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Research Article

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Generalized Derivations of BiHom-Lie Algebras

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Abstract

BiHom-Lie algebra is a generalized Hom-Lie algebra endowed with two commuting multiplicative linear maps. This paper is devoted to investigate the generalized derivation of BiHom-Lie algebra. We generalize the main results of Leger and Luks to the case of BiHom-Lie algebra. Firstly we review some concepts associated with BiHom-Lie algebra *L*. Furthermore, we give the definitions of the generalized derivation *GDer*(*L*), quasiderivations *QDer*(*L*), center derivation *Z*(*L*), centroid *C*(*L*) and quasicentroid *QC*(*L*). Later one, we give some useful proprieties and connections between these derivations. In particular, we prove that *GDer*(*L*)=*QDer*(*L*)+*QC*(*L*). We also prove that *QDer*(*L*) can be embedded as derivations in larger BiHom-Lie algebra.

Keywords: BiHom-Lie algebras; Generalized derivations; Quasiderivations; Centroids; Quasicentroids

Introduction

The motivations to study Hom-Lie structures are related to physics and to deformations of Lie algebras, in particular Lie algebras of vector fields. Hom-Lie algebra, introduced by Hartwig, Larson and Silvestrov in ref. [1], is a triple $(L,[,],\alpha)$ consisting of a vector space L, a bilinear map $[,]:L\times L\to L$ and a vector space homomorphism $\alpha:L\to L$ satisfying the following conditions:

- $[x,y]=-[y,x]$,
- $\circlearrowleft x,y,z[\alpha(x),[y,z]]=0$,

for all x,y,z from A, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x,y,z.

The main feature of these algebras is that the identities defining the structures are twisted by homomorphisms. The paradigmatic examples are q-deformations of Witt and Virasoro algebras, Heisenberg-Virasoro algebra and other algebraic structure constructed in pioneering works [2-5].

In ref. [6], the authors introduced a generalized algebraic structure endowed with two commuting multiplicative linear maps, called BiHom-algebras. When the two linear maps are same, then BiHom-algebras will be return to Hom-algebras. These algebraic structures include BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras. Hom-Lie algebras are a generalization of Lie algebras, where the classical Jacobi identity is twisted by a linear map. In the particular case that the twisted map is the identity map, Hom-Lie algebras become Lie algebras. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the structures on certain deformations of the Witt algebras and the Virasoro algebras [1]. Hom-Lie algebras are also related to deformed vector fields, the various versions of the Yang-Baxter equations, braid group representations, and quantum groups [1,5,7]. Recently, Hom-Lie algebras were studied in refs. [8,9]. More applications of the Hom-Lie algebras, Hom-algebras and Hom-Lie superalgebras can be found in refs. [10,11].

As is well known, derivation and generalized derivation algebra are very important subjects both in the research of Lie algebras and Lie superalgebras. In the study of Levi factors in derivation algebra of nilpotent Lie algebras, the generalized derivations, quasiderivations, centroids, and quasicentroids play key roles [12]. The most important and systematic research on the generalized derivation algebra of a Lie algebra was due to Leger and Luks [13]. In ref. [12], some nice properties of the generalized derivation algebra and their subalgebras, for example, of the quasiderivation algebra and of the centroid have been obtained. In particular, they investigated the structure of the generalized derivation algebra and characterized the Lie algebras satisfying certain conditions. Meanwhile, they also pointed that there exist some connections between quasiderivations and cohomology of Lie algebras.

The purpose of this paper is to generalize some beautiful results in ref. [13] to the generalized derivation algebra of a BiHom-Lie algebra. In this paper, we mainly study the derivation algebra $Der(L)$, the center derivation algebra $ZDer(L)$, the quasi derivation algebra $QDer(L)$, and the generalized derivation algebra $GDer(L)$ of a BiHom-Lie algebra L. We proceed as follows. Firstly we recall some basic definitions and propositions which will be used in what follows. Then we give some basic properties of the generalized derivation algebra and their BiHom-subalgebras, show that the quasicentroid of a BiHom-Lie algebra is also a BiHom-Lie algebra if only and if it is a BiHom-associative algebra. Finally we prove that the quasiderivations of L can be embedded as derivations in a larger Hom-Lie algebra and obtain a direct sum decomposition of $Der(L)$ when the annihilator of L is equal to zero.

Preliminaries

Definition 2.1

A Hom-Lie algebra is a triple $(A,[,],\alpha)$ consisting of a vector space A, a bilinear map $[,]$ and a linear map $\alpha:A\to A$ such that [1,14]:

- $[x,y]=-[y,x]$,
- $\circlearrowleft_{x,y,z}[\alpha(x),[y,z]]=0$,

for all x,y,z from A, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x,y,z.

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In particular, if α preserves the bracket, (i.e. $\alpha[x,y]=[\alpha(x),\alpha(y)], \forall x,y \in A$), then we call $(A,[,],\alpha)$ a multiplicative Hom-Lie algebra.

Definition 2.2

1. A BiHom-Lie algebra over a field $\mathbb{K}$ is a 4-tuple $(L,[,],\alpha,\beta)$, where L is a $\mathbb{K}$-linear space, $\alpha: L \to L$, $\beta: L \to L$ and $[,]: L \times L \to L$ are linear maps, with notation $[,](a,a')=[a,a']$, satisfying the following conditions, for all $a,a',a'' \in L$ [15]:

$$\alpha \circ \beta = \beta \circ \alpha$$

$$[\beta(a'),\alpha(a'')] = -[\beta(a''),\alpha(a')] \ (skewsymmetry) \quad (1)$$

$$[\beta^2(a),[\beta(a'),\alpha(a'')]] + [\beta^2(a'),[\beta(a''),\alpha(a)]] + [\beta^2(a''),[\beta(a),\alpha(a')]] = 0. \quad (2)$$

2. A BiHom-Lie algebra is called a multiplicative BiHom-Lie algebra if α and β are algebraic morphisms, i.e. for any $a',a'' \in L$, we have:

$$\alpha([a',a'']) = [\alpha(a'),\alpha(a'')] \quad \text{and} \quad \beta([a',a'']) = [\beta(a'),\beta(a'')]. \quad (3)$$

3. A multiplicative BiHom-Lie algebra is called a regular BiHom-Lie algebra if α, β are bijective maps.

Definition 2.3

1. A sub-vector space $H \subset L$ is a BiHom-Lie sub-algebra of $(L,[,],\alpha,\beta)$ if $\alpha(H) \subset H, \beta(H) \subset H$ and H is closed under the bracket operation $[,]$, i.e. [15]:

$$[u,u'] \in H, \forall u,u' \in H.$$

2. A sub-vector space $I \subset L$ is a BiHom ideal of $(L,[,],\alpha,\beta)$ if $\alpha(I) \subset I, \beta(I) \subset I$ and

$$[u,u'] \in I, \forall u \in I, u' \in L.$$

A morphism $f:(L,[,],\alpha,\beta) \to (L',[,]',\alpha',\beta')$ of BiHom-Lie algebras is a linear map $f: L \to L'$ such that $\alpha' \circ f = f \circ \alpha$,

$\beta' \circ f = f \circ \beta$ and $f([u,v]) = [f(u),f(v)]'$, for all $u,v \in L$.

