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Generalized Robertson-Walker Space-Times with W_1-Curvature Tensor

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Abstract

The fluid space-times carrying a W_1-curvature tensor are studied. It is proved that the energy momentum tensor of the space-time is of Codazzi type if and only if the W_1-curvature tensor is divergence free. We also show that the pseudo Ricci symmetric space-time with divergence free W_1-curvature tensor is a GRW space-time. A necessary and sufficient condition for a perfect fluid space-time to be a GRW space-time is given.

Keywords: Space-times; Curvature tensor; Semi-Riemannian geometry; Einstein's field equations

Introduction

Due to applications of the semi-Riemannian geometry in sciences, engineering, medical, etc. (specially, in the mathematical physics to study the general theory of relativity and cosmology), its study became most popular and interesting area among the researchers (specially for geometers and physicist). A smooth connected n-dimensional semi-Riemannian manifold along with the Lorentzian metric g of signature (+, +, +, . . . , +) is known as an n-dimensional Lorentzian manifold. To start the study of Lorentzian manifold the causal character of vectors played a significant role and therefore it become a suitable choice for the researcher to study the general theory of relativity and cosmology. A space-time means a Lorentzian manifold of dimension 4. The Einstein's field equations showed that the energy momentum tensor is divergence free [1]. Due to this requirement, the energy momentum tensor should be covariantly constant. In [2], Chaki et al. considered a general relativistic space-time and proved that if the energy momentum tensor on a space-time is covariantly constant, then the space-times is Ricci symmetric, that is, S=0, where and S denote the covariant derivative and the Ricci tensor corresponding to the metric g, respectively. In [3], De et al. taken a general relativistic space-time and proved that the energy momentum tensor is semisymmetric if and only if the space-time is Ricci semisymmetric. For more details about the space-times, we refer [4-7] and their references. On the other hand, if S=0 then the space-time may be called as weakly Ricci symmetric [8]. The notion of the pseudo Ricci symmetric Riemannian manifold has been introduced by Chaki [9]. Let M be an n-dimensional semi-Riemannian manifold, then M is said to be pseudo Ricci symmetric, denoted by (P RS)n, if it's the non-vanishing Ricci tensor S satisfies the tensorial relation

$$(\nabla U\ S)(V, Z)=2A(U)S(V, Z)+A(V)S(U, Z)+A(Z)S(U, V) \qquad (1.1)$$

for all vector fields U, V and Z on M , where A is the 1-form corresponding to the vector field ρ, that is, $g(U, \rho)=A(U)$. As a particular case if the 1-form A is zero, then the $(P\ RS)_n$ manifold converts in to the Ricci symmetric manifold.

Alias [10] introduced the notion of generalized Robertson-Walker (GRW) space-time, which is a generalization of the Robertson-Walker (RW) space-time. A GRW space-time of dimension n is an n-dimensional Lorentzian manifold M , that is, $M=-I\times f^2M^*$, where I is an open interval of the real line R, M^*, a Riemannian manifold of dimension (n-1) and f (>0), a smooth warping function (or scale factor). In [11], it is observed that the GRW space-times have applications in inhomogeneous space-times admitting an isotropic radiation. An n-dimensional Lorentzian manifold M with the metric (in local shape)

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta=-(dt)^2+f(t)^2g^*_{\alpha\beta}\ dx^\alpha dx^\beta,$$

where $g^*_{\alpha\beta}=g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only ($\alpha, \beta, \gamma=2, 3, \ldots, n$) and f, the warping function of t only, is known as GRW space-time. In particular, if $g\alpha^*\ \beta$ has dimension 3 and constant curvature, then the space-time converts into the RW space-time. For instance, we refer [12-17].

Lorentzian manifolds with a non-vanishing Ricci tensor S are known as the perfect fluid space-times if

$$S=ag+bA\otimes A, \qquad (1.2)$$

where a and b are scalar fields and $g(\rho, \rho)=-1$. O'Neill [1] in his book listed that a Robertson-Walker space-time is a perfect fluid space-time. It is also noticed that a GRW space-time (for n=4) is a perfect fluid if and only if it is RW space-time. If the energy-matter content of space-time is a perfect fluid with fluid velocity ρ, then the Einstein's field equations reflect that the Ricci tensor assumes the form (1.2) and the scalars a and b are linearly related to the pressure p and the energy density μ measured in the locally commoving inertial frame [16]. Many authors studied the properties of space-time but few are [18-20].

