} & \text{for } x \in S_k \setminus S_{k+1}, \\ 1 & \text{for } x \in \bigcap S_k, \end{cases}$$

where  $k=0,1,2,\dots$  and  $S_0=L$ , is a fuzzy ideal with an infinite number of values.

If  $[x_1, \dots, x_n] \in \bigcap S_k$ , then obviously

$$\mu([x_1, \dots, x_n]) = 1 \geq \max\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\}.$$

If  $[x_1, \dots, x_n] \notin \bigcap S_k$ , then  $[x_1, \dots, x_n] \in S_p \setminus S_{p+1}$  for some  $p \geq 0$  and there exists at least one  $i=1,2,\dots,n$  such that  $x_i \notin \bigcap S_k$ , because  $x_1, \dots, x_n \in \bigcap S_k$  implies  $[x_1, \dots, x_n] \in \bigcap S_k$ .

Let  $S_m$  be a maximal ideal of  $L$  such that at least one of  $x_1, \dots, x_n$  belongs to  $S_m \setminus S_{m+1}$ . Then  $m \leq p$ . Indeed, for  $m > p$  we have  $x_1, x_2, \dots, x_n \in S_m \subseteq S_{p+1} \subseteq S_p$  and, consequently  $[x_1, \dots, x_n] \in S_{p+1}$ , which is impossible. Thus  $m \leq p$  and

$$\mu([x_1, \dots, x_n]) = \frac{p}{p+1} \geq \max\{\mu(x_1), \dots, \mu(x_n)\} = \frac{m}{m+1}.$$

This proves that  $\mu$  is a fuzzy ideal and has an infinite number of different values. This is a contradiction. Hence every descending chain of ideals terminates at finite step.

For subalgebras the proof is analogous.

**Theorem 3.16**

Every ascending chain of subalgebras (ideals) of an  $n$ -Lie algebra  $L$  terminates at finite step if and only if the set of values of any fuzzy subalgebra (ideal) of  $L$  is a well-ordered subset of  $[0,1]$ .

**Proof.** If the set of values of a fuzzy subalgebra (ideal)  $\mu$  is not well-ordered, then there exists a strictly decreasing sequence  $\{t_i\}$  such that  $t_i = \mu(x_i)$  for some  $x_i \in L$ . But in this case  $\overline{\mu_{t_i}}$  form a strictly ascending chain of subalgebras (ideals) of  $L$ , which is a contradiction.

In order to prove the converse suppose that there exists a strictly ascending chain  $S_1 \subset S_2 \subset S_3 \subset \dots$  of subalgebras (ideals) of  $L$ . Then  $M = \bigcup_{i \in \mathbb{N}} S_i$  is a subalgebra (ideal) of  $L$  and  $\mu$  defined by

$$\mu(x) = \begin{cases} 0 & \text{for } x \notin M, \\ \frac{1}{k} & \text{where } k = \min\{i \mid x \in S_i\} \end{cases}$$

is a fuzzy subalgebra (ideal) on  $L$ .

Indeed, for every  $x_1, \dots, x_n \in M$  there exist a minimal number  $k_i$  such that  $x_i \in S_{k_i}$ , and a minimal number  $p$  such that  $[x_1, \dots, x_n] \in S_p$ . If all  $S_i$  are subalgebras, then for  $k = \max\{k_1, k_2, \dots, k_n\}$  all  $x_1, \dots, x_n$  and  $[x_1, \dots, x_n]$  are in  $S_k$ . Thus  $k \geq p$ . Consequently,

$$\mu([x_1, \dots, x_n]) = \frac{1}{p} \geq \frac{1}{k} = \min\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\}.$$

The case when at least one of  $x_1, x_2, \dots, x_n$  is not in  $M$  is obvious. Hence  $\mu$  is a fuzzy subalgebra.

Now, if all  $S_i$  are ideals, then  $[x_1, \dots, x_n] \in S_m$  for  $m = \min\{k_1, \dots, k_n\}$ . Thus  $p \leq m$ . Hence

$$\mu([x_1, \dots, x_n]) = \frac{1}{p} \geq \frac{1}{m} = \max\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\},$$

which means that in this case  $\mu$  is a fuzzy ideal.

