Global Classical Solutions to the Mixed Initial-boundary Value Problem for a Class of Quasilinear Hyperbolic Systems of Balance Laws

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Abstract
It is proven that the mixed initial-boundary value problem for a class of quasilinear hyperbolic systems of balance laws with general nonlinear boundary conditions in the half space \(|\{(t, x) | t \geq 0, x \geq 0\}| \) admits a unique global C1 solution \(u = u(t, x)\) with small C1 norm, provided that each characteristic with positive velocity is weakly linearly degenerate. This result is also applied to the flow equations of a model class of fluids with viscosity induced by fading memory.

MSC: 35L45; 35L50; 35Q72.

Keywords: Mixed initial-boundary value problem; Global classical solution; Quasilinear hyperbolic; systems of balance laws; Weakly linearly degenerate characteristics

Introduction and Main Result
Consider the following quasilinear hyperbolic system of balance laws in one space dimension:
\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + Lu = 0
\]
(1.1)
where \(L > 0\) is a constant; \(u=(u_1, \ldots, u_n)^T\) is the unknown vector function of \((t, x)\), \(f(u)\) is a given \(C^3\) vector function of \(u\).

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given \(u\) on the domain under consideration, the Jacobian \(A(u) = \nabla f(u)\) has \(n\) real distinct eigenvalues.
\[
\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)
\]
(1.2)

Let \(l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))\) (resp. \(r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))\))^T be a left (resp. right) eigenvector corresponding to \(\lambda_i(u)(i=1, \ldots, n)\)
\[
l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \text{resp.} \quad A(u)r_i(u) = \lambda_i(u)r_i(u)
\]
(1.3)
than we have
\[
\det |l_j(u)| \neq 0 \quad \text{(Equivalently, } \det |l_j(u)| \neq 0).
\]
(1.4)
Without loss of generality, we may assume that on the domain under consideration
\[
l_i(u)r_j(u) = \delta_{ij}(i, j = 1, \ldots, n)
\]
(1.5)
And
\[
r_i^T(u)r_j(u) = l_i(i=1, \ldots, n)
\]
(1.6)
Where \(\delta_{ij}\) stands for the Kronecker's symbol.

Clearly, all \(\lambda_i(u), l_i(u)\) and \(r_i(u)(i, j = 1, \ldots, n)\)
\[
have the same regularity as \(A(u)\), i.e., \(C^2\) regularity.
\]
We assume that on the domain under consideration, each characteristic with positive velocity is weakly linearly degenerate and the eigenvalues of \(A(u) = \nabla f(u)\)
\[
\lambda r(u) < 0 < \lambda s(u)
\]
(1.9)
satisfy the non-characteristic condition.
\[
(r = l_1, \ldots, m; s = m+1, \ldots, n)
\]
(1.10)
We are concerned with the existence and uniqueness of global C1 solutions to the mixed initial-boundary value problem for system (1.1) in the half space
\[
D = \{(t, x) | t \geq 0, x \geq 0\}
\]
(1.11)
with the initial condition:
\[
t = 0 : u = \phi(x)(x \geq 0)
\]
(1.12)
and the nonlinear boundary condition:
\[
x = 0 : v_r = G_r(\alpha(t), v_{1m} + h(t), s = m+1, \ldots, n(t \geq 0)
\]
(1.13)
Where
\[
v_r = l_i(u)u(i=1, \ldots, n)
\]
(1.14)
And
\[
\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))
\]
Here, \(\phi = (\phi_1, \ldots, \phi_m)^T, \alpha_r, G_r\) and \(h(s = m+1, \ldots, n)\) are all \(C^1\) functions with respect to their arguments, which satisfy the conditions of \(C^1\) compatibility at the point \((0; 0)\). Also, we assume that there exists a constant \(M > 0\) such that
\[
0 \leq \max_{x \geq 0} \sup_{x \geq 0} |(1+x)l + \mu| |\phi(x)| + |\phi'(x)| + \mu (1+t)|l + \mu| |\phi(t)| + |h(t)| + |\alpha(t)| + |\alpha'(t)| + |h'(t)|) < +\infty
\]
(1.15)
in which
\[
h(t) = (h_{m1}(t), \ldots, h_n(t))
\]

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Without loss of generality, we assume that
\[ G_i(\alpha(t),0) = 0 (s=m+1,\ldots,n) \quad (1.16) \]

For the special case where (1.1) is a quasilinear hyperbolic system of
conservation laws, i.e., \( \Lambda = 0 \), such kinds of problems have been
extensively studied (for instance, [1-8] and the references therein).
In particular, Li and Wang proved the existence and uniqueness of
global \( C^1 \) solutions to the mixed initial boundary value problem for
first order quasilinear hyperbolic systems with general nonlinear
boundary conditions in the half space \( \{(t,x) | t \geq 0, x \geq 0\} \). On the
other hand, for quasilinear hyperbolic systems of balance laws, many
results on the existence of global solutions have also been obtained
by Liu, et al., (for instance, see [8-14] and the references therein),
and some methods have been established. So the following question arises
naturally: when can we obtain the existence and uniqueness of semi-
global \( C^1 \) solutions for quasilinear hyperbolic systems of balance laws?
It is well known that for first-order quasilinear hyperbolic systems of
balance laws, generically speaking, the classical solution exists only
locally in time and the singularity will appear in a finite time even if
the data are sufficiently smooth and small [15-20]. However, in some
cases global existence in time of classical solutions can be obtained.
In this paper, we will generalize the results in [21] to a nonhomogeneous
quasilinear hyperbolic system, the analysis relies on a careful study of
the interaction of the nonhomogeneous term. Our main results can be
stated as follows:

**Theorem 1.1.** Suppose that the non-characteristic condition
(1.10) holds and system (1.1) is strictly hyperbolic. Suppose
furthermore that for \( j = m + 1,\ldots, n \) each \( j \)-characteristic field
with positive velocity is weakly linearly degenerate. Suppose finally
that \( \varphi, \alpha, G_i, h_i (s = m+1,\ldots, n) \) are all \( C^1 \) functions with respect to
their arguments, satisfying (1.15)-(1.16) and the conditions of \( C^1 \)
compatibility at the point \( (0,0) \). Then there exists a sufficiently small
\( \theta_0 > 0 \) such that for any given \( 0 \in [0, \theta_0] \) the mixed initial-boundary
value problem (1.1) and (1.12)-(1.13) admits a unique global \( C^1 \)
solution \( u = u(t,x) \) in the half space \( \{(t,x) | t \geq 0, x \geq 0\} \) in the half space
\( \{(t,x) | t \geq 0, x \geq 0\} \).