Proposition 2.1: Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra and define the following subvector space of $End(L)$ consisting of linear maps u on L as follows:

$$\mho = \{u \in End(L) \mid u \circ \alpha = \alpha \circ u, u \circ \beta = \beta \circ u\} \quad \text{and}$$

$$\tilde{\alpha}, \tilde{\beta}: \mho \to \mho; \ \tilde{\alpha}(D) = \alpha \circ D, \tilde{\beta}(D) = \beta \circ D.$$

Then $(\mho,[\,,\,],\tilde{\alpha},\tilde{\beta})$ is a BiHom-Lie algebra over $\mathbb{K}$ with the bracket:

$$[D_1,D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

for all $D_1,D_2 \in \mho$.

Proof. Let $D_1,D_2,D_3 \in \mho$. Then we have:

o $\tilde{\alpha} \circ \tilde{\beta}(D_1) = \tilde{\alpha}(\beta \circ D_1) = \alpha \circ \beta \circ D_1 = \beta \circ \alpha \circ D_1 = \tilde{\beta} \circ \tilde{\alpha} \circ D_1$

o

$$\begin{aligned}
[\tilde{\beta}(D_1),\tilde{\alpha}(D_2)] &= [\beta \circ D_1, \alpha \circ D_2] \\
&= \beta \circ D_1 \circ \alpha \circ D_2 - \alpha \circ D_2 \circ \beta \circ D_1 \\
&= \alpha \circ \beta \circ D_1 \circ D_2 - \alpha \circ \beta \circ D_2 \circ D_1 \\
&= \alpha \circ D_1 \circ \beta \circ D_2 - \beta \circ D_2 \circ \alpha \circ D_1 \\
&= -[\beta \circ D_2, \alpha \circ D_1] = -[\tilde{\beta}(D_2),\tilde{\alpha}(D_1)].
\end{aligned}$$

o It is easy to show that:

$$[\tilde{\beta}^2(D_1),[\tilde{\beta}(D_2),\tilde{\alpha}(D_3)]] + [\tilde{\beta}^2(D_2),[\tilde{\beta}(D_3),\tilde{\alpha}(D_1)]] + [\tilde{\beta}^2(D_3),[\tilde{\beta}(D_1),\tilde{\alpha}(D_2)]] = 0.$$

Now, let $(L,[,],\alpha,\beta)$ be a regular BiHom-Lie algebra. For any integer k, l, denote by α^k the k-times composition of α and β^l the l-times composition of β, i.e.

$$\alpha^k = \underbrace{\alpha \circ \ldots \circ \alpha}_{(k-times)}, \quad \beta^l = \underbrace{\beta \circ \ldots \circ \beta}_{(l-times)}.$$

Since the maps α, β commute, we denote by:

$$\alpha^k \beta^l = \underbrace{\alpha \circ \ldots \circ \alpha}_{(k-times)} \circ \underbrace{\beta \circ \ldots \circ \beta}_{(l-times)}.$$

In particular, $\alpha^0\beta^0 = Id, \alpha^1\beta^1 = \alpha\beta, \alpha^{-k}\beta^{-l}$ is the inverse of $\alpha^{-k}\beta^{-l}\alpha^{-k}\beta^{-l}$.

Definition 2.4

Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra. A linear map $D: L \to L$ is said to be an $\alpha^k\beta^l$– derivation of L, where $k,l \in \mathbb{N}$, if it satisfies [15]

$$D \circ \alpha = \alpha \circ D,$$

$$D \circ \beta = \alpha \circ \beta,$$

$$[D(x),\alpha^k\beta^l(y)] + [\alpha^k\beta^l(x),D(y)] = D([x,y]), \quad (4)$$

$$\forall x,y \in L.$$

We denote the set of all $\alpha^k\beta^l$– derivations by $Der_{\alpha^k\beta^l}(L)$, then $Der(L) := \bigoplus_{k\geq 0, l\geq 0} Der_{\alpha^k\beta^l}(L)$ provided with the commutator and the following map:

$$\tilde{\alpha}: Der(L) \to Der(L), \quad \tilde{\alpha}(D) = D \circ \alpha$$

is a BiHom-subalgebra of $\mho$ and is called the derivation algebra of L.

Definition 2.5

An endomorphism $D \in End(L)$ is said to be a generalized $\alpha^k\beta^l$– derivation of L, if there exist two endomorphisms $D',D'' \in End(L)$ such that:

$$D \circ \beta = \beta \circ D = 0, D' \circ \beta = \beta \circ D' = 0, D'' \circ \beta = \beta \circ D'' = 0$$

$$D \circ \beta = \beta \circ D = 0, D' \circ \beta = \beta \circ D' = 0, D'' \circ \beta = \beta \circ D'' = 0$$

$$[D(x),\alpha^k\beta^l(y)] + [\alpha^k\beta^l(x),D'(y)] = D''([x,y]), \quad (5)$$

for all $x,y \in L$.

Definition 2.6

An endomorphism $D \in End(L)$ is said to be a $\alpha^k\beta^l$– quasiderivation, if there exists an endomorphism $D' \in End(L)$ such that:

$$D \circ \alpha = \alpha \circ D = 0, D' \circ \alpha = \alpha \circ D' = 0,$$

$$D \circ \beta = \beta \circ D = 0, D' \circ \beta = \beta \circ D' = 0,$$

$$[D(x),\alpha^k\beta^l(y)] + [\alpha^k\beta^l(x),D(y)] = D'([x,y]), \quad (6)$$

for all $x,y \in L$.

Let $GDer_{\alpha^k\beta^l}(L)$ and $QDer_{\alpha^k\beta^l}(L)$ be the sets of generalized $\alpha^k\beta^l$– derivations and of $\alpha^k\beta^l$– quasiderivations, respectively. That is:

$$GDer(L) := \bigoplus_{k\geq 0, l\geq 0} GDer_{\alpha^k\beta^l}(L), \quad QDer(L) := \bigoplus_{k\geq 0, l\geq 0} QDer_{\alpha^k\beta^l}(L).$$

It is easy to verify that both $GDer(L)$ and $QDer(L)$ are BiHom-subalgebras of $\mho$ (see Proposition (3.1)).