According to the geometers, a semi-Riemannian manifold with a non-zero Ricci tensor S, satisfies the equation (1.2), is known as a quasi-Einstein manifold. For instance [21-23]. Deszcz [23] investigated that a quasi-Einstein Rie-mannian manifold under certain conditions is a warped product $(+1)\times f^{\,2}M^*$, where M^* is an (n-1)-dimensional Riemannian manifold of constant curvature.

In [24], Derdzinski et al. discussed the existence of a non-trivial Codazzi tensor on a Riemannian manifold and listed its geometrical and topological consequences. The parallel tensors are the simplest example of the Codazzi tensors. A non-vanishing (0, 2)-type tensor field P on an n-dimensional semi-Riemannian manifold M is said to be of Codazzi type if

$$(\nabla_U P)(V, Z) = (\nabla_V P)(U, Z)$$

for all vector fields U, V and Z on M. As a particular case, if we replace P with S

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then M possesses a Codazzi type Ricci tensor, that is,

$$(\nabla_U S)(V, Z) = (\nabla_V S)(U, Z). \tag{1.3}$$

It is well known that the conformal curvature tensor C plays a lead role in the theory of relativity and cosmology. It is defined on an n-dimensional semi-Riemannian manifold M by

$$C(U,V)Z = R(U,V)Z - \frac{1}{n-2}$$
$$\left\{S(V,Z)U - S(U,Z)V + g(V,Z)QU - g(U,Z)QV - \frac{r}{(n-1)}[g(V,Z)U - g(U,Z)V]\right\}$$

for all vector fields U, V and Z on M [25]. Here R denotes the curvature tensor with respect to∇, Q, the Ricci operator corresponding to S and r is the scalar curvature of the manifold.

Motivated by the projective curvature tensor, Pokhariyal [26], defined and studied the relativistic significance of W_1-curvature tensor on an n-dimensional semi-Riemannian manifold, defined by

$$W_1(U,V)Z = R(U,V)Z + \frac{1}{n-1}\{S(V,Z)U - S(U,Z)V\} \tag{1.4}$$

for all vector fields U, V and Z on M. The properties of the same curvature tensor on the Lorentzian para-Sasakian manifolds have been studied by Pokhariyal [27].

The above studies motivate us to study the properties of space-time admitting W_1-curvature tensor. We plan our present work as follows: After introduction part in Section 1, we study the properties of the space-time endowed with the divergence free W_1-curvature tensor in Section 2. It is proved that a W_1-curvature tensor on a space-time is divergence free if and only if the Ricci tensor is of Codazzi type. If the divergence of W_1-curvature tensor on a space-time is zero, then the 1-form A is closed and the integral curves generated by a unit timelike vector field ρ are geodesics and also ρ is irrotational. We summarize this section by the result "A perfect fluid space-time with divergence free W_1-curvature tensor is a GRW space-time". Section 3 deals with the study of pseudo Ricci symmetric space-time admitting the divergence free W_1-curvature tensor. We show that the pseudo Ricci symmetric space-time satisfying divergence free W_1-curvature tensor is a GRW space-time. In Section 4, we consider a perfect fluid space-time satisfying the Einstein's field equations without cosmological term and investigate many results. A space-time admits a divergence free W_1-curvature tensor if and only if the energy momentum tensor is of Codazzi type. It is also proved that a perfect fluid space-time satisfying the Einstein's field equations without cosmological term has divergence free W_1-curvature tensor if and only if the space-time is a GRW space-time.

Divergence Free W_1-Curvature Tensor

This section deals with the study of fluid space-time carrying the divergence free W_1-curvature tensor. Let V_4 is a space-time, and then the W_1-curvature tensor, from (1.4), takes the form

$$W_1(U,V)Z = R(U,V)Z + \frac{1}{3}\{S(V,Z)U - S(U,Z)V\}. \tag{2.1}$$

The covariant derivative of (2.1) with respect to the vector field X gives

$$(\nabla_X W_1)(U,V)Z = (\nabla_X R)(U,V)Z + \frac{1}{3}\left\{(\nabla_X S)(V,Z)U - (\nabla_X S)(U,Z)V\right\}. \tag{2.2}$$