The rest of this paper is organized as follows. In Section 2, we give the
main tools of the proof that is several formulas on the decomposition of
waves for system (1.1), which will play an important role in our discussion.

**Lemma 2.1.**
\[
\frac{d(e^{\lambda u} w_i)}{dt} = \sum_{j=1}^{n} e^{\lambda_j u} w_j w_i + \sum_{j=1}^{n} e^{\lambda_j u} \gamma_{ij} (u) v_j w_i (i = 1,\ldots,n) \quad (2.6)
\]

Where
\[
\gamma_{ij} (u) = (\lambda_i (u) - \lambda_j (u))^T (u) v_j (u) - \nabla \lambda_i (u) (\nabla \lambda_j (u) v_j (u) \delta_{ij}) \quad (2.7)
\]

Hence, we have
\[
\tilde{\gamma}_{ij} (u) = 0, \forall j \neq i, i = 1,\ldots,n \quad (2.9)
\]

Moreover, in the normalized coordinates,
\[
\tilde{\gamma}_{ij} (u,e) = 0, \forall u_i | small, \forall i, j \quad (2.11)
\]

while, when the \( j \)-characteristic \( \lambda_j (u) \) is weakly linearly degenerate,
in the normalized coordinates,
\[
\gamma_{ij} (u,e) = 0, \forall | u_i | small, \forall i. \quad (2.12)
\]

**Lemma 2.2.**
\[
\frac{d(e^{\lambda u} v_i)}{dt} = \sum_{j=1}^{n} e^{\lambda_j u} v_j w_i + \sum_{j=1}^{n} e^{\lambda_j u} \beta_{ij} (u) v_j v_i (i = 1,\ldots,n) \quad (2.13)
\]

Where
\[
\beta_{ij} (u) = (\lambda_i (u))^T (u) \nabla v_i (u) \quad (2.14)
\]

Thus, we have
\[
\tilde{\beta}_{ij} (u) = 0, \forall i, j (i, j = 1,\ldots,n) \quad (2.16)
\]

Moreover, by (2.1), in the normalized coordinates we have
is given by (2.8) and (2.22) that

$$\Gamma_{ij}(u) = (\lambda_k(u) - \lambda_i(u)) r_{ij}(u) V_1(u) r_{ij}(u)$$

(2.29)

Hence,

$$\Gamma_{ij}(u) = 0 \quad \forall i,j$$

(2.30)

Proof. Differentiating the first equation of (2.27) with respect to $y$ gives

$$\frac{d}{dt} \tilde{\chi}(t,y) = \nabla \lambda(u(t, \tilde{s}(t,y))) \frac{\partial u}{\partial y} \tilde{\chi}(t,y) \frac{\partial}{\partial y}$$

(2.31)

Then, noting (2.6), it follows from (2.31) that

$$\frac{d}{dt} \tilde{\chi}(t,y) = \frac{\partial u}{\partial y} \tilde{\chi}(t,y) + \frac{\partial w}{\partial y} \tilde{\chi}(t,y) = \frac{\partial u}{\partial y} \tilde{\chi}(t,y)$$

(2.32)

Thus, from (2.4), (2.7) and (2.32), we immediately get (2.28)-(2.30).

The proof of Lemma 2.4 is finished.

Similarly, noting (2.4), by (2.13) and (2.31), we have

$$\tilde{\chi}(t,y) = \frac{\partial u}{\partial y} \tilde{\chi}(t,y) + \frac{\partial w}{\partial y} \tilde{\chi}(t,y)$$

(2.33)

Thus along the $i$th characteristic

$$\lambda(x,y) = \tilde{s}(t,y)$$

(2.34)

By (2.16), it is easy to see that

$$\tilde{B}_i(u) = \tilde{B}_i(u) + \nabla \lambda(u) r_i(u) \delta_0$$

(2.35)

Moreover, by (2.17), in the normalized coordinates we have

$$B_i(u) = 0 \quad \forall |u_j| \text{ small}, \forall j \neq i$$

(2.36)

while, when the $i$th characteristic $\lambda_i(u)$ is weakly linearly degenerate, in the normalized coordinates,

$$B_i(u) = 0 \quad \forall |u_j| \text{ small, } \forall i$$

(2.37)

Lemma 2.6. Let $\tilde{z}(t,x)$ be defined by $\tilde{z}(t,s(t,y)) = u_i(t, \tilde{s}(t,y)) \frac{\partial \tilde{\chi}(t,y)}{\partial y}$

Then along the $i$th characteristic

$$\tilde{z}(t,x) = \tilde{s}(t,y)$$

(2.38)

where $\tilde{z}(t,x)$ is given by (2.8) and

$$\frac{d(\tilde{z}_i)}{dt} = \sum_{i=1}^{n} \epsilon_i \Gamma_{ij}(u) \frac{\partial \tilde{\chi}(t,y)}{\partial y} w_i w_j + \sum_{i=1}^{n} \epsilon_i \tilde{r}_{ij}(u) \frac{\partial \tilde{\chi}(t,y)}{\partial y} v_i w_j$$

(2.39)

where $\tilde{r}_{ij}(u)$ is given by (2.8) and

$$\frac{d(\epsilon_i)}{dt} = \sum_{i=1}^{n} \epsilon_i \Gamma_{ij}(u) \frac{\partial \tilde{\chi}(t,y)}{\partial y} w_i w_j + \sum_{i=1}^{n} \epsilon_i \tilde{r}_{ij}(u) \frac{\partial \tilde{\chi}(t,y)}{\partial y} v_i w_j$$

(2.40)