Definition 2.7

If $C(L) := \bigoplus_{k\geq 0, l\geq 0} C_{\alpha^k\beta^l}(L)$, with $C_{\alpha^k\beta^l}(L)$ consisting of $D \in End(L)$

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satisfying:

$D \circ \alpha = \alpha \circ D$,

$D \circ \beta = \beta \circ D$,

$$[D(x), \alpha^k\beta^l(y)] = [\alpha^k\beta^l(x), D(y)] = D([x,y]), \tag{7}$$

$\forall x \in L$ and $y \in L$, then $C(L)$ is called an $\alpha^k\beta^l$– centroid of L.

Definition 2.8

If $QC(L) := \bigoplus_{k\geq 0, l\geq 0} QC_{\alpha^k\beta^l}(L)$, with $QC_{\alpha^k\beta^l}(L)$ consisting of $D \in End(L)$ such that:

$$[D(x), \alpha^k\beta^l(y)] = [\alpha^k\beta^l(x), D(y)], \tag{8}$$

$\forall x \in L$ and $y \in L$, then $QC(L)$ is called an $\alpha^k\beta^l$– quasicentroid of L.

Define $ZDer(L) := \bigoplus_{k\geq 0, l\geq 0} Der_{\alpha^k\beta^l}(L)$, where $Der_{\alpha^k\beta^l}(L)$ consists of $D \in End(L)$ such that:

$$[D(x), \alpha^k\beta^l(y)] = D([x,y]) = 0,$$

$\forall x \in L$ and $y \in L$, then $ZDer(L)$ is called an $\alpha^k\beta^l$– -central derivation of L.

It is easy to verify that:

$ZDer(L) \subseteq Der(L) \subseteq QDer(L) \subseteq GDer(L) \subseteq End(L)$

$C(L) \subseteq QC(L) \subseteq QDer(L)$.

Definition 2.9

Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra. If $Z(L)=\{x \in L [x,y]=0, y \in L\}$, then $Z(L)$ is called the center of L.

Generalized Derivation Algebra and their BiHom-Subalgebras

First, we give some basic properties of center derivation algebra, quasiderivation algebra and the generalized derivation algebra of a BiHom-Lie algebra.

Proposition 3.1

Let $(A,[,],\alpha)$ be a multiplicative Hom-Lie algebra. Then the following statements hold:

1. $GDer(L)$, $QGer(L)$ and $C(L)$ are BiHom-subalgebras of $\mho$.

2. $ZDer(L)$ is a BiHom-ideal of $Der(L)$.

Proof.

1. Assume that $D_1 \in GDer_{\alpha^k\beta^l}(L)$, $D_2 \in GDer_{\alpha^s\beta^t}(L)$, $\forall x \in L$ and $y \in L$. We have:

- $[(\tilde{\alpha}(D_1)(x)), \alpha^{k+1}\beta^l(y)] = [D_1 \circ \alpha(x), \alpha^{k+1}\beta^l(y)] = \alpha([D_1(x), \alpha^k\beta^l(y)])$
$= \alpha(D_{1''}([x,y]) - [\alpha^k\beta^l(x), D_{1'}(y)])$
$= \tilde{\alpha}(D_{1''}[x,y]) - [\alpha^{k+1}\beta^l(x), \tilde{\alpha}(D_{1'}(y))]$.

Since both $\tilde{\alpha}(D_{1''})$ and $\tilde{\alpha}(D_{1'})$ are in $End(L)$, $\tilde{\alpha}(D_1) \in GDer_{\alpha^{k+1}\beta^l}(L)$.

- $[(\tilde{\beta}(D_2)(x)), \alpha^k\beta^{l+1}(y)] = [D_2 \circ \beta(x), \alpha^k\beta^{l+1}(y)] = \beta([D_2(x), \alpha^k\beta^l(y)])$
$= \beta(D_{2''}([x,y]) - [\alpha^k\beta^l(x), D_{2'}(y)])$
$= \tilde{\beta}(D_{2''}[x,y]) - [\alpha^k\beta^{l+1}(x), \tilde{\beta}(D_{2'}(y))]$

Since both $\tilde{\beta}(D_{2''})$ and $\tilde{\beta}(D_{2'})$ are in $End(L)$, $\tilde{\beta}(D_2) \in GDer_{\alpha^k\beta^{l+1}}(L)$. We also have:

- $[D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] = D_{1''}([D_2(x), \alpha^s\beta^t(y)]) - [\alpha^k\beta^l(D_2(x)), D_{1'}(\alpha^s\beta^t(y))]$
$= D_{1''}(D_{2''}([x,y]) - [\alpha^s\beta^t(x), D_{2'}(y)]) - [\alpha^k\beta^l(D_2(x)), D_{1'}(\alpha^s\beta^t(y))]$
$= D_{1''}(D_{2''}([x,y])) - D_{1''}([\alpha^s\beta^t(x), D_{2'}(y)]) - [\alpha^k\beta^l(D_2(x)), D_{1'}(\alpha^s\beta^t(y))]$
$= D_{1''}(D_{2''}([x,y])) - [D_1(\alpha^s\beta^t(x)), \alpha^k\beta^l(D_2'(x))] - [\alpha^{k+s}\beta^{l+t}(x), D_{1'}D_{2'}(y)]$
$-[\alpha^k\beta^l(D_2(x)), D_{1'}(\alpha^s\beta^t(y))]$.

And,

- $[D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)] = D_{2''}([D_1(x), \alpha^k\beta^l(y)]) - [\alpha^s\beta^t(D_1(x)), D_{2'}(\alpha^k\beta^l(y))]$
$= D_{2''}(D_{1''}([x,y]) - [\alpha^k\beta^l(x), D_{1'}(y)]) - [\alpha^s\beta^t(D_1(x)), D_{2'}(\alpha^k\beta^l(y))]$
$= D_{2''}(D_{1''}([x,y])) - D_{2''}([\alpha^k\beta^l(x), D_{1'}(y)]) - [\alpha^s\beta^t(D_1(x)), D_{2'}(\alpha^k\beta^l(y))]$
$= D_{2''}(D_{1''}([x,y])) - [D_2(\alpha^k\beta^l(x)), \alpha^s\beta^t(D_1'(x))] - [\alpha^{k+s}\beta^{l+t}(x), D_{2'}D_{1'}(y)]$
$-[\alpha^s\beta^t(D_1(x)), D_{2'}(\alpha^k\beta^l(y))]$.

Thus for all $x \in A$ and $y \in A$, we have:

$[[D_1, D_2], \alpha^{k+s}\beta^{l+t}(y)] = [D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] - [D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)]$

$= [D_{1''}, D_{2''}]([x,y]) - [\alpha^{k+s}\beta^{l+t}(x), [D_{1'}, D_{2'}](y)]$.

Since both $[D_{1''}, D_{2'}]$ and $[D_{1''}, D_{2''}]$ are in $End(L)$, so $[D_1, D_2] \in GDer_{\alpha^{k+s}\beta^{l+t}}(l)$, $\forall x,y \in L$, $GDer(L)$ is a BiHom-subalgebra of $\mho$.

Similarly, $QDer(L)$ is a BiHom-subalgebra of $\mho$.