Since the chain  $S_1 \subset S_2 \subset S_3 \subset \dots$  is not terminating,  $\mu$  has a strictly descending sequence of values. This contradicts that the set of values of any fuzzy subalgebra (ideal) is well-ordered. The proof is complete.

**Definition 3.17**

A fuzzy subset  $\mu$  of an  $n$ -Lie algebra  $L$  is said to be normal if  $\mu(0) = 1$ .

The following lemma is obvious.

**Lemma 3.18**

If  $\mu$  is a fuzzy subalgebra (ideal) of an  $n$ -Lie algebra  $L$ , then  $\mu^+$  defined by

$$\mu^+(x) = \mu(x) + 1 - \mu(0)$$

is a normal fuzzy subalgebra (ideal) of  $L$ .

**Corollary 3.19**

Any fuzzy subalgebra (ideal) of an  $n$ -Lie algebra  $L$  is contained in some normal fuzzy subalgebra (ideal) of it.

**Proof.** Indeed,  $\mu(x) \leq \mu(x) + 1 - \mu(0) = \mu^+(x)$  for every  $x \in L$ .

**Proposition 3.20:** A maximal normal fuzzy subalgebra of an  $n$ -Lie algebra  $L$  takes only two values: 0 and 1.

**Proof.** If  $\mu(x) = 1$  for all  $x \in L$ , then obviously  $\mu$  is a maximal normal fuzzy subalgebra of  $L$ . If  $\mu$  is a maximal normal fuzzy subalgebra of  $L$  and  $0 < \mu(a) < 1$  for some  $a \in L$ , then a fuzzy subset  $\nu$  defined by  $\nu(x) = \frac{1}{2}(\mu(x) + \mu(a))$  is a fuzzy subalgebra of  $L$ . Moreover,  $\nu^+$  is a non-constant normal fuzzy subalgebra of  $L$  such that  $\mu(x) \leq \nu^+(x)$  for

every  $x \in L$ . Thus,  $\mu$  is not maximal. Obtained contradiction shows that  $\mu(a)=0$  for all  $\mu(a)<1$ .

**Proposition 3.21:** Let  $\mu$  be a fuzzy subalgebra (ideal) of an  $n$ -Lie algebra  $L$ . If  $h:[0,\mu(0)] \rightarrow [0,1]$  is an increasing function, then a fuzzy subset  $\mu_h$  defined on  $L$  by  $\mu_h(x)=h(\mu(x))$  is a fuzzy subalgebra (ideal). Moreover,  $\mu_h$  is normal if and only if  $h(\mu(0))=1$ .

**Proof.** Straightforward.

If  $\mu$  is a fuzzy subset of an  $n$ -Lie algebra  $L$ , and  $f$  is a function defined on  $L$ , then the fuzzy subset  $\nu$  of  $f(L)$  defined by  $\nu(y) = \sup_{x \in f^{-1}(y)} \{\mu(x)\}$ , for all  $y \in f(L)$  is called the image of  $\mu$  under  $f$ . Similarly, if  $\nu$  is a fuzzy subset in  $f(L)$ , then the fuzzy set  $\mu = \nu \circ f$  in  $L$  is called preimage of  $\nu$  under  $f$ .

**Theorem 3.22**

An  $n$ -Lie algebra homomorphic preimage of a fuzzy ideal is a fuzzy ideal.

**Proof.** Let  $\varphi:L_1 \rightarrow L_2$  be an  $n$ -Lie algebra homomorphism, and  $\nu$  be a fuzzy ideal of  $L_2$  and  $\mu$  be the preimage of  $\nu$  under  $\varphi$ . Then, as it is not difficult to see,  $\mu$  is a fuzzy subspace of  $L$  and

$$\begin{aligned} \mu([x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n]) &= \nu(\varphi([x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n])) \\ &= \nu([\varphi(x_1), \dots, \varphi(x_{i-1}), \varphi(y), \varphi(x_{i+1}), \dots, \varphi(x_n)]) \\ &\geq \nu(\varphi(y)) = \mu(y), \end{aligned}$$

for all  $x_1, \dots, x_n, y \in L$  and  $\alpha \in F$ .

A fuzzy set  $\mu$  of a set  $X$  is said to possess sup property if for every non-empty subset  $S$  of  $X$ , there exists  $x_0 \in S$  such that  $\mu(x_0) = \sup_{x \in S} \{\mu(x)\}$ .