By (2.20) and (2.22), it is easy to see that

$$F_{ij}(u) = 0 \quad \forall i \neq j, \forall j = 1, \ldots, n$$

(2.41)

$$F_{ij}(u) = 0 \quad \forall i \neq j, \forall j = 1, \ldots, n$$

(2.42)
And
\[ F_i(u) = \nabla \lambda_i(u) \tau_i(u) \quad \forall i = 1, \ldots, n \] (2.43)

Proof. Differentiating the first equation of (2.27) with respect to \( y \) gives
\[ \frac{d}{dt} \left( \frac{\partial \lambda_i}{\partial y}(t, y) \right) = \nabla \lambda_i(u(t, \tilde{x}(t, y))) \frac{\partial u}{\partial y}(t, \tilde{x}(t, y)) + \frac{\partial \lambda_i}{\partial y}(t, y) \] (2.44)

Then, noting (2.19), it follows from (2.44) that
\[ \frac{d}{dt} \left( \frac{\partial \lambda_i}{\partial y}(t, y) \right) = \sum_{i=1}^{m} \left[ \frac{c_i}{p_0(u(t, \tilde{x}(t, y)))} u_i W_i(t, x) + c_i^{\mu} \nabla \lambda_i(u(t, \tilde{x}(t, y))) \frac{\partial u_i}{\partial y}(t, \tilde{x}(t, y)) \right] \] (2.45)

Thus, from (2.4), (2.20)-(2.22) and (2.45), we immediately get (2.39)-(2.43). The proof of Lemma 2.6 is finished.

Proof of Theorem 1.1

By the existence and uniqueness of a local \( C^1 \) solution for quasilinear hyperbolic systems [22], there exists \( T_0 > 0 \) such that the mixed initial-boundary value problem (1.1) and (1.12)-(1.13) admits a unique \( C^1 \) solution \( u = u(t; x) \) on the domain
\[ D(T_0) \]

Thus, in order to prove Theorem 1.1 it suffices to establish a uniform a priori estimate for the \( C^1 \) norm of \( u \) and \( u_x \) on any given domain of existence of the \( C^1 \) solution \( u = u(t; x) \).

Noting (1.2) and (1.10), we have
\[ \lambda_i(0) < \ldots < \lambda_l(0) < \lambda_{l+1}(0) < \ldots < \lambda_n(0) \] (3.2)

Thus, there exist sufficiently small positive constants \( \delta \) and \( \delta_0 \) such that
\[ \lambda_{l+1}(u) - \lambda_l(u) \geq 4\delta_0 \quad \forall \ l \quad |v| \leq \delta(1, \ldots, n-l) \] (3.3)

\[ \lambda_l(u) - \lambda_l(u) \leq \frac{\delta_0}{2} \quad \forall \ l \quad |v| \leq \delta(1, \ldots, n) \] (3.4)

And
\[ |\lambda_i(0)| \geq \delta_0 \quad (i = 1, \ldots, n) \] (3.5)

For the time being it is supposed that on the domain of existence of the \( C^1 \) solution \( u = u(t; x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have
\[ |u(t, x)| \leq \delta \] (3.6)

At the end of the proof of Lemma 3.3, we will explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.1, we only need to establish a uniform a priori estimate for the piecewise \( C^1 \) norm of \( v \) and \( w \) defined by (1.14) and (2.1) on the domain of existence of the \( C^1 \) solution \( u = u(t; x) \).

For any fixed \( T > 0 \), let
\[ D^T = \{(t, x) | 0 \leq t \leq T, x \in (\lambda_{l+1}(0) + \delta_0) t\} \] (3.7)

\[ D^{T'} = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq (\lambda_{m+1}(0) + \delta_0) t\} \] (3.8)

\[ D^T = \{(t, x) | 0 \leq t \leq T, (\lambda_{m+1}(0) - \delta_0) t \leq x \leq (\lambda_n(0) + \delta_0) t\} \] (3.9)

and for \( i = m + 1, \ldots, n \), let
\[ D^T_i = \{(t, x) | 0 \leq t \leq T, [-\delta_i + n(\lambda_i(0) - \lambda_{m+1}(0))] t \leq x - \lambda_i(0) t \leq [-\delta_i + n(\lambda_i(0) + \delta_0)] t\} \] (3.10)

where \( n > 0 \) is suitably small (Figure 1).

Noting that \( n > 0 \) is small, by (3.3), it is easy to see that
\[ D^T \cap D^{T'} = \emptyset, \quad \forall i \neq j \] (3.11)

And
\[ \bigcup_{i=m+1}^{n} D^T_i \subset D^T \] (3.12)

By the definitions of \( D^T \) and \( D^{T'} \), it is easy to get the following lemma. Lemma 3.1. For each \( i = m + 1, \ldots, n \), on the domain \( D^T_i / D^{T'}_i \) we have
\[ c t \leq x - \lambda_i(0) t \leq C t, \quad c x \leq x - \lambda_i(0) t \leq C x \] (3.13)

where \( c \) and \( C \) are positive constants independent of \( T \).

Let
\[ V(D^T_i) = \max_{i=1}^{n} \| (1 + x)^{\mu} u(t, x) \|_{L^\infty(D^T_i)} \] (3.14)

\[ W(D^T_i) = \max_{i=1}^{n} \| (1 + t)^{\mu} u(t, x) \|_{L^\infty(D^T_i)} \] (3.15)

\[ V(D^{T'}) = \max_{i=1}^{n} \| (1 + t)^{\mu} u(t, x) \|_{L^\infty(D^{T'})} \] (3.16)

\[ W(D^{T'}) = \max_{i=1}^{n} \| (1 + t)^{\mu} u(t, x) \|_{L^\infty(D^{T'})} \] (3.17)

\[ V^\iota(T) = \max_{i=1}^{n} \sup_{i=m+1}^{n} \| (1 + T)^{\mu} V_i(t, x) \|_{L^\infty(D^T_i)} \] (3.18)

\[ \max_{i=m+1}^{n} \sup_{i=m+1}^{n} \| (1 + T)^{\mu} V_i(t, x) \|_{L^\infty(D^{T'})} \] (3.19)

\[ W^\iota(T) = \max_{i=1}^{n} \sup_{i=m+1}^{n} \| (1 + T)^{\mu} W_i(t, x) \|_{L^\infty(D^T_i)} \] (3.20)

\[ \max_{i=m+1}^{n} \sup_{i=m+1}^{n} \| (1 + T)^{\mu} W_i(t, x) \|_{L^\infty(D^{T'})} \] (3.21)
\[ \hat{W}_i(T) = \max_{i=m+1, \ldots, n} \max_{j=1}^{n} \int_{c_j} |w_i(t, x)| \, dt \quad (3.22) \]

\[ \hat{U}_i(T) = \max_{i=m+1, \ldots, n} \max_{j=1}^{n} \int_{c_j} |U_i(t, x)| \, dt \quad (3.23) \]