Considering a frame field and contracting (2.2), we find

$$(divW_1)(U,V)Z = (divR)(U,V)Z + \frac{1}{3}\left\{(\nabla_U S)(V,Z)\right\}, \tag{2.3}$$

where $(divW_1)(U, V)Z = \sum_{i=1}^{4} \in g\ ((\nabla_{ei}\ W_1)(U, V)Z, e_i)$, $\in_i = g(e_i, e_i)$. Here {ei, i=1, 2, 3, 4} denotes an orthonormal frame field of the space-time. It is well known that $(divR)(U, V)Z=(\nabla_U S)(V, Z) - (\nabla_V S)(U, Z)$ and therefore the equation (2.3) becomes

$$(divW_1)(U,V)Z = \frac{4}{3}\{(\nabla_U S)(V,Z) - (\nabla_V S)(U,Z)\}. \tag{2.4}$$

Thus we can state:

Theorem 2.1. *The space-time admits a Codazzi type Ricci tensor if and only if the W_1-curvature tensor is divergence free.*

If the space-time is Einstein, that is, $S(U, V)=\kappa_1 g(U, V)$, where κ_1 is a constant.

Then we have

$\nabla S=0$.

Thus we have the following theorem.

Theorem 2.2. *The divergence of W_1-curvature tensor on an Einstein space-time is always zero.*

Next, we are going to study the properties of divergence free W_1-curvature tensor on a perfect fluid space-time. A space-time V_4 is said to be a perfect fluid space-time if the non-vanishing Ricci tensor S of V_4 satisfies the equation (1.2).

The covariant derivative of (1.2) gives

$$(\nabla_U S)(V, Z)=da(U)g(V, Z) + db(U)A(V)A(Z)+b\{(\nabla_U A)(V)A(Z) + A(V)(\nabla_U A)(Z)\}. \tag{2.5}$$

if we suppose that the W1-curvature tensor of the space-time is divergence free, that is, (divW1) (U, V) Z=0. Then the Theorem 2.1 shows that the Ricci tensor S of V_4 to be of Codazzi type, that is, the equation (1.3) is satisfied on V4.

From (1.3) and (2.5), we have

$$da(U)g(V,Z)+db(U)A(V)A(Z)-da(V)g(U,Z)-db(V)A(U)A(Z)+b\{(\nabla_U A)(V)A(Z)+A(V)(\nabla_U A)(Z)-(\nabla_V A)(U)A(Z)-A(U)(\nabla_V A)(Z)\}=0. \tag{2.6}$$

Considering a frame field and then contracting (2.6) over U and Z we get

$$-3da(V)+db(V)+db(\rho)A(V)+b\{(\nabla_\rho A)(V)+A(V)\delta A\}=0, \tag{2.7}$$

where $\delta A = \sum_{i=1}^{4} \in_i \left(\nabla_{ei} A\right)\left(e_i\right)$ and $\in_i = g(e_i, e_i)$. Replacing U and Z with ρ in (2.6), we find

$$b(\nabla_\rho A)(V) = -3da(\rho)A(V) - 3da(V). \tag{2.8}$$

Taking a frame field and then contracting the equation (1.3) for U and V, we have

$dr(Z)=0 \Leftrightarrow r=\text{constant}$.

Again, considering a frame field and contracting (1.2) over U and V we conclude that

$r=4a- b$,

which gives

$$4da(U)=db(V). \tag{2.9}$$

In consequence of the equations (2.8) and (2.9), equation (2.7) takes the form

$$-2da(V)+da(\rho)A(V)+bA(V)\delta A=0. \tag{2.10}$$

Substituting V=ρ in (2.10), we obtain

$$\delta A=-\frac{3}{b}da(p), \tag{2.11}$$

Provided b≠0. We have from (2.9) and (2.11)

$$da(V)=-da(\rho)A(V),\ db(V-4da(\rho)A(V). \tag{2.12}$$

Replacing Z with ρ in (2.6), we get

$da(U)A(V)-db(U)A(V)-da(V)A(U)+db(V)A(U)-b\{(\nabla_U A)(V)-(\nabla_V A)(U)\}=0.$

In view of the equation (2.12), above equation assumes the form

$$b\{(\nabla_U A)(V)-(\nabla_V A)(U)\}=0, \tag{2.13}$$

which shows that the 1-form A is closed, provided b≠0. Thus we can state the following theorem.