Assume that $D_2 \in C_{\alpha^s\beta^t}(L)$, $D_2 \in C_{\alpha^s\beta^t}(L)$. For all $x,y \in L$, we have:

$\tilde{\alpha}(D_1)([x,y]) = D_1[\alpha(x), \alpha(y)]$

$= [\alpha^{k+1}\beta^l(x), D_1(\alpha(y))] = [\alpha^{k+1}\beta^l(x), \tilde{\alpha}(D_1)(y)]$,

and,

$\tilde{\beta}(D_2)([x,y]) = D_2[\beta(x), \beta(y)]$

$= [\alpha^s\beta^{t+1}(x), D_2(\beta(y))] = [\alpha^s\beta^{t+1}(x), \tilde{\beta}(D_2)(y)]$.

So $\tilde{\alpha}(D_1) \in C_{\alpha^{k+1}\beta^l}(L)$ and $\tilde{\beta}(D_2) \in C_{\alpha^k\beta^{l+1}}(L)$. Note that:

$[[D_1, D_2](x), \alpha^{k+s}\beta^{l+t}(y)] = [D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] - [D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)]$
$= D_1([D_2(x), \alpha^s\beta^t(y)]) - D_2([D_1(x), \alpha^k\beta^l(y)])$
$= D_1D_2([x,y]) - D_2D_1([x,y])$
$= [D_1, D_2]([x,y])$.

Similarly,

$[\alpha^{k+s}\beta^{l+t}(x), [D_1, D_2](y)] = [D_1, D_2]([x,y])$.

Then $[D_1, D_2] \in C_{\alpha^{k+s}\beta^{l+t}}(L)$. Thus $C(L)$ is a BiHom-subalgebra of $\mho$.

2. Assume that $D_2 \in Der_{\alpha^s\beta^t}(L)$, $D_2 \in Der_{\alpha^s\beta^t}(L)$, $\forall x \in L$, $y \in L$. Then:

$[\tilde{\alpha}(D_1)(x), \alpha^{k+1}\beta^l(y)] = \alpha([D_1(x), \alpha^k\beta^l(y)]) = \alpha \circ D_1([x,y]) = \tilde{\alpha}(D_1)([x,y]) = 0$,

and,

$[\tilde{\beta}(D_2)(x), \alpha^s\beta^{t+1}(y)] = \beta([D_2(x), \alpha^s\beta^t(y)]) = \beta \circ D_2([x,y]) = \tilde{\beta}(D_2)([x,y]) = 0$.

So $\tilde{\alpha}(D_1) \in ZDer_{\alpha^{k+1}\beta^l}(L)$ and $\tilde{\beta}(D_2) \in ZDer_{\alpha^s\beta^{t+1}}(L)$. Note that:

$[[D_1, D_2]([x,y])] = D_1D_2([x,y]) - D_2D_1([x,y])$

$= D_1([D_2(x), \alpha^s\beta^t(y)]) - D_2([D_1(x), \alpha^k\beta^l(x)])$

$= 0$.

And,

$[[D_1, D_2](x), \alpha^{k+s}\beta^{l+t}(y)] = [D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)]$

$= D_1([D_2(x), \alpha^s\beta^t(y)]) - D_2([D_1(x), \alpha^k\beta^l(y)]) = 0$.

Then $[D_1, D_2] \in ZDer_{\alpha^{k+s}\beta^{l+t}}(L)$. Thus $ZDer(L)$ is a BiHom-ideal of $Der(L)$.

Lemma 3.1: Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra. Then:

- $[Der(L),C(L)]\subseteq C(L)$.
- $[QDer(L),QC(L)]\subseteq QC(L)$.
- $[QC(L),QC(L)]\subseteq QDer(L)$.
- $C(L)\subseteq QDer(L)$.

Proof.

- Assume that $D_1 \in GDer_{\alpha^k\beta^l}(L), D_2 \in C_{\alpha^s\beta^t}(L)$. Forall $x,y\in L$, we have:

$$[D_1, D_2]([x,y]) = D_1D_2([x,y]) - D_2D_1([x,y])$$
$$= D_1([D_2(x), \alpha^s\beta^t(y)]) - D_2([D_1(x), \alpha^k\beta^l(y)] + [\alpha^k\beta^l(x), D_1(y)])$$
$$= [D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] + [\alpha^k\beta^l(D_2(x)), D_1(\alpha^s\beta^t(y))] - [D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)]$$
$$-[D_2(\alpha^k\beta^l(x)), \alpha^s\beta^t(D_1(y))]$$
$$= [[D_1, D_2](x), \alpha^{k+s}\beta^{l+t}(y)].$$

Similarly,

$$[D_1,D_2]([x,y])=[\alpha^{k+s}\beta^{l+t}(x),[D_1,D_2](y)].$$

Hence $[D_1, D_2]([x,y]) \in C_{\alpha^{k+s}\beta^{l+t}}$, so $[Der(T),C(T)]\subset C(T)$.

- Similar to the proof of (1).
- Assume that $D_1 \in QC_{\alpha^k\beta^l}(L), D_2 \in C_{\alpha^s\beta^t}(L)$. For all $x,y\in L$, we have:

$$[[D_1, D_2](x), \alpha^{k+s}\beta^{l+t}(y)] + [\alpha^{k+s}\beta^{l+t}(x), [D_1, D_2](y)] =$$
$$[D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] - [D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)]$$
$$+[\alpha^{k+s}\beta^{l+t}(x), D_2D_1(y)] - [\alpha^{k+s}\beta^{l+t}(x), D_1D_2(y)].$$

It is easy to verify:

$$[D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] = [\alpha^k\beta^l(D_2(x)), D_1(\alpha^s\beta^t(y))]$$
$$= [\alpha^{k+s}\beta^{l+t}(x), D_2D_1(y)],$$

and,

$$[\alpha^{k+s}\beta^{l+t}(x), D_1D_2(y)] = [\alpha^s\beta^t(D_1(x)), D_2(\alpha^k\beta^l(y))]$$
$$= [D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)].$$

Hence $[[D_1,D_2](x),\alpha^{k+s}\beta^{l+t}(y)]+[\alpha^{k+s}\beta^{l+t}(x),[D_1,D_2](y)]=0$ and $[D_1, D_2] \in QDer_{\alpha^{k+s}\beta^{l+t}}(L)$.

- Assume that $D \in QC_{\alpha^k\beta^l}(L)$. Then for all $x,y\in L$, we have:

$$D([x,y])=[D(x),\alpha k\beta l(y)]=[\alpha^k\beta^l(y),D(y)].$$

So $2D([x,y])=[D(x),\alpha k\beta l(y)]+[\alpha^k\beta^l(y),D(y)]$, which means $D\in QDer(L)$.

Proposition 3.2

Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra. Then:

$GDer(L)=QDer(L)+QC(L)$.