Where \( C_j \) denotes any given \( j \)-th characteristic in

\[ D'_j(j \neq i, i = m+1, \ldots, n) \]

\[ V_l(T) = \max_{i=m+1, \ldots, n} \max_{t \in [0, T]} \int_{D'_j(t)} |v_l(t, x)| \, dx \quad (3.24) \]

\[ W_l(T) = \max_{i=m+1, \ldots, n} \max_{t \in [0, T]} \int_{D'_j(t)} |w_l(t, x)| \, dx \quad (3.25) \]

\[ U_l(T) = \max_{i=m+1, \ldots, n} \max_{t \in [0, T]} \int_{D'_j(t)} |u_l(t, x)| \, dx \quad (3.26) \]

Where \( D'_j(t)(t \geq 0) \) denotes the \( t \)-section of \( D'_j \)

\[ D'_j(t) = \{(t, x) \mid T = t, (t, x) \in D'_j \} \quad (3.27) \]

\[ V_r(T) = \max_{i=m+1, \ldots, n} \max_{t \in [0, T]} |v_r(t, x)| \quad (3.28) \]

\[ V_n(T) = \max_{i=m+1, \ldots, n} \max_{t \in [0, T]} |w_n(t, x)| \quad (3.29) \]

Clearly, \( V_n(T) \) is equivalent to

\[ U_n(T) = \max_{i=m+1, \ldots, n} \max_{t \in [0, T]} |U_i(t, x)| \quad (3.30) \]

In the present situation, similar to the corresponding result in [24,30-33], we have

**Lemma 3.2.** Suppose that in a neighborhood of \( u=0 \): \( A(u) \in C^2 \) system \((1.1)\) is strictly hyperbolic and \((1.10)\) holds. Suppose furthermore that \( \varphi(x) \) satisfies \((1.15)\). Then there exists a sufficiently small \( \delta_0 > 0 \) such that for any fixed \( \theta_0 \in [0, \theta_1] \) on any given existence domain \( \{t(x), y(x) \mid 0 \leq t \leq T, x \in \Sigma \} \) of the \( C' \) solution \( u = u(t, x) \) to the mixed initialboundary value problem \((1.1)\) and \((1.12)-(1.13)\), we have the following uniform a priori estimates:

\[ V(D'_i), W(D'_i) \leq k, \theta, T \quad (3.31) \]

where here and henceforth, \( k_0(i=1, 2, \ldots) \) are positive constants independent of \( \theta \) and \( T \).

**Proof.** We first estimate \( W(D'_i) \)

(i) For \( i = m+1, \ldots, n \), let \( \xi_i = \chi_i(s, y)(0 \leq s \leq t) \) be the \( i \)-th characteristic passing through any fixed point \( (t, x) \in D'_i \) and intersecting the \( x \)-axis at a point \( (0, y) \). Noting \((3.6)\), by \((3.3)-(3.4)\), it is easy to see that the whole characteristic \( \xi_i = \chi_i(s, y)(0 \leq s \leq t) \) is included in \( D'_i \)

\[ \rho(t) = \max_{i=m+1, \ldots, n} \max_{j=1}^{n} |\lambda_i(0) - \delta_0 / 2| \quad (3.32) \]

By \((3.4)\), it is easy to see that

\[ s \leq t \leq t_0 \quad (3.33) \]

where \( t_0 \) denotes the \( s \)-coordinate of the intersection point of the straight line \( x = (\lambda_i(0) + \delta_0) t \) with the straight line \( x = y + (\lambda_i(0) + \delta_0 / 2) t \) passing through the point \( (0, y) \). Clearly,

\[ t_0 = \frac{y}{\lambda_i(0) - \lambda_i(0) + \delta_0 / 2} \quad (3.34) \]

Therefore it follows from \((3.32)-(3.34)\) that

\[ \frac{\lambda_i(0)}{\lambda_i(0) - \delta_0 / 2} y \leq \chi_i(s, y) \leq y, \quad \forall s \in [0, t] \quad (3.35) \]

By integrating \((2.6)\) along this \( i \)-th characteristic, we have

\[ w_i(t, x) = e^{-(t - s)} w_i(0, y) + \int_0^t e^{-l(t - s)} \sum_{k=1}^n \gamma_{ik}(u) w_j \, ds \quad (3.36) \]

Thus, noting \((3.6)\) and the fact that \( L > 0 \), using \((3.33)-(3.35)\), it follows from \((3.36)\) that

\[ \frac{(t + x)^\delta}{|w(t, x)|} \leq C_1(t + y)^\delta |w_i(0, y)| \quad (3.37) \]

where here and henceforth, \( C(1 \equiv 1, \ldots) \) will denote positive constants independent of \( \theta \) and \( T \).