Theorem 2.3. *If the W_1-curvature tensor is divergence free on a perfect fluid space-time, then the 1-form A is closed.*

Since b≠0 (in general), therefore the equation (2.13) gives

$(\nabla_U A)(V)=(\nabla_V A)(U).$

Setting V=ρ in the above equation, we get

$g(\nabla_U\rho, \rho)=g(\nabla_\rho\rho, U).$

Since $g(\rho, \rho)=-1\Rightarrow g(\nabla_U \rho, \rho)=0$ and hence $\nabla_\rho\rho=0$. This reflects that the integral curves generated by the velocity vector field ρ are geodesics and the vector field ρ is an irrotational vector field. Thus we are in position to state the following:

Theorem 2.4. *If the W_1-curvature tensor on a perfect fluid space-time is diver-gence free, then the integral curves generated by the unit timelike vector field ρ are geodesics and the vector field ρ is irrotational.*

Yano [28], studied the geometrical properties of a torse-forming vector field on a Riemannian manifold. This study on the semi-Riemannian manifolds has been extended by Sinyukov [29], Mikes [30] and many others.

Deftnition 2.5. *A vector field U_j on a semi-Riemannian manifold is said to be a torse-forming vector field if $\nabla_k U_j=\omega_k U_j+\varphi g_{kj}$, where φ is a scalar function and ωk, a non-vanishing 1-form .*

It is noticed that a unit time like torse-forming vector field u_i on a semi-Riemannian manifold M takes the following form:

$$\nabla_k u_j=\varphi\ (u_k u_j+g_{kj}). \tag{2.14}$$

For more details, we refer [31].

Motivated by the beautiful result of Chen [32]

Theorem 2.6. *[32] A Lorentzian n-manifold with n ≥ 3 is a generalized Robert-son-Walker space-time if and only if it admits a timelike concircular vector field.*

Mantica [31] proved the necessary and sufficient conditions for the Lorentzian manifold to be GRW space-time as:

Theorem 2.7. *A Lorentzian manifold of dimension n ≥ 3 is a GRW space-time if and only if it admits a unit timelike torse-forming vector, $\nabla_k u_j=\varphi(u_k u_j + g_{kj})$, that is also an eigen vector of the Ricci tensor [31].*

In consequence of (2.6) and (2.12), we obtain

$A(V)(\nabla_U A)(Z)-A(U)(\nabla_V A)(Z)=da(\rho)\{A(U)g(V,Z)-A(V)g(U,Z)\}.$

Changing U by ρ in the above equation and then using the equations (2.8) and (2.12) we find

$$(\nabla_V A)(Z)=-da(\rho)\{g(V,Z) + A(V)A(Z)\}, \tag{2.15}$$

which shows, from (2.14), that a unit timelike vector field ρ is a torse-forming vector field. Again in view of r=4a−b, equation (1.2) becomes

$S(U,V)=\alpha g(U,V)+(4\alpha - r)A(U)A(V),$

where r is a constant scalar curvature. Setting V=ρ in the above equation, we have

$S(U, \rho)=(r - 3a)g(U, \rho),$

provided r≠3a. This informs that the unit timelike torse-forming vector field ρ is an eigen vector of S corresponding to the eigen value r-3a. Above discussions along with the Theorem 2.7 state the following:

Theorem 2.8. *A perfect fluid space-time with divergence free W_1-curvature tensor is a GRW space-time.*

Let V_4 is a perfect fluid space-time endowed with a divergence free W_1-curvature tensor. We suppose that ρ⊥ is an orthonormal 3-dimensional distribution to ρ in V_4, then g(U, ρ)=g(V, ρ)=0. It is well known that

$(\nabla_U g)(V, \rho)=U\, g(V, \rho)-g(\nabla_U V, \rho)-g(V, \nabla_U \rho).$

In consequence of the equation (2.15), we have

$g(\nabla_U V, \rho)=da(\rho)g(V, U).$

In the same fashion, we can also show that

$g(\nabla_V U, \rho)=da(\rho)g(U,V).$

Since ∇ is a Levi-Civita connection, then g([U, V], ρ)=0. Hence [U, V] is orthog-onal to ρ and therefore [U, V] belongs to ρ⊥. In the light of [33], we can observe that ρ⊥ is an involute and thus the Frobineous theorem [33] implies that ρ⊥ is in-tegrable. This shows that the perfect fluid space-time along with a divergence free W_1-curvature tensor is locally a product space. Hence we can state:

Theorem 2.9. *A perfect fluid space-time together with the divergence free W_1-curvature tensor is locally a product space.*

Proposition 2.10. *If a perfect fluid space-time admits a divergence free W_1-curvature tensor, then we have*

I. $R(U, V)\rho=\varphi\{A(U)V - A(V)U \},$

II. $A(R(U, V)Z)=\varphi\{A(V)g(U, Z) - A(U)g(V, Z)\},$

III. $R(\rho, U)V=\varphi\{A(V)U - g(U, V)\rho\},$

IV. $S(U, \rho)=-3\varphi A(U),$

where $\varphi=d^2a(\rho, \rho)-(da(\rho))^2$.

Proof. From equation (2.12), we have

$d^2a(U, V)=-d^2a(\rho, V)A(U)-da(\rho)(\nabla_V A)(U),$

which gives

$$d^2a(\rho, V)A(U)=d^2a(\rho, U)A(V). \tag{2.16}$$

In consequence of (2.15), we find that

$$\nabla_U \nabla_V \rho=-d^2 a(\rho, U)\{V+A(V)\rho\}-da(\rho)\{\nabla_U V+(\nabla_U A)(V)\rho+A(\nabla_U V)\rho+A(V)\nabla_U \rho\}. \quad (2.17)$$

Interchanging U and V in (2.17), we get

$$\nabla_V \nabla_U \rho=-d^2 a(\rho, V)\{U+A(U)\rho\}-da(\rho)\{\nabla_V U+(\nabla_V A)(U)\rho+A(\nabla_V U)\rho+A(U)\nabla_V \rho\}. \quad (2.18)$$

Equations (2.15)-(2.18) give

$$R(U, V)\rho=\{d^2a(\rho, \rho)-(da(\rho))^2\}\{A(U)V-A(V)U\}. \quad (2.19)$$

This proves the Proposition 2.10 (i). Other parts can be easily obtained from (2.19).

Now, we prove the following theorem.

Theorem 2.11. *On a space-time, the W_1-curvature tensor is divergence free if and only if the divergence of the conformal curvature tensor is zero.*

Proof. Let us suppose that the W_1-curvature tensor on a space-time is divergence free, then Theorem 2.1 says that the space-time possesses a Codazzi type Ricci tensor, that is, the equation (1.3) is satisfied and hence r = constant. Also, we have

$$(div\ C)(U,V)Z=\frac{1}{2}\{(\nabla_U S)(V,Z)-(\nabla_V S)(U,Z)\}+\frac{1}{6}\{dr(U)g(V,Z)-dr(V)g(U,Z)\}. \quad (2.20)$$

In consequence of the above discussions, equation (2.20) gives div C=0. Con-versely, we assume that the space-time is conformally divergence free and thus the equation (2.20) becomes

$$(\nabla_U S)(V,Z)-(\nabla_V S)(U,Z)=-\frac{1}{3}\{dr(U)g(V,Z)-dr(V)g(U,Z)\}.$$

Considering a frame field and contracting for V and Z we get dr(U)=0 $\Rightarrow$ r=constant. Hence the above equation shows that the Ricci tensor is of Codazzi type and thus the W1-curvature tensor is divergence free. Hence the statement of the Theorem 2.11 is proved.

Pseudo Ricci Symmetric Space-time Admitting Divergence free W_1-curvature Tensor

This section deals with the study of pseudo Ricci symmetric space-time with vanishing divergence of W_1-curvature tensor.

Let us suppose that the space-time is pseudo Ricci symmetric, then the equations (1.1) and (2.4) give

$$(divW_1)(U, V)Z=\frac{4}{3}\{A(U)S(V, Z)-A(V)S(U, Z)\}. \quad (3.1)$$

Suppose that the W_1-curvature tensor on the pseudo Ricci symmetric space-time is divergence free. Then the equation (3.1) becomes

$$A(U)S(V, Z)-A(V)S(U, Z)=0. \quad (3.2)$$

Setting V=ρ in (3.2), we get

$$S(U, Z)=-A(U)S(\rho, Z). \quad (3.3)$$

Considering a frame field and contracting over the vector fields V and Z in (3.2) we have

$$S(U, \rho)=rA(U). \quad (3.4)$$

In the light of the equation (3.4), equation (3.3) becomes

$$S(U, Z)=-rA(U)A(Z). \quad (3.5)$$

This shows that the space-time under consideration is a special type of perfect fluid space-time.