**Theorem 3.23**

An  $n$ -Lie algebra homomorphism image of a fuzzy ideal having the sup property is a fuzzy ideal.

**Proof.** Suppose that  $\varphi:L_1 \rightarrow L_2$  is an  $n$ -Lie algebra homomorphism,  $\mu$  is a fuzzy ideal of  $L_1$  with the sup property and  $\nu$  is the image of  $\mu$  under  $\varphi$ . Suppose that  $\varphi(x), \varphi(y) \in \varphi(L)$ . Let  $x_0 \in \varphi^{-1}(\varphi(x))$  and  $y_0 \in \varphi^{-1}(\varphi(y))$  be such that  $\mu(x_0) = \sup_{t \in \varphi^{-1}(\varphi(x))} \{\mu(t)\}$  and  $\mu(y_0) = \sup_{t \in \varphi^{-1}(\varphi(y))} \{\mu(t)\}$ , respectively. Then,

$$\begin{aligned} \nu(\varphi(x) + \varphi(y)) &= \sup_{t \in \varphi^{-1}(\varphi(x) + \varphi(y))} \{\mu(t)\} \geq \mu(x_0 + y_0) \geq \min\{\mu(x_0), \mu(y_0)\} \\ &= \min \left\{ \sup_{t \in \varphi^{-1}(\varphi(x))} \{\mu(t)\}, \sup_{t \in \varphi^{-1}(\varphi(y))} \{\mu(t)\} \right\} \\ &= \min\{\nu(\varphi(x)), \nu(\varphi(y))\}, \end{aligned}$$

and

$$\begin{aligned} \nu(\varphi(-x)) &= \sup_{t \in \varphi^{-1}(\varphi(-x))} \{\mu(t)\} \geq \mu(-x_0) = \mu(x_0) = \nu(\varphi(x)), \\ \nu(\alpha\varphi(x)) &= \sup_{t \in \alpha\varphi^{-1}(\varphi(x))} \{\mu(t)\} \geq \mu(\alpha x_0) \geq \mu(x_0) = \nu(\varphi(x)). \end{aligned}$$

Finally, let  $\varphi(x_1), \dots, \varphi(x_n), \varphi(y) \in \varphi(L)$  and let  $a_1 \in \varphi^{-1}(\varphi(x_1)), \dots, a_n \in \varphi^{-1}(\varphi(x_n)), b \in \varphi^{-1}(\varphi(y))$  be such that

$$\begin{aligned} \mu(a_1) &= \sup_{t \in \varphi^{-1}(\varphi(x_1))} \{\mu(t)\}, \dots, \mu(a_n) = \sup_{t \in \varphi^{-1}(\varphi(x_n))} \{\mu(t)\}, \\ \mu(b) &= \sup_{t \in \varphi^{-1}(\varphi(y))} \{\mu(t)\}. \end{aligned}$$

Then,

$$\begin{aligned} &\nu([\varphi(x_1), \dots, \varphi(x_{i-1}), \varphi(y), \varphi(x_{i+1}), \dots, \varphi(x_n)]) \\ &= \nu(\varphi([\varphi(x_1), \dots, \varphi(x_{i-1}), \varphi(y), \varphi(x_{i+1}), \dots, \varphi(x_n)])) \\ &= \sup_{t \in \varphi^{-1}(\varphi([\varphi(x_1), \dots, \varphi(x_{i-1}), \varphi(y), \varphi(x_{i+1}), \dots, \varphi(x_n)]))} \{\mu(t)\} \\ &\geq \mu([a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n]) \geq \mu(b) = \nu(\varphi(y)). \end{aligned}$$

This proves that  $\nu$  is a fuzzy ideal of  $\varphi(L)$ .

**Fuzzy Quotient  $n$ -Lie Algebras**

If  $I$  is an ideal of an  $n$ -Lie algebra  $L$ , then we can define a new  $n$ -Lie algebra on the quotient space  $L/I$  with the  $n$ -linear map

$$[x_1 + I, \dots, x_n + I] := [x_1, \dots, x_n] + I,$$

for all  $x_1, \dots, x_n \in L$ .

If  $I$  is an ideal of an  $n$ -Lie algebra  $L$ , then the quotient space  $L/I$  is also an  $n$ -Lie algebra and is the quotient  $n$ -Lie algebra.