(ii) For \( i = m+1, \ldots, n \); let \( \xi = \chi(s, y) \) be the \( i \)-th characteristic passing through any fixed point \( (t, x) \in D'_i \) and intersecting the \( x \)-axis at a point \( (0, y) \). Noting \((3.6)\), by \((3.3)-(3.4)\), it is easy to see that the whole characteristic \( \xi = \chi(s, y)(0 \leq s \leq t) \) is included in \( D'_i \)

\[ \rho(t) = \max_{i=m+1, \ldots, n} \max_{j=1}^{n} |\lambda_i(0) - \delta_0 / 2| \quad (3.32) \]

By \((3.4)\), it is easy to see that

\[ s \leq t \leq t_0 \quad (3.33) \]

where \( t_0 \) denotes the \( s \)-coordinate of the intersection point of the straight line \( x = (\lambda_i(0) + \delta_0) t \) with the straight line \( x = y + (\lambda_i(0) + \delta_0 / 2) t \) passing through the point \( (0, y) \). Clearly,

\[ t_0 = \frac{y}{\lambda_i(0) - \lambda_i(0) + \delta_0 / 2} \quad (3.40) \]

Therefore it follows from \((3.38)-(3.40)\) that

\[ \frac{\lambda_i(0)}{\lambda_i(0) - \delta_0 / 2} y \leq \chi_i(s, y) \leq y, \forall s \in [0, t] \quad (3.41) \]

Then, similar to \((3.37)\), we have

\[ (1 + x)^\delta |w_i(t, x)| \leq C_1(1 + y)^\delta |w_i(0, y)| \quad (3.42) \]

Combining \((3.37)\) and \((3.42)\), we obtain

\[ W(D'_i) \leq C_1(t + W(D'_i) + V(D'_i)) \quad (3.43) \]

Similarly, we have

\[ V(D'_i) \leq C_1(t + W(D'_i) + V(D'_i)) \quad (3.44) \]

By \((3.43)\) and \((3.44)\), it is easy to prove that for \( \mu > 0 \) suitably small, there exists a positive constant \( k \) independent of \( \theta \) and \( T \), such that for any fixed \( T > T_0 \) if
W(D^k), V(D^k) \leq 2k, \theta \quad (3.45)

Then

W(D^k), V(D^k) \leq k, \theta \quad (3.46)

Hence, noting (1.15), by continuity we immediately get (3.31). The proof of Lemma 3.2 is finished. Lemma 3.3. Under the assumptions of Lemma 3.2, suppose furthermore that system (1.1) is weakly linearly degenerate. Then in the normalized coordinates there exists a sufficiently small \( \delta_0 > 0 \) such that for any fixed \( \theta \in [0, \delta_0] \) on any given existence domain \( \{(t, x)|0 \leq t \leq T, x \geq 0\} \) of the C1 solution \( u = u(t; x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have the following uniform a priori estimates:

\[
W(D^j) \leq k, \theta \quad (3.47)
\]

\[
V(D^j) \leq k, \theta \quad (3.48)
\]

\[
U^c(T) \leq k, \theta \quad (3.49)
\]

\[
W^c(T), V^c(T) \leq k, \theta \quad (3.50)
\]

\[
W_i(T), W_j(T), V_i(T), V_j(T), U_i(T) \leq k, \theta \quad (3.51)
\]

\[
U_i(T), V_i(T) \leq k, \theta \quad (3.52)
\]

And

\[
W^c(T) \leq k, \theta \quad (3.53)
\]

Proof. We first estimate \( W(D^j) \).

For \( j = 1, \ldots, m \), passing through any fixed point \( (t, x) \in D^j \) we draw the \( j \)-th characteristic \( C_j(x) = \xi_j(t, x) \) which must intersect the boundary \( \Gamma = (\lambda_j(t) + \delta_j) t \) of \( D^j \) at a point \((0, y)\).

Proposition 3.1. On this \( j \)-th characteristic \( C_j(x) = \xi_j(t, x) \) it follows that

\[
t \geq t_0 \geq \frac{\lambda_j(0) - \delta_j}{\lambda_j(0) - \lambda_j(t)} t
\]

(3.54)

This is easy to see that

\[
x = (\lambda_j(0) + \delta_j) t \leq y (\lambda_j(0) + \lambda_j(t)) t_0
\]

(3.55)

On the other hand, from (3.8), we have

\[
x \geq 0
\]

(3.56)

Since

\[
y = (\lambda_j(0) + \delta_j) t_0
\]

we conclude from (3.55)-(3.57) that

\[
t_0 \geq \frac{\delta_j}{\lambda_j(0) \lambda_j(t) - \delta_j}
\]

(3.58)

Noting the fact that \( t \geq t_0 \), we immediately get (3.54).

By integrating (2.6) along \( \xi_j(x,t) \) and noting (2.9) and (2.11), we have

\[
w_i(t, x) = e^{-L^n_{iso}} w_i(t_0, y)
\]

where

\[
\int_{t_0}^{t} e^{-\lambda_j(s)} \left( \sum_{k=1}^{m} \sum_{|I|=n} \sum_{|J|=n} \sum_{k=1}^{m} \sum_{|I|=n} \right) \gamma_{ij}(u) w_i(s, \xi_j(s, t), x) ds
\]

(3.59)

Thus, noting the fact that \( L > 0 \), and using (3.13) and (3.54), we obtain from (3.59) that

\[
|w_i(t_0, y)| \leq k, \theta (1 + y) \leq C, \theta (1 + t_0) \leq C, \theta (1 + t) \leq C (3.60)
\]

By Hadamard's formula, we have

\[
\gamma_{ij}(u) - \tilde{\gamma}_{ij}(u(e)) = \int_{t_0}^{t} \frac{\partial}{\partial s} \left( \sum_{k=1}^{m} \sum_{|I|=n} \sum_{|J|=n} \sum_{k=1}^{m} \sum_{|I|=n} \right) \gamma_{ij}(u(s, \xi_j(s, t), x)) ds
\]

(3.61)

Thus, noting the fact that \( L > 0 \), and using (3.13) and (3.54), we obtain from (3.59) that

\[
(1 + t)^{\alpha} |w_i(t, x)| \leq C, \theta (1 + t)^{\alpha} \leq C, \theta (1 + t) \leq C (3.60)
\]

Similar to Lemma 3.2 in [21], differentiating the nonlinear boundary condition (1.13) with respect to \( t \), we get

\[
x = 0 \quad \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial u_i} (\alpha(t, v_1, \ldots, v_m)) \frac{\partial v_i}{\partial t}
\]

(3.63)

By (1.1), (1.3) and (2.4), it is easy to see that

\[
\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial u_i} (l, u) u = -\lambda_i(u) w_i + \sum_{i=1}^{n} \sum_{i=1}^{n} a_{ii}(u) w_i - \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n}
\]