In [15], Mantica et al. proved the following:

"Let (M, g) be an n (> 3)-dimensional Lorentzian manifold. If the Ricci tensor has the form S(U,V)=-rA(U)A(V) and the divergence of the conformal curvature tensor vanishes, then there exists a suitable coordinate domain u of M such that on this set the space is a GRW space-time with Einstein fibers".

After considering these facts, we can state the following theorem.

Theorem 3.1. *A pseudo Ricci symmetric space-time satisfying divergence free W1-curvature tensor is a GRW space-time.*

A space-time is said to be Ricci simple if its Ricci tensor satisfies (3.5). Thus we can state:

Corollary 3.2. *A pseudo Ricci symmetric space-time with the divergence free W_1-curvature tensor is Ricci-simple.*

From (3.3), we can state:

Theorem 3.3. *Let W_1-curvature tensor is divergence free on a pseudo Ricci sym- metric space-time. Then the unit timelike vector field ρ is an eigen vector of the Ricci tensor S corresponding to the eigen value −r.*

Perfect Fluid Space-times without Cosmological Terms

In this section, we consider a perfect fluid space-time satisfying the Einstein's field equations without cosmological term and the divergence of W_1-curvature ten-sor is zero.

The Einstein's field equations without cosmological term are given by

$$S(U,V)-\frac{1}{2}rg(U,V)=\kappa T\ (U,V) \quad (4.1)$$

for all vector fields U and V , where κ is a non-zero gravitational constant and T , the energy momentum tensor of the space-time. It is obvious from equation (4.1) that

$$-r = \kappa T, \quad (4.2)$$

$$(\nabla_U S)(V, Z)-(\nabla_V S)(U, Z)=\kappa\{(\nabla_U T)(V, Z)-(\nabla_V T)(U, Z)\}+\frac{1}{2}\{dr(U)g(V, Z)-dr(V)g(U, Z)\}, \quad (4.3)$$

where T is the trace of T. If we suppose that the energy momentum tensor

T of the space-time is of Codazzi type, then the equation (4.3) leads to

$$2\{(\nabla_U S)(V, Z)-(\nabla_V S)(U, Z)\}=dr(U)g(V, Z)-dr(V)g(U, Z). \quad (4.4)$$

Considering a frame field and contracting for V and Z we get dr(U)=0, which implies that r=constant. Thus, like others, we can also state the following:

Theorem 4.1. *If a perfect fluid space-time admits the Codazzi type energy momentum tensor, then it possesses a constant scalar curvature.*

Next, we prove the following theorem.

Theorem 4.2. *The W_1-curvature tensor on a perfect fluid space-time is divergence free if and only if the energy momentum tensor is of Codazzi type.*

Proof. We suppose that the W_1-curvature tensor of the perfect fluid space-time is divergence free, and then the Theorem 2.1 shows that the Ricci tensor of the space-time is of Codazzi type, that is, the equation (1.3) is satisfied. Taking a frame field and contracting (1.3) for V and Z, we conclude that dr(U)=0. This result along with (4.3) gives

$$(\nabla_U T)(V,Z)=(\nabla_V T)(U,Z), \tag{4.5}$$

provided $\kappa\neq 0$. This shows that the energy momentum tensor is of Codazzi type. Conversely, we assume the perfect fluid space-time admits (4.5), and then equation (4.3) leads to the equation (4.4). Hence the equation (1.3), Theorem 2.1 and Theorem 4.1 prove the converse part of the Theorem 4.2.