**Theorem 4.1**

Let  $L$  be an  $n$ -Lie algebra.

- Let  $\mu$  be a fuzzy ideal of  $L$  and let  $t = \mu(0)$ . Then the fuzzy subset  $\mu^*$  of  $L / \overline{\mu_t}$  defined by  $\mu^*(x + \overline{\mu_t}) = \mu(x)$  for all  $x \in L$ , is a fuzzy ideal of  $L / \overline{\mu_t}$ .

- If  $I$  is an ideal of  $L$  and  $\nu$  is a fuzzy ideal of  $L/I$  such that  $\nu(x+I) = \nu(I)$  only when  $x \in I$ , then there exists a fuzzy ideal  $\mu$  of  $L$  such that  $\mu_t = I$ , where  $t = \mu(0)$ ; and  $\nu = \mu^*$ .

**Proof.** (1). Since  $\mu$  is a fuzzy ideal of  $L$ ,  $\overline{\mu_t}$  is an ideal of  $L$ . Now,  $\mu^*$  is well-defined, because if  $x + \overline{\mu_t} = y + \overline{\mu_t}$  for  $x, y \in L$ , then  $x - y \in \overline{\mu_t}$  and so  $\mu(x-y) = \mu(0)$ . Hence,  $\mu(x) = \mu(y)$  which implies that  $\mu^*(x + \overline{\mu_t}) = \mu^*(y + \overline{\mu_t})$ .

Now, we show  $\mu^*$  is a fuzzy ideal of  $L$ . Let  $x, y \in L$  and  $\alpha \in F$ . Then, we have

$$\begin{aligned} \mu^*((x + \overline{\mu_t}) + (y + \overline{\mu_t})) &= \mu^*((x + y) + \overline{\mu_t}) = \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \\ &= \min\{\mu^*(x + \overline{\mu_t}), \mu^*(y + \overline{\mu_t})\}, \end{aligned}$$

and

$$\begin{aligned} \mu^*(-x + \overline{\mu_t}) &= \mu(-x) = \mu(x) = \mu^*(x + \overline{\mu_t}), \\ \mu^*(\alpha(x + \overline{\mu_t})) &= \mu^*(\alpha x + \overline{\mu_t}) = \mu(\alpha x) \geq \mu(x) = \mu^*(x + \overline{\mu_t}). \end{aligned}$$

Finally, for  $x_1, \dots, x_n \in L$ , we have

$$\begin{aligned} \mu^*([x_1 + \overline{\mu_t}, \dots, x_{i-1} + \overline{\mu_t}, y + \overline{\mu_t}, x_{i+1} + \overline{\mu_t}, \dots, x_n + \overline{\mu_t}]) \\ &= \mu^*([x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n] + \overline{\mu_t}) \\ &= \mu([x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n]) \geq \mu(y) = \mu^*(y + \overline{\mu_t}). \end{aligned}$$

(2). We define a fuzzy subset  $\mu$  of  $L$  by  $\mu(x) = \nu(x+I)$  for all  $x \in L$ . A routine computation shows that  $\mu$  is a fuzzy ideal of  $L$ . Now,  $\overline{\mu_t} = I$ , because

$$x \in \overline{\mu_t} \Leftrightarrow \mu(x) = t = \mu(0) \Leftrightarrow \nu(x+I) = \nu(I) \Leftrightarrow x \in I.$$

Finally,  $\mu^* = \nu$ , since  $\mu^*(x + I) = \mu^*(x + \overline{\mu_t}) = \mu(x) = \nu(x + I)$ .

Let  $\mu$  be any fuzzy ideal of an  $n$ -Lie algebra  $L$  and let  $x \in L$ . The fuzzy subset  $\mu_x^*$  of  $L$  defined by  $\mu_x^*(a) = \mu(a - x)$  for all  $a \in L$  is called the

fuzzy coset determined by  $x$  and  $\mu$ .

Let  $I$  be an ideal of  $L$ . If  $\chi_I$  is the characteristic function of  $I$ , then it is easy to see that  $(\chi_I)_x^*$  is the characteristic function of  $x+I$ .