Where

\[
a_{ii}(u) = -\lambda_i(u) \frac{\partial}{\partial u_i} (l, u) u
\]

Therefore it follows from (3.63)-(3.65) that

\[
x = 0 \quad \begin{pmatrix}
(I_{n-m} - B_1) (u) \\
\vdots \\
(I_{n-m} - B_3) (u)
\end{pmatrix} = B_2 (u)
\]

(3.66)

where \( B_1(u) \) is a matrix whose elements are all C1 functions of \( u \), which satisfy

\[
In-m - B_1(u) \text{ is invertible; for sufficiently small } |u|
\]

\[
B_2 \text{ is an } (n-m) \times m \text{ matrix independent of } w(t) (t = 1, \ldots, n)
\]

(3.67)

\[
\tilde{B}_2 = \sum_{i=1}^{n} \frac{\partial}{\partial u_i} (l, u) u
\]

(3.68)

in which \( F \) is a continuous function of \( t \) and \( u \).

Thus, noting (3.6), for \( \theta > 0 \) small enough, by (3.66)-(3.68) we easily get
\[ x = 0 : \text{ws} = \sum_{j=1}^{n} f_j(t,u)w_j + \sum_{j=1}^{n} f_j(t,u)u_j(t) \quad (3.69) \]

\[ + \sum_{i=1}^{n} f_i(t,u)h_i(t) + f_j(t,u)u(s = m+1, \ldots, n) \]

Where \( f_j, f_i, \bar{f}_j, \text{and} \bar{f}_i \) are continuous functions of \( t \) and \( u \).

For \( j = m+1; \ldots; n \), passing through any fixed point \( (t,x) \in D^T \) we draw the \( j \)th characteristic \( \xi_j = \xi_j(s;t,x) \) which must intersect the \( t \)-axis at a point \((t_0,0)\). Then, we have

**Proposition 3.2.** On this \( j \)th characteristic \( \xi_j = \xi_j(s;t,x) \), it follows that

\[ t \geq t_0 \geq \frac{\lambda_{j0}(0) - \lambda_{m0}(0) + \delta_0}{\lambda_{j0}(0) - \lambda_{m0}(0) + \delta_0} t \quad (3.70) \]

Noting the fact that \( t \geq t_0 \), we immediately get (3.70).

By integrating (2.6) along \( \xi_j = \xi_j(s;t,x) \) we have

\[ w_j(t,x) = e^{-\lambda_{j0}(0)t} w_j(t_0,0) + \int_0^t e^{-\lambda_{j0}(0)s} \sum_{i=1}^{n} [f_{ji}(u) w_i] ds \quad (3.74) \]

\[ + f_j(t,u) u_j(t_0,0) \quad (3.75) \]

**Proposition 3.2.** On this \( j \)th characteristic \( \xi_j = \xi_j(s;t,x) \), it follows that

\[ t \geq t_0 \geq \frac{\lambda_{j0}(0) - \lambda_{m0}(0) + \delta_0}{\lambda_{j0}(0) - \lambda_{m0}(0) + \delta_0} t \quad (3.70) \]

Noting the fact that \( t \geq t_0 \), we immediately get (3.70).

By integrating (2.6) along \( \xi_j = \xi_j(s;t,x) \) we have

\[ w_j(t,x) = e^{-\lambda_{j0}(0)t} w_j(t_0,0) + \int_0^t e^{-\lambda_{j0}(0)s} \sum_{i=1}^{n} [f_{ji}(u) w_i] ds \quad (3.74) \]

\[ + f_j(t,u) u_j(t_0,0) \quad (3.75) \]

By employing the same arguments as in (i), we can obtain

\[ (1 + t_j)^{-\mu} \lesssim c_0 (0 + (W(D))^2 + W(D)^2) \quad (3.76) \]

Thus, noting (1.15), (3.6) and (3.70), it follows from (3.75) and (3.76) that

\[ (1 + t_j)^{\mu} \lesssim c_0 (0 + (W(D))^2 + W(D)^2) \quad (3.77) \]

Hence, noting the fact that \( L > 0 \), we obtain from (3.74) that

\[ (1 + t_j)^{\mu} \lesssim c_0 (0 + (W(D))^2 + W(D)^2) + W(T) \quad (3.78) \]

Combing (3.62) with (3.78), we get

\[ W(D)^2 \lesssim c_0 (0 + (W(D))^2) + W(D)^2 + W^2(T) \quad (3.79) \]

We next estimate \( W(T) \)

Let \( \bar{c} : x = \bar{x}_i(t) (0 \leq t \leq T) \)

be any given \( j \)th characteristic in \( D^T (j \neq i, i = m+1, \ldots, n) \) By (3.4), the whole \( i \)th characteristic \( x = \bar{x}_i(t) \) passing through \( O(0,0) \) is included in \( D^T \). Let \( (0, \bar{x}_i(t_0)) \) be the intersection point of this characteristic with \( \bar{C}_i \). Passing through any given point \((t, \bar{x}_i(t)) \) on \( \bar{C}_i \), we draw the \( i \)th characteristic \( \bar{x}_i(t) \) which intersects one of the boundaries of \( DT \). Let \( x = (\lambda_0(n)\theta + \delta_0) t \) (resp. \( x = (\lambda_{m0}(0) - \delta_0) t \)) at a point \( A_j (\lambda_{j0}(0) + \delta_0) \) (resp. \( B_j (\lambda_{j0}(0) - \delta_0) \)) if \( t \leq t \leq t \leq t \). Clearly, we have

\[ \bar{x}_i(t) = \bar{x}_i(t) \quad (3.80) \]

which gives a one-to-one correspondence \( t = t \) (between the segment \( \bar{C}_i \) (resp. \( \bar{B}_i \)) and \( \bar{C}_i (t \leq t \leq t) \) (resp. \( \bar{C}_i (t \leq t \leq t) \)). Thus, the integral on \( \bar{C}_i \), with respect to \( t \) can be reduced to the integral with respect to \( y \). Differentiating (3.80) with respect to \( t \) gives

\[ dt = \frac{1}{\lambda_j(u(t, \bar{x}_i(t), y)) - \lambda_j(u(t, \bar{x}_i(t), y))} dy \quad (3.81) \]

in which \( t = t(y) \). Then, noting (3.3) and (3.6), it is easy to see that in order to estimate

\[ \int_{t_j}^{t} \left| w(t, x(t)) \right| dt = \int_{t_j}^{t_j} \left| w(t, x(t)) \right| dt + \int_{t_j}^{t} \left| w(t, x(t)) \right| dt \]

it suffices to estimate

\[ \int_{x(\bar{t})}^{x(t)} \left| q(t, \bar{x}_i(t), y) \right| dy + \int_{x(\bar{t})}^{x(t)} \left| q(t, \bar{x}_i(t), y) \right| dy \quad (3.83) \]