Let us suppose that the perfect fluid space-time is dust. Therefore

$$T(V,Z)=\mu A(V)A(Z), \tag{4.6}$$

where μ is the energy density and A is a 1-form associated with the velocity vector field ρ of the perfect fluid. Covariant derivative of (4.6) along U gives

$$(\nabla_U T)(V,Z)=(\nabla_U \mu)A(V)A(Z)+\mu\{(\nabla_U A)(V)A(Z)+(\nabla_U A)(Z)A(V)\}. \tag{4.7}$$

If we assume that the dust has divergence free 1-curvature tensor. From equations (4.6) and (4.7), we obtain

$$T=-\mu,\ dT(U)=-d\mu(U). \tag{4.8}$$

In view of (4.2), equation (4.8) leads to μ =constant. Hence the equation (4.8) takes the form

$$(\nabla_U T)(V,Z)=\mu\{(\nabla_U A)(V)A(Z)+(\nabla_U A)(Z)A(V)\}. \tag{4.9}$$

Theorem 4.2 and equation (4.9) lead to

$$(\nabla_U A)(V)A(Z)+(\nabla_U A)(Z)A(V)-(\nabla_V A)(U)A(Z)-(\nabla_V A)(Z)A(U)=0, \tag{4.10}$$

provided $\mu\neq 0$. Changing Z with ρ in (4.10), we have

$$(\nabla_U A)(V)-(\nabla_V A)(U)=0. \tag{4.11}$$

This shows that Curl ρ=0 and thus ρ is an irrotational. Hence the vorticity of the fluid vanishes. Again changing U with ρ in (4.11), we obtain

$$\nabla_\rho \rho=0.$$

This means that the integral curves generated by the unit timelike velocity vector field ρ are geodesic curves. Hence we are in condition to state the following:

Theorem 4.3. *If a dust is satisfying the Einstein's field equations without cos-mological term and admits a divergence free W_1-curvature tensor, then the perfect fluid is vorticity free and the integral curves generated by the unit timelike velocity vector field are geodesics.*

Next, we suppose that the energy momentum tensor of the perfect fluid space- time without cosmological term takes the form

$$T_{ji}=(\mu+p)u_j u_i+pg_{ji}, \tag{4.12}$$

where μ is defined in (4.6) and p, isotropic pressure of the perfect fluid. Here T_{ji}, u_i and g_{ij} denote, respectively, the energy momentum tensor, velocity vector field of the fluid and the Lorentzian metric of the fluid. Multiplying (4.12) with g^{ji}, we obtain

$$T=-\mu+3p,$$

which gives

$$r=\kappa(\mu-3p), \tag{4.13}$$

where equation (4.2) is used.

Suppose that the perfect fluid space-time along with the energy momentum tensor defined by (4.12) possesses a divergence free W_1-curvature tensor, then we have r=constant and hence the equation (4.13) leads to

$$\mu-3p=\text{constant}. \tag{4.14}$$

Thus we state:

Theorem 4.4. *If a perfect fluid space-time equipped with the energy momentum tensor defined by (4.12) satisfies the Einstein's field equations without cosmological term, and W_1-curvature tensor is free from divergence, then the state equation is μ=3p+constant.*

In particular, if the constant in (4.14) becomes zero, then we have $\mu=3p$ and hence the following:

Corollary 4.5. *Suppose a perfect fluid space-time satisfies the Einstein's field equa- tions without cosmological term, and has divergence free W_1-curvature tensor, then it is filled with radiation.*

From (4.12), we have

$$T_{ji;k}=(\mu+p)_{;k}u_j u_i+p_{;k}g_{ji}+(\mu+p)\{u_{j;k}u_i+u_j u_{i;k}\}. \tag{4.15}$$

We consider that the W_1-curvature tensor is divergence free on the perfect fluid, then the Theorem 4.2 says that the energy momentum tensor to be of Codazzi type and then the equation (4.15) leads to

$$T_{ij;k}-T_{ik;j}=(\mu+p)_{;k}u_j u_i+p_{;k}g_{ji}+(\mu+p)\{u_{j;k}u_i+u_j u_{i;k}\}-(\mu+p)_{;j}u_k u_i-p_{;j}g_{ik}-(\mu+p)\{u_{k;j}u_i+u_k u_{i;j}\}=0. \tag{4.16}$$

Multiplication of (4.16) with u^k gives

$$(\mu+p)u_j u_i+p\, g_{ji}+(\mu+p)\{u_j u_i+u_j u_i\}+(\mu+p)_{;i}u_j-p_{;j}u_j+(\mu+p)u_{j;i}=0, \tag{4.17}$$