Proof. Let $D_1 \in GDer_{\alpha^k\beta^l}(L)$. Then there exist $D_{1'},D_{1''}\in End(A)$ such that for all $x,y\in L$,

$$[D_1(x), \alpha^k\beta^l(y)] + [\alpha^k\beta^l(x), D_{1'}(y)] = D_1''([x,y]).$$

Du to $[\alpha^k\beta^l(y), D_1(x)] + [D_{1'}(y), \alpha^k\beta^l(x)] = D_1''([y,x])$, and,

$$[D_{1'}(y), \alpha^k(x)] + [\alpha^k(y), D_1(x)] = D_{1''}([y,x]).$$

Therefore $D_{1'} \in GDer_{\alpha^k\beta^l}(L)$. For all $x,y\in L$, there are:

$$[\frac{D_1 + D_{1'}}{2}(x), \alpha^k(y)] + [\alpha^k(x), \frac{D_1 + D_{1'}}{2}(y)] = D_{1''}([x,y]),$$

and,

$$[\frac{D_1 - D_{1'}}{2}(x), \alpha^k(y)] - [\alpha^k(x), \frac{D_1 - D_{1'}}{2}(y)] = 0,$$

which imply that $\frac{D_1 + D_{1'}}{2} \in QDer_{\alpha^k\beta^l}(L)$ and $\frac{D_1 - D_{1'}}{2} \in QC_{\alpha^k\beta^l}(L)$. Hence:

$$D_1 = \frac{D_1 + D_{1'}}{2} + \frac{D_1 - D_{1'}}{2} \in QDer_{\alpha^k}(A) + QC_{\alpha^k}(A).$$

Also we have:

$GDer(L)\subseteq QDer(L)+QC(L)$.

Therefore $QDer(L)+QC(L)=GDer(L)$.

Theorem 3.1: Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra, α, β are surjections and $Z(L)$ the center of L. Then $[C(L),QC(L)]\subseteq End(L,Z(L))$. Moreover, if $Z(L)=\{0\}$, then $[C(L),QC(L)]=\{0\}$.

Proof. Assume that $D_1 \in C_{\alpha^k\beta^l}(L)$, $D_2 \in QC_{\alpha^s\beta^t}(L)$. For all $x\in L$, since α and β are surjections, $\forall y\in L$, $\exists y'\in L$, such that $y=\alpha^{k+s}\beta^{l+}t(y')$, then:

$$[[D_1, D_2](x), y] = [[D_1, D_2](x), \alpha^{k+s}\beta^{l+t}(y)]$$
$$= [D_1D_2(x), \alpha^{k+s}\beta^{l+t}(y)] - [D_2D_1(x), \alpha^{k+s}\beta^{l+t}(y)]$$
$$= D_1([D_2(x), \alpha^s\beta^t(y)]) - [\alpha^s\beta^t(D_1(x)), D_2(\alpha^k\beta^l(y))]$$
$$= D_1([D_2(x), \alpha^s\beta^t(y)]) - D_1([\alpha^s\beta^t(x), D_2(y)])$$
$$= D_1([D_2(x), \alpha^s\beta^t(y)] - D_1[\alpha^s\beta^t(x), D_2(y)]) = 0.$$

Hence $[D_1,D_2](x)\in Z(L)$, and $[D_1,D_2]\in End(L,Z(L))$ as desired. Furthermore, if $Z(L)=\{0\}$, it is clear that $[C(L),QC(L)]=\{0\}$.

Theorem 3.2: If $(L,[,],\alpha,\beta)$ is a multiplicative BiHom-Lie algebra, then $ZDer(L)=C(L)\cap Der(L)$.

Proof. Assume that $D \in C_{\alpha^k\beta^l}(L) \cap Der_{\alpha^k\beta^l}(L)$. For all $x,y\in L$, we have:

$$D([x,y])=[D(x),\alpha^k\beta^l(y)]+[\alpha^k\beta^l(x),D(y)],$$

and,

$$D([x,y])=[D(x),\alpha^k\beta^l(y)]=[\alpha^k\beta^l(x),D(y)].$$

So $D([x,y])=0$. Then $D \in ZDer_{\alpha^k\beta^l}(L)$ and $C(L)\cap Der(L)\subset ZDer(L)$.

Now, we assume that $D \in ZDer_{\alpha^k\beta^l}(L)$, for all $x,y\in L$ we have:

$$D([x,y])=[D(x),\alpha^k\beta^l(y)]=0$$

It is easy to verify $D \in C_{\alpha^k\beta^l}(L) \cap Der_{\alpha^k\beta^l}(L)$ and $ZDer(L)\subset C(L)\cap Der(L)$.

By Theorem 2.3 in [16], if $(A,[,])$ is a Lie algebra with $Z(A)=\{0\}$, where:

$Z(A)$is the center of A, then $C(A)=QDer(A)\cap QC(A)$. But it is not true in case that $(A,[,],\alpha)$ is a multiplicative Hom-Lie algebra.

Example 3.1: Let $\{x_1,x_2,x_3\}$ be a basis of a 3-dimensional linear space L over . The following bracket and linear maps and β on L define a BiHom-Lie algebra over $\mathbb{K}$:

$[x_1,x_2]=x_1$, $\alpha(x_1)=x_1$, $\beta(x_1)=x_1$

$[x_1,x_3]=x_2$, $\alpha(x_2)=2x_2$, $\beta(x_2)=x_2$

$[x_3,x_3]=2x_3,\ \alpha(x_1)=2x_3,\ \beta(x_3)=x_3$

with $[x_2,x_1],[x_3,x_1],[x_3,x_2]$ defined via skew symmetry.

Define $D:L\to L$ satisfying:

$D(x_1)=x_1, D(x_2)=2^kx_2, D(x_3)=2^kx_3.(k\in \mathbb{Z}_+)$

It is obvious that $Z(L)=0$. $\forall\ y\in L$, suppose $y=ax_1+bx_2+cx_3$. Define $D'\in End(L)$ by:

$D'(x_1=2^{k+1}x_1, D'(x_2)=2^{k+1}x_2, D'(x_3)=2^{k+1}2kx_3$.

It is obvious that for $i=1,2,3$,

$D'([x_i,ax_1+bx_2+cx_3])=[D(x_i),\alpha\beta(ax_1+bx_2+cx_3)]+[\alpha\beta(x_i),D(ax_1+bx_2+cx_3)]$,

and,

$[D(x_i),\alpha\beta(ax_1+bx_2+cx_3)]=[\alpha\beta(x_i),D(ax_1+bx_2+cx_3)]$.

So,

$D\in QDer(A)\cap QC(A)$.

But for all $t,l\in \mathbb{Z}$, while:

$D([x_1,ax_1+bx_2+cx_3])=D(bx_1+cx_2)=bx_1+c2^kx_2$.