We now estimate \( \int_{t_j}^{t_j} \left| q(t, \bar{x}_i(t), y) \right| dy \)

By integrating (2.28) along \( \xi_j = \xi_j(s,y) \) and noting (2.30) and the fact that \( \bar{x}_i(y/\lambda_j(0)\theta + \delta_0) = \bar{x}_i \) we obtain

\[ q(t, x(t), y(x(t))), y(x(t))) = e^{-\lambda_j(u(t, \bar{x}_i(t), y))} \left| \frac{\lambda_j(u(t, \bar{x}_i(t), y))}{\lambda_j(0) - \lambda_j} \right| \quad (3.84) \]

By Hadamard's formula and (2.11), we have
Clearly, we have
given by the definition of the boundary of \(DT\)

\[
\xi_u(v, (s, (t, x))) = \int_{\mathbb{R}^n} \frac{\partial \xi_v}{\partial y}(s, (t, x)) d\sigma_y
\]

Therefore we obtain

where \(y_1\) and \(y_2\) are shown in Figure 2.

\[
|w(t_0, y)| \leq k(1 + y)^{-\alpha s} \leq C_{\alpha s} \theta(1 + y)^{-\alpha s} \leq C_{\alpha s} \theta(1 + y)^{-\alpha s}
\]

Similar to (3.90), it follows from (3.91) that

\[
W_j(T) \leq C_{\alpha s} \theta \left[ W_j(T) \right] + W_j(T) + \{W_{j+1}(T) + \} + \} + \} + \} + \} + \} + \} + \} + \}
\]

We next estimate \(W_1(T)\).

(i) For \(r = 1; \ldots; m\), passing through any fixed point \((t, x) \in D^r\) we draw the \(r\)th characteristic \(C_r: \xi = \xi(s, t, x)\) which must intersect the boundary \(x = (\lambda_r(0) + \delta_r(t))\) of \(D^r\) at a point \((0, y)\). Then, we have

\[
\begin{align*}
\text{Proposition 3.3.} & \quad \text{On this \(r\)th characteristic \(C_r: \xi = \xi(s, t, x)\) it follows that} \\
\lambda_r(0) & \leq \lambda_r(0) - \frac{3\delta_r}{2} t
\end{align*}
\]


\[
|w(t_0, y)| \leq k(1 + y)^{-\alpha s} \leq C_{\alpha s} \theta(1 + y)^{-\alpha s} \leq C_{\alpha s} \theta(1 + y)^{-\alpha s}
\]

By Hadamard's formula, we have

\[
\xi_v(u) - \xi_v(v) = \int_{t_0}^t \frac{\partial \xi_v}{\partial u}(u, (s, (t, x))) dT
\]

Thus, noting (3.93) and the fact that \(L > 0\), we obtain from (3.98) that

\[
(t + y)^{-\alpha s} \leq C_{\alpha s} \theta (t + y)^{-\alpha s} \leq C_{\alpha s} \theta (1 + y)^{-\alpha s}
\]

By the definition of \(D^r\) for fixing the idea we may assume that

\[
C_{\alpha s} \theta \left[ W_1(T) \right] + W_1(T) + \{W_{j+1}(T) + \} + \} + \} + \} + \} + \} + \} + \} + \}
\]
\[ x - \lambda(0)t > \frac{\delta_0 + \eta(\lambda(0) - \lambda_i(0))}{2} \quad (3.102) \]

which implies \( i < n \). Let \( \xi = \xi(s; t, x) \) be the \( i \)th characteristic passing through \((t, x)\), which intersects the boundary \( x = (\lambda_i(0) + \delta_i) t \) of \( D^T \) at a point \((0; y)\) (Figure 3).

Recalling (3.4), it is easy to see that
\[ x - (\lambda_i(0) + \frac{\delta_i}{2}) t \leq y - (\lambda_i(0) + \frac{\delta_i}{2}) \eta \quad (3.103) \]

Since
\[ y = (\lambda_i(0) + \delta_i) \eta, \]
recalling (3.102) and the fact that \( t \geq t_0 \) it follows from (3.103) that \( t \geq t_0 \geq y t. \quad (3.105) \)

By integrating (2.6) along \( \xi = \xi(s; t, x) \) and noting (2.9) and (2.11), we have
\[ w(s; t, x) = e^{-\lambda_i(s) t} w(t_0, y) \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) + \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \tilde{\gamma}_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
(3.106)

By Lemma 3.2 and observing (3.104)-(3.105), it is easy to see that
\[ |w(t, x)| \leq k(1 + y)^{t - \alpha} \leq C_{\alpha} \theta(1 + t^\alpha) \leq C_{\alpha} \theta(1 + t)^{t - \alpha} \quad (3.107) \]

By Hadamard’s formula, we have
\[ \tilde{\gamma}_{ij}(u) - \gamma_{ij}(u) = \int_0^t \frac{\partial \tilde{\gamma}_{ij}(u, s) \partial \eta}{\partial \eta} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \tilde{\gamma}_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
(3.108)

Thus, recalling (3.13) and (3.105), and noting the fact that \( L > 0 \), it follows from (3.106) that
\[ \Omega = |x - \lambda(0) t| \leq \left| w(t, x) \right| \leq C_{\alpha} \theta(1 + t^\alpha) \leq C_{\alpha} \theta(1 + t)^{t - \alpha} \quad (3.109) \]

Next, we assume that
\[ x - \lambda_i(0) t < -\frac{\delta_0 + \eta(\lambda(0) - \lambda_{i+1}(0))}{2} \quad (3.110) \]

which implies \( i > m + 1 \). Let \( \xi = \xi(s; t, x) \) (t, x), which intersects the boundary \( x = (\lambda_{i+1}(0) - \delta_{i+1}) t \) of \( D^T \) at a point \((0; y)\).