Here overhead dot represents the covariant derivative along the fluid flow vector u_i ($u^i u_i=-1$, $u^k u_{i;k}=u_i$). It is obvious that the divergence equation $T^{ij}_{\ ;ij}=0$

gives

$$(\mu+p)u_i=-p_{;i}-pu_i, \tag{4.18}$$

$$\mu=(\mu+p)u^i_{\ ;i}=-(\mu+p)\vartheta, \tag{4.19}$$

where ϑ is the expansion scalar. The equations (4.18) and (3.20) represent the force equation and the energy equation, respectively. Also, we have [34]

$$u_{i;j}=\frac{\vartheta}{3}\left(g_{ij}+u_i u_j\right)-u_i u_j+\sigma_{ij}+\omega_{ij}, \tag{4.20}$$

where u_i is the acceleration vector, σ_{ij}, a symmetric shear tensor and ω_{ij}, the vorticity or rotation tensor. Using (4.17) and (4.19), we get

$$(\mu-p)\, u_j u_i+(\mu-p)_{;i}u_j-p_{;j}u_i+p\, g_{ji}+(\mu+p)u_{j;i}=0. \tag{4.21}$$

After multiplication with u^j, it reduces to

$$(\mu-p)_{;i}=-(\mu-p)\, u_i. \tag{4.22}$$

The equation (4.18) along with (4.22) give

$$(\mu+p)u_i=\mu\, u_i-\mu_{;i}. \tag{4.23}$$

In consequence of (4.22), equation (4.21) assumes the form

$$-p_{;j}u_i+pg_{ji}+(\mu+p)u_{j;i}=0. \tag{4.24}$$

The contraction of (4.24) with gji gives

$$p=-\frac{1}{3}(\mu+p)\vartheta. \tag{4.25}$$

Equation (4.25) together with the energy equation (4.19) becomes

$$3p=\mu. \tag{4.26}$$

From force equation, $p_{;i}=pu_i-(\mu+p)u_i$ and hence the equation (4.24) takes the form

$$p(u_ju_i+g_{ji})+(\mu+p)u_ju_i+(\mu+p)u_{j;i}=0,$$

which, with the help of (4.25), leads to

$$(\mu+p)\{-\frac{\vartheta}{3}(u_ju_i+g_{ji})+u_ju_i+u_{j;i}\}=0. \tag{4.27}$$

Hawking [35] listed in his book that $\mu>0$, $\mu+p\neq0$ and therefore by equation (4.27) we get

$$u_{j;i}=\frac{\vartheta}{3}(u_ju_i+g_{ji})-u_ju_i. \tag{4.28}$$

In view of (4.26), we have

$$3p-\mu=\text{constant}.$$

Next, we suppose that $\mu+p\neq0$ and the equation (4.28) is satisfied. The equations (4.20) and (4.28) give

$$\sigma_{ij}+\omega_{ij}=0,$$

which holds if $\sigma_{ij}=0$ and $\omega_{ij}=0$. With the help of (4.16), (4.18), (4.22), (4.23) and (4.28), it is no hard to get

$$(\mu+p)u_k=0,$$

which gives $u_k=0$ and hence the equation (4.28) takes the form

$$u_{j;i}=\frac{\vartheta}{3}(u_ju_i+g_{ji}).$$

This shows that the unit timelike vector field u_i is a torse-forming vector field. In consequence of (4.1) and (4.5), we get the following expression for the Ricci tensor in index free notation as:

$$S(U,V)=(\kappa p+\frac{r}{2})g(U,V)+\kappa(\mu+p)A(U)A(V). \tag{4.29}$$

Setting $V=\rho$ in (4.29), we get

$$S(U,\rho)=\frac{1}{2}(r-2\kappa\mu)g(U,\rho).$$

This implies that the unit time like torse-forming vector field ρ is an eigen vector of S corresponding to the eigen value $\frac{1}{2}(r-2\kappa\mu)$. Thus the Theorem 2.7 shows that the space-time under consideration is a GRW space-time. To prove the converse part, we suppose that the perfect fluid space-time satisfies the equation (4.28) and $\mu + p\neq0$. If the perfect fluid is shear-free, rotation-free, and acceleration-free and the energy density and pressure are constants, then we can notice that the energy momentum tensor is of Codazzi type and hence the W_1-curvature tensor is divergence free. Thus we have the following:

Theorem 4.6. *A perfect fluid space-time satisfying the Einstein's field equations without cosmological term has divergence free W_1-curvature tensor if and only if the space-time is a GRW space-time.*

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