$[\alpha^t\beta^l(x_1),D(ax_1+bx_2+cx_3)]=[x1,ax_1+b2^kx_2+c2^kx_3]=b2^kx_1+c2^kx_2$.

That means $D\notin C(A)$.

Now, we recall the definition of Hom-Jordan algebra as [17].

Definition 3.1: Let (L,μ,α) be a Hom-algebra.

• The Hom-associator of L is the bilinear map as $as\alpha:L\times L\to L$ defined as:

$as\alpha=\mu\circ(\mu\otimes\alpha-\alpha\otimes\mu)$.

In terms of elements, the map $as\alpha$ is given by:

$as\alpha(x,y,z)=\mu(\mu(x,y),\alpha(z))-\mu(\alpha(x),\mu(y,z))$

for all $x,y,z\in L$.

• Let A be a Hom-algebra over a field $\mathbb{K}$ of characteristic $\neq 2$ with an bilinear multiplication $\circ$. If $\alpha:A\to A$ an linear map, then $(A,\circ,\alpha)$ is a Hom-Jordan algebra if the following identities:

$x\circ y=y\circ x$,

$as\alpha(x\circ y,\alpha(z),\alpha(w))+as\alpha(y\circ w,\alpha(z),\alpha(x))+as\alpha(w\circ x,\alpha(z),\alpha(y))=0$,

hold for all $x,y,z\in L$.

Definition 3.2: Let (L,μ,α,β) be a BiHom-algebra.

• The BiHom-associator ($(,\beta)$ -associator) of L is the bilinear map as $asa_\beta:L\times L\to L$ defined as:

$as\alpha\beta=\mu\circ(\mu\otimes\beta-\alpha\otimes\mu)$.

In terms of elements, the map $as\alpha\beta$ is given by:

$as\alpha\beta(x,y,z)=\mu(\mu(x,y),\beta(z))-\mu(\alpha(x),\mu(y,z))$

for all $x,y,z\in L$.

• Let A be a Hom-algebra over a field $\mathbb{K}$ of characteristic $\neq 2$ with an bilinear multiplication $\circ$. If $\alpha,:A\to A$ are two linear maps, then $(A,\circ,\alpha,\beta)$ is a BiHom-Jordan algebra if the following identities:

$x\circ y=y\circ x$,

$as\alpha(x\circ y,\alpha(z),\beta(w))+as\alpha(y\circ w,\alpha(z),\beta(x))+as\alpha(w\circ x,\alpha(z),\beta(y))=0$,

hold for all $x,y,z\in A$.

Proposition 3.3

Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra, with the operation $D_1\bullet D_2=D_1\circ D_2+D_2\circ D_1$, for all $D_1,D_2\in\mho$. Then $(\mho,\bullet,\alpha,\beta)$ is a BiHom-Jordan algebra.

Proof. Assume that $D_1,D_2,D_3,D_4\in\mho$. We have:

$D_1\bullet D_2=D_1\circ D_2+D_2\circ D_1$

$=D_2\circ D_1+D_1\circ D_2=D_2\bullet D_1$.

Since,

$$
\begin{aligned}
&((D_1\bullet D_2)\bullet\alpha(D_3))\bullet\beta(D_4)=((D_1\circ D_2+D_2\circ D_1)\bullet\alpha(D_3))\bullet\beta(D_4)\\
&=((D_1\circ D_2+D_2\circ D_1)\circ\alpha(D_3))\bullet\beta(D_4)+(\alpha(D_3)\circ(D_1\circ D_2+D_2\circ D_1))\bullet\beta(D_4)\\
&=((D_1\circ D_2)\circ\alpha(D_3))\bullet\beta(D_4)+((D_2\circ D_1)\circ\alpha(D_3))\bullet\beta(D_4)\\
&+(\alpha(D_3)\circ(D_1\circ D_2))\bullet\beta(D_4)+(\alpha(D_3)\circ(D_2\circ D_1))\bullet\beta(D_4)\\
&=((D_1\circ D_2)\circ\alpha(D_3))\circ\beta(D_4)+\beta(D_4)\circ((D_1\circ D_2)\circ\alpha(D_3))\\
&+((D_2\circ D_1)\circ\alpha(D_3))\circ\beta(D_4)+\beta(D_4)\circ((D_2\circ D_1)\circ\alpha(D_3))\\
&+(\alpha(D_3)\circ(D_1\circ D_2))\circ\beta(D_4)+\beta(D_4)\circ(\alpha(D_3)\circ(D_1\circ D_2))\\
&+(\alpha(D_3)\circ(D_2\circ D_1))\circ\beta(D_4)+\beta(D_4)\circ(\alpha(D_3)\circ(D_2\circ D_1))
\end{aligned}
$$

and,

$$
\begin{aligned}
\alpha(D_1\bullet D_2)\bullet(\alpha(D_3)\bullet\beta(D_4))&=\alpha(D_1\circ D_2+D_2\circ D_1)\bullet(\alpha(D_3)\circ\beta(D_4)+\alpha(D_3)\circ\beta(D_4))\\
&=\alpha(D_1\circ D_2))\bullet(\alpha(D_3)\circ\beta(D_4)+\beta(D_4)\circ\alpha(D_3))\\
&+\alpha(D_2\circ D_1)\bullet(\alpha(D_3)\circ\beta(D_4)+\beta(D_4)\circ\alpha(D_3))\\
&=\alpha(D_1\circ D_2)\circ(\alpha(D_3)\circ\beta(D_4))+\alpha(D_1\circ D_2)(\beta(D_4)\circ\alpha(D_3))\\
&+(\alpha(D_3)\circ\beta(D_4))\circ\alpha(D_1\circ D_2)+(\beta(D_4)\circ\alpha(D_3))\circ\alpha(D_1\circ D_2)\\
&+\alpha(D_2\circ D_1)\circ(\alpha(D_3)\circ\beta(D_4))+\alpha(D_2\circ D_1)(\beta(D_4)\circ\alpha(D_3))\\
&+(\alpha(D_3)\circ\beta(D_4))\circ\alpha(D_2\circ D_1)+(\beta(D_4)\circ\alpha(D_3))\circ\alpha(D_2\circ D_1),
\end{aligned}
$$

then, we have:

$$
\begin{aligned}
as_\alpha(D_1\bullet D_2,\alpha(D_3),\beta(D_4))&=\beta(D_4)\circ((D_1\circ D_2)\circ\alpha(D_3))+\beta(D_4)\circ((D_2\circ D_1)\circ\alpha(D_3))\\
&+\beta(D_4)\circ(\alpha(D_3)\circ(D_1\circ D_2))+\beta(D_4)\circ(\alpha(D_3)\circ(D_2\circ D_1))\\
&-\alpha(D_1\circ D_2)(\beta(D_4)\circ\alpha(D_3))-(\beta(D_4)\circ\alpha(D_3))\circ\alpha(D_1\circ D_2)\\
&-\alpha(D_2\circ D_1)(\beta(D_4)\circ\alpha(D_3))-(\beta(D_4)\circ\alpha(D_3))\circ\alpha(D_2\circ D_1).
\end{aligned}
$$

Therefore, we get:

$$
\begin{aligned}
&as_\alpha(D_1\bullet D_2,\alpha(D_3),\beta(D_4))+as_\alpha(D_2\bullet D_4,\alpha(D_3),\beta(D_1))\\
&+as_\alpha(D_4\bullet D_1,\alpha(D_3),\beta(D_2))=0,
\end{aligned}
$$

and so the statement holds.