**Theorem 4.2**

Let  $\mu$  be any fuzzy ideal of an  $n$ -Lie algebra  $L$ . Then the set of all fuzzy cosets of  $\mu$  in  $L$ , i.e., the set  $L[\mu] = \{\mu_x^* | x \in L\}$ , is an  $n$ -Lie algebra under the following operations:

$$\begin{aligned} \mu_x^* + \mu_y^* &= \mu_{x+y}^* && \text{for all } x, y \in L, \\ \alpha \mu_x^* &= \mu_{\alpha x}^* && \text{for all } x \in L, \alpha \in F, \\ [\mu_{x_1}^*, \dots, \mu_{x_n}^*] &= \mu_{[x_1, \dots, x_n]}^* && \text{for all } x_1, \dots, x_n \in L. \end{aligned}$$

**Theorem 4.3**

If  $\mu$  is any fuzzy ideal of an  $n$ -Lie algebra  $L$ , then the map  $\phi: L \rightarrow L[\mu]$  defined by  $\phi(x) = \mu_x^*$  for all  $x \in L$ , is a homomorphism with kernel  $\overline{\mu_t}$ , where  $t = \mu(0)$ .

**Proof.** It is easy to see that  $f$  is a homomorphism. We show  $\mu(x) = \mu(0)$  implies  $\mu_x^* = \mu_0^*$ . For this, let  $a \in L$ . Then,  $\mu(a) \leq \mu(0) = \mu(x)$ . If  $\mu(a) < \mu(x)$ , then  $\mu(a-x) = \mu(a)$ , by Lemma 2.2. On the other hand, if  $\mu(a) = \mu(x)$ , then  $a, x \in \{y \in L | \mu(y) = \mu(0)\}$ . Hence,  $\mu(a-x) = \mu(0) = \mu(x) = \mu(a)$ . Therefore, in either case, we have shown that  $\mu(a-x) = \mu(a)$  for all  $a \in L$ . Consequently  $\mu_x^* = \mu_0^*$ . Also,  $\mu_x^* = \mu_0^*$  implies that  $\mu(x) = \mu(0)$ . Hence,  $\mu_x^* = \mu_0^*$  if and only if  $\mu(x) = \mu(0)$ . Now, we have

$$\ker \phi = \{x \in L | \phi(x) = \mu_0^*\} = \{x \in L | \mu_x^* = \mu_0^*\} = \{x \in L | \mu(x) = \mu(0)\} = \overline{\mu_t},$$

where  $t = \mu(0)$   $t = \mu(0)$ .

**Theorem 4.4**

Given a homomorphism of  $n$ -Lie algebras  $:L \rightarrow L'$  and fuzzy ideal  $\mu$  of  $L$  and  $\mu'$  of  $L'$  such that  $\phi(\mu) \subseteq \mu'$ . Then, there is a homomorphism of  $n$ -Lie algebras  $\phi^*: L[\mu] \rightarrow L'[\mu']$ , where  $\phi^*(\mu_x^*) = \mu'_{\phi(x)}$ , such that the following diagram is commutative.

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ \downarrow & & \downarrow \\ L[\mu] & \xrightarrow{\phi^*} & L'[\mu'] \end{array}$$

**Proof.** If  $\mu_x^* = \mu_y^*$ , then  $\mu(x-y) = \mu(0)$ . So

$$\mu'(\phi(x) - \phi(y)) = \mu'(\phi(x-y)) = \phi^{-1}(\mu')(x-y) \geq \mu(x-y) = \mu(0),$$

and so  $\mu'(\phi(x) - \phi(y)) = \mu(0)$ . Hence,  $\mu'(\phi(x)) = \mu'(\phi(y))$  holds. Thus,  $\phi^*$  is well-defined. It is easily seen that  $\phi^*$  is a homomorphism.