Corollary 3.1: Let $(L,[,],\alpha)$ be a multiplicative BiHom-Lie algebra, with the operation $D_1\bullet D_2=D_1D_2+D_2D_1$, for all elements $D_1,D_2QC(L)$. Then the quadruplet $(QC(L),\bullet,\alpha,\beta)$ is a biom-Jordan algebra.

Proof. We need to show that $D_1\bullet D_2\in QC(L)$, for all $D_1\in QC_{\alpha^k\beta^l}(L)$, $D_2\in QC_{\alpha^s\beta^t}(L)$ and $x,y\in L$, we have:

$$
\begin{aligned}
&[D_1\bullet D_2,\alpha^{k+s}\beta^{l+t}(y)]=[D_1D_2,\alpha^{k+s}\beta^{l+t}(y)]+[D_2\bullet D_1,\alpha^{k+s}\beta^{l+t}(y)]\\
&=[\alpha^k\beta^l(D_2(x)),D_1(\alpha^s\beta^t(y))]+[\alpha^s\beta^t(D_1(x)),D_2(\alpha^k\beta^l(y))]\\
&=[D_2(\alpha^k\beta^l(x)),D_1(\alpha^s\beta^t(y))]+[D_1(\alpha^s\beta^t(x)),D_2(\alpha^k\beta^l(y))]\\
&=[\alpha^{k+s}\beta^{l+t}(x),D_1D_2(y)]+[\alpha^{k+s}\beta^{l+t}(x),D_2D_1(y)]=[\alpha^{k+s}\beta^{l+t}(x),D_1\bullet D_2(y)].
\end{aligned}
$$

Then $D_1\bullet D_2\in QC_{\alpha^{k+s}\beta^{l+t}}$ and $QC(L)$ is a BiHom-Jordan algebra.

Theorem 3.3: Let $(L,[,],\alpha,\beta)$ be a multiplicative BiHom-Lie algebra over $\mathbb{K}$. Then we have:

1. $QC(L)$ is a BiHom-Lie algebra with $[D_1,D_2]=D_1D_2-D_2D_1$, if

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and only if $QC(L)$ is also a BiHom-associative algebra with respect to composition.

2. If $Char\mathbb{K}\neq 2, \alpha,\beta$ are surjections and $Z(L)=\{0\}$, then $QC(L)$ is a BiHom-Lie algebra if $[QC(L), QC(L)]=0$.

Proof.

1. ("") For $D_1,D_2\in QC(L)$, we have $D_1 D_2\in QC(L)$ and $D_2 D_1\in QC(L)$, so $[D_1,D_2]=D_1D_2-D_2D_1\in QC(L)$. Hence, $QC(L)$ is a BiHom-Lie algebra.

("⇒") Note that $D_1D_2 = D_1 \bullet D_2 + \frac{[D_1,D_2]}{2}$ and by previous corollary, we have $D_1\bullet D_2\in QC(L)$, $[D_1,D_2]\in QC(L)$. It follows that $D_1D_2\in QC(L)$.

2. Assume that $D_1 \in QC_{\alpha^k\beta^l}(L)$, $D_2 \in QC_{\alpha^s\beta^t}(L)$, for all $x,y\in L$ there exists $y=\alpha^{k+s}\beta^{l+t}(y')$ since α and β are surjections. $QC(L)$ is a BiHom-Lie algebra, so $[D_1,D_2]\in QC_{\alpha^{k+s}\beta^{l+t}}(L)$, then:

$$[[D_1,D_2](x),y]=[[D_1,D_2](x),\alpha^{k+s}\beta^{l+t}(y')]$$

$$=[\alpha^{k+s}\beta^{l+t}(x),[D_1,D_2](y')].$$

From the proof of lemma (3.1), we have $2[[D_1,D_2](x),y]=0$. So $[D_1,D_2]=0$ since $Char\mathbb{K}\neq 2$.

The Quasiderivations of BiHom-Lie Algebras

In this section, we will prove that the quasiderivations of L can be embedded as derivations in a larger BiHom–Lie algebra and obtain a direct sum decomposition of $Der(L)$ when the annihilator of L is equal to zero.

Proposition 4.1

Let $(L,[,],\alpha,\beta)$ be a BiHom-Lie algebra over $\mathbb{K}$ and t an indeterminate. We define $\breve{L}=\{\sum(x\otimes t+y\otimes t^2)\,|\,x,y\in L\}$, $\breve{\alpha}(\breve{L})=\{\sum(\alpha(x)\otimes t+\alpha(y)\otimes t^2)\,|\,x,y\in L\}$ and $\breve{\beta}(\breve{L})=\{\sum(\beta(x)\otimes t+\beta(y)\otimes t^2)\,|\,x,y\in L\}$. Then $\breve{L}$ is BiHom-Lie algebra with the operation $[x\otimes t^i, y\otimes t^j]=[x,y]\otimes t^{i+j}$, for all $x,y\in L$, $i,j\in\{1,2\}$.

Proof. $\forall x_1,x_2,x_3\in L$, $i,j,k\in\{1,2\}$, we have:

$$[\breve{\beta}(x_1\otimes t^i),\breve{\alpha}(x_2\otimes t^j)]=[\beta(x_1),\alpha(x_2)]\otimes t^{i+j}$$

$$=-[\beta(x_2),\alpha(x_1)]\otimes t^{i+j}$$

$$=-[\breve{\beta}(x_2\otimes t^j),\breve{\alpha}(x_1\otimes t^i)].$$

Also,

$$[\breve{\beta}^2(x_1\otimes t^i),[\breve{\beta}(x_2\otimes t^j),\breve{\alpha}(x_3\otimes t^k)]]=[\beta^2(x_1),[\beta(x_2),\alpha(x_3)]]\otimes t^{i+j+k}$$

$$=-[\beta^2(x_3),[\beta(x_1),\alpha(x_2)]]\otimes t^{i+j+k}-[\beta^2(x_2),[\beta(x_3),\alpha(x_1)]]\otimes t^{i+j+k}$$

$$=-[\breve{\beta}^2(x_3\otimes t^k),[\breve{\beta}(x_1\otimes t^i),\breve{\alpha}(x_2\otimes t^j)]]$$

$$-[\breve{\beta}^2(x_2\otimes t^j),[\breve{\beta}(x_3\otimes t^k),\breve{\alpha}(x_1\otimes t^i)]].$$

Hence $\breve{L}$ is a BiHom-Lie algebra.

For convenience, we write $xt(xt^2)$ in place $x\otimes t(x\otimes t^2)$. If U is a subspace of L such that $L=U\oplus[L,L]$, then:

$$\breve{L}=Lt+Lt^2=Lt\oplus[L,L]t^2\oplus Ut^2.$$

Now, we define a map $\varphi:QDer(L)\to End(L)$ satisfying:

$$\varphi(D)(at+ut^2+bt^2)=D(a)t+D'(b)t^2$$

where $D\in QDer_{\alpha^k\beta^l}(L)$, D' is defined in equation, $a\in L, u\in U$ and $b\in[L,L]$.

Proposition 4.2

Let L, $\breve{L}$ and φ are defined as above.

1. φ is injective and $\varphi(D)$ does not depend on the choice of D'.

2. $\varphi(QDer(L))\subset Der(\breve{L})$.

Proof.

1. If $\varphi(D_1)=\varphi(D_2)$, then for all $a\in L$, $b\in[L,L]$, and $u\in U$, we have:

$\varphi(D_1)(at+ut^2+bt^2)=\varphi(D_2)(at+ut^2+bt^2)$,

that is:

$D_1(a)t+D_{1'}(b)^2=D_2(a)t+D_{2'}(b)^2$.

So $D_1(a)=D_2(a)$. Hence $D_1=D_2$, and φ est injective.

Suppose that there exists D'' such that:

$\varphi(D)(at+ut^2+bt^2)=D(a)t+D''(b)^2$,

and,

$[D(x),\alpha^k\beta^l(y)]+[\alpha^k\beta^l(x),D(y)]=D''([x,y])$.

Then we have $D'([x,y])=D''([x,y])$, thus $D'(b)=D''(b)$. Hence:

$\varphi(D)(at+ut^2+bt^2)=D(a)t+D'(b)t^2=D(a)t+D''(b)t^2$.

Which implies $\varphi(D)$ is determined by D.

2. We have $[xt^i,yt^j]=xyt^{i+j}=0$, for all $i+j\geq 3$. Thus, to show that $\varphi(D)\in Der(\breve{L})$, we need only to check the validness of the following equation:

$$\varphi(D)([xt,yt])=[\varphi(D)(xt),\breve{\alpha}^k\breve{\beta}^l(yt)]+[\breve{\alpha}^k\breve{\beta}^l(xt),\varphi(D)(yt)].$$

For all $x,y\in L$, we have:

$$\varphi(D)([xt,yt])=\varphi(D)([x,y]t^2)=D'([x,y])t^2$$

$$=([D(x),\alpha^k\beta^l(y)]+[\alpha^k\beta^l(x),D(y)])t^2$$

$$=[D(x)t,\alpha^k\beta^l(y)t]+[\alpha^k\beta^l(x)t,D(y)t]$$

$$=[\varphi(D)(xt),\breve{\alpha}^k\breve{\beta}^l(yt)]+[\breve{\alpha}^k\breve{\beta}^l(xt),\varphi(D)(yt)].$$

Therefore for all $D\in QDer_{\alpha^k\beta^l}(L)$, we have $\varphi(D)\in Der_{\breve{\alpha}^k\breve{\beta}^l}(\breve{L})$.

Proposition 4.3

Let L be a BiHom-Lie algebra. $Z(L)=\{0\}$ and $\breve{L}$, φ are defined as above. Then:

$$Der(\breve{L})=\varphi(QDer(L))\oplus ZDer(\breve{L}).$$

Proof. Since $Z(L)=\{0\}$, we have $Z(\breve{L})=Lt^2$. For all $g\in Der(\breve{L})$, we have $g(Z(\breve{L}))\subset Z(\breve{L})$, hence $g(Ut^2)\subset g(Z(\breve{L}))\subset Z(\breve{L})=Lt^2$. Now we define a map $f:Lt+Ut^2+[L,L]t^2\to Tt^2$ by:

$$f(x)=\begin{cases} g(x)\cap Lt^2, & x\in Lt\\ g(x), & x\in Ut^2\\ 0, & x\in[L,L]t^2\end{cases}$$

It is clear that f is linear map. Note that:

$$f([\breve{L},\breve{L}])=f([L,L]t^2)=0,$$

$$[f(\breve{L}),\breve{\alpha}^k\breve{\beta}^l(\breve{L})]\subset[Lt^2,\alpha^k\beta^l(L)t+\alpha^k\beta^l(T)t^2]=0,$$

hence $f\in ZDer_{\breve{\alpha}^k\breve{\beta}^l}(\breve{L})$. Since:

$(g-f)(Lt)=g(Lt)-g(Lt)\cap Lt^2=g(Lt)-Lt^2\subset Lt,(g-f)(Ut^2)=0,$

and,

$(g-f)([L,L]t^2) = g([\breve{L},\breve{L}]) \subset [\breve{L},\breve{L}] = [L,L]t^2,$

there exist D,D' in Equation (4) such that for all $a\in L$, $b\in[L,L]$,

$(g-f)(at)=D(a)t, (g-f)(bt^2)=D'(b)t^2$

Since $(g-f)\in Der(\breve{L})$ and by the definition of $Der(\breve{L})$, we have:

$[(g-f)(a_1t), \breve{\alpha}^k\breve{\beta}^l(a_2t)] + [\breve{\alpha}^k\breve{\beta}^l(a_1t),(g-f)(a_2t)] = (g-f)([a_1t,a_2t])$

for all $a_1,a_2\in L$. Hence,

$[D(a_1),\alpha^k\beta^l(a_2)t]+[\alpha^k\beta^l(a_1)t,D(a_2)]=D'([a_1,a_2])t^2$

Thus $D\in Der_{\alpha^k\beta^l}(L)$. Therefore $(g-f)=\varphi(D)\in\varphi(QDer(L))$, so $Der(\breve{L})\subset\varphi(QDer(L))+ZDer(\breve{L})$ By Proposition (4.2) (2), we have $Der(\breve{L})=\varphi(QDer(L))+ZDer(\breve{L})$.

For all $f\in\varphi(QDer(L))\cap ZDer(\breve{L})$, there exists an element $D\in QDer(L)$ such that $f=\varphi(D)$. Then:

$f(at+ut^2+bt^2)=\varphi(D)(at+ut^2+bt^2)=D(a)t+D'(b)t^2,$

for all $a\in L, b\in[L,L]$.

On the other hand, since $f\in ZDer(\breve{L})$, we have:

$f(at+ut^2+bt^2)\in Z(\breve{L})=Lt^2.$

That is to say, $D(a)=0$, for all $a\in L$ and so $D=0$. Hence $f=0$.

Therefore $Der(\breve{L})=\varphi(QDer(L))\oplus ZDer(\breve{L})$ as desired.

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