Let  $\mu$  be a fuzzy ideal of an  $n$ -Lie algebra  $L$ . For any  $x, y \in L$ , we define a binary relation  $\sim$  on  $L$  by  $x \sim y$  if and only if  $\mu(x-y) = \mu(0)$ . Then  $\sim$  is a congruence relation on  $L$ . We denote  $[x]_\mu$  the equivalence class containing  $x$ , and  $L/\mu = \{[x]_\mu | x \in L\}$  the set of all equivalence classes of  $L$ . Then,  $L/\mu$  is an  $n$ -Lie algebra under the following operations:

$$\begin{aligned} [x]_\mu + [y]_\mu &= [x+y]_\mu && \text{for all } x, y \in L, \\ \alpha [x]_\mu &= [cx]_\mu && \text{for all } x \in L, \alpha \in F, \\ [[x_1]_\mu, \dots, [x_n]_\mu] &= [[x_1, \dots, x_n]_\mu] && \text{for all } x, y \in L. \end{aligned}$$

**Theorem 4.5 (Fuzzy first isomorphism theorem)**

Let  $\phi: L \rightarrow L'$  be an epimorphism of  $n$ -Lie algebras and  $\lambda$  be a fuzzy

ideal of  $L'$ . Then  $L/\phi^{-1}(\lambda) \cong L'/\lambda$ .

Let  $I$  be an ideal and  $\mu$  a fuzzy ideal of an  $n$ -Lie algebra  $L$ . If  $\mu$  is restricted to  $I$ , then  $\mu$  is a fuzzy ideal of  $I$  and  $I/\mu$  is an ideal of  $L/\mu$ .

**Theorem 4.6 (Fuzzy second isomorphism theorem)**

Let  $\mu$  and  $\lambda$  be two fuzzy ideals of an  $n$ -Lie algebra  $L$  with  $\mu(0) = \lambda(0)$ .

Then  $\frac{L_\mu + L_\lambda}{\lambda} \cong \frac{L_\mu}{\mu \cap \lambda}$ .

**Theorem 4.7 (Fuzzy third isomorphism theorem)**

Let  $\mu$  and  $\lambda$  be two fuzzy ideals of an  $n$ -Lie algebra  $L$  with  $\lambda \subseteq \mu$  and  $\mu(0) = \lambda(0)$ . Then  $\frac{L/\lambda}{L_\mu/\lambda} \cong L/\mu$ .

**Conclusion**

Methods of construction fuzzy ideals are presented. Connections with various fuzzy quotient  $n$ -Lie algebras are proved. Properties of fuzzy subalgebras and ideals of  $n$ -ary Lie algebras are described.

**References**

- Filippov VT (1985)  $n$ -ary Lie algebras. (Russian) Sibirsk Mat Zh 26: 126-140.
- Takhtajan LA (1994) On foundation of generalized Nambu mechanics. Commun Math Phys 160: 295-315.
- Nambu Y (1973) Generalized Hamiltonian mechanics. Phys Rev D7: 2405-2412.
- Filippov VT (1988) On  $n$ -Lie algebra of Jacobians. Siberian Math J 39: 573-581
- Bremner MR, Hentzel I (2000) Identities for generalized Lie and Jordan products on totally associative triple systems. J Algebra 231: 387-405.
- Bremner MR, Peresi LA (2006) Ternary analogues of Lie and Malcev algebras. Linear Algebra Appl 414: 1-18.
- Faulkner JR, Ferrar JC (1977) Exceptional Lie algebras and related algebraic and geometric structures. Bull London Math Soc 9: 1-35.
- Yehia SE (1996) Fuzzy ideals and fuzzy subalgebras of Lie algebras. Fuzzy Sets Syst 80: 237-244.
- Kim CG, Lee DS (1998) Fuzzy Lie ideals and fuzzy Lie subalgebras. Fuzzy Sets Syst 94: 101-107.
- Akram M (2008) Redefined fuzzy Lie subalgebras. Quasigroups Relat Syst 16: 119-132.
- Davvaz B (2001) Fuzzy Lie algebras. Intern J Appl Math 6: 449-461.
- Jun YB, Chul H (2007) Intuitionistic fuzzy Lie ideals of Lie algebras. Honam Math J 29: 259-268.
- Dudek WA (2000) Fuzzification of  $n$ -ary groupoids. Quasigroups Relat Syst 7: 45-66.
- Davvaz B, Dudek WA (2009) Fuzzy  $n$ -ary groups as a generalization of Rosenfeld's fuzzy groups. J Mult-Valued Logic and Soft Computing 15: 451-469.
- Katsaras AK, Liu BD (1977) Fuzzy vector spaces. J Math Anal Appl 58: 135-146.
- Kondo M, Dudek WA (2005) On the transfer principle in fuzzy theory. Mathware and Soft Computing 12: 41-55.