Recalling (3.4), it is easy to see that
\[ x - (\lambda_i(0) - \frac{\delta_i}{2}) t > y - (\lambda_i(0) - \frac{\delta_i}{2}) \eta \quad (3.111) \]

Since
\[ y = (\lambda_i(0) - \delta_i) \eta, \]
noting (3.110) and the fact that \( t \geq t_0 t \) it follows from (3.111) that
\[ t \geq t_0 \geq y t. \quad (3.112) \]

By integrating (2.6) along \( \xi = \xi(s; t, x) \) and noting (2.9) and (2.11), we have
\[ w(t, x) = e^{-\lambda_i(s) t} w(t_0, y) \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \tilde{\gamma}_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
(3.114)

By (3.113), it is easy to see that
\[ |w(t, x)| \leq C_{\alpha} \theta(1 + t)^{t - \alpha} \leq C_{\alpha} \theta(1 + t)^{t - \alpha} \quad (3.115) \]

Thus, using (3.13), (3.108) and (3.113), and noting the fact that \( L > 0 \), it follows from (3.114) that
\[ (1 + s - \lambda(0) s) \leq \left| w(t, x) \right| \leq C_{\alpha} \theta(1 + t^\alpha) \leq C_{\alpha} \theta(1 + t)^{t - \alpha} \quad (3.116) \]

Combining (3.101) and (3.109), (3.116), we obtain
\[ W_i^C(T) \leq C_{\alpha} \left| \partial + W(D^T) + W_i^C(T) \right| \leq C_{\alpha} \left| \partial + \tilde{V}(T) \right| \leq C_{\alpha} \left| \partial + \tilde{V}(T) \right| \]
\[ = C_{\alpha} \left| \partial + \tilde{V}(T) \right| \leq C_{\alpha} \left| \partial + \tilde{V}(T) \right| \quad (3.117) \]

We next estimate \( V(D^T) \) for \( j = 1, \ldots, m \), for any fixed point \((t, x) \in D^T \) similar to (3.59), by integrating (2.13) along \( \xi = \xi(s; t, x) \) and noting (2.17)-(2.18), we have
\[ v_j(t, x) = e^{-\lambda_j(s) t} v_j(t_0, y) \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \tilde{\gamma}_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
\[ + \int_0^t e^{-\lambda_i(s) t} \left( \sum_{j=1}^n \sum_{i=1}^m \gamma_{ij}(u) w_i(s, \xi(s; t, x)) \right) ds \]
(3.118)

By using Lemma 3.2 and noting (3.54) and (3.57), it is easy to see that
By Hadamard’s formula, we have
\[ \beta_i(u) - \beta_j(u) = \int_0^1 \frac{\partial^2 \beta}{\partial t^2} \left( T_{ij} \right) (\nu_t, \nu_0) \, dt \, \partial t \, \partial t \] (3.120)

And
\[ \beta_i(u) - \beta_j(u) = \int_0^1 \frac{\partial^2 \beta}{\partial t^2} \left( T_{ij} \right) (\nu_t, \nu_0) \, dt \, \partial t \, \partial t \] (3.121)

Thus, noting the fact that \( L > 0 \), and using (3.13) and (3.54), we obtain from (3.118) that
\[ (1 + t)^{\gamma + 1} v_j(t, x) \leq C_0 \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) + U_{v}(D^T) \left( D^T \right) \] (3.122)

For \( j = m + 1, \ldots, n \), for any fixed point \( t, x \in D_T \), similar to (3.74), we have
\[ v_j(t, x) = e^{-t} \int_0^{t} v_j(t, x) \, dt + \int_0^{t} e^{-t-s} \sum_{k=1}^{n} \beta_k(u) v_k \right) \] (3.123)

\[ v_j(t, x) = \sum_{i=1}^{n} g_{ij}(t) v_i(t, x) + h_{ij}(t) \] (3.124)

Where
\[ g_{ij}(t) = \int_0^1 \frac{\partial^2 \beta}{\partial t^2} \left( \alpha(t, x), T_i(t, x) \right) \, dt \] (3.125)

By employing the same arguments as in (1), we can obtain
\[ (1 + t)^{\gamma + 1} v_j(t, x) \leq C_0 \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) + U_{v}(D^T) \left( D^T \right) \] (3.126)

Thus, noting (1.15), (3.6) and (3.70), it follows from (3.124)-(3.126) that
\[ \beta_j(u) - \beta_j(u) = \beta_j(u) - \beta_j(u) ] \] (3.127)

Hence, noting the fact that \( L > 0 \), we obtain from (3.123) that
\[ \beta_j(u) - \beta_j(u) = \int_0^1 \frac{\partial^2 \beta}{\partial t^2} \left( \alpha(t, x), T_i(t, x) \right) \, dt \] (3.128)

Combining (3.122) and (3.128), we get
\[ \beta_j(u) - \beta_j(u) = \int_0^1 \frac{\partial^2 \beta}{\partial t^2} \left( \alpha(t, x), T_i(t, x) \right) \, dt \] (3.129)

We next estimate \( \bar{V}_i(T) \) and \( V_i(T) \).

For \( \iota = m + 1, \ldots, n \), for any given \( j \) characteristic \( \bar{C}_j \) in \( D_T^j (j \neq 1) \) as in the proof of (3.90), in order to estimate \( \bar{V}_i(T) \) it suffices to estimate
\[ \int_0^1 \left| p_i(t, x) \right| (x) \, dx \] (3.130)

By integrating (2.33) along \( \xi = \bar{x}_i(t, x) \), similar to (3.84), we have
\[ p_i(t, x) \left( x \right) \right) \leq C_0 \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) + U_{v}(D^T) \left( D^T \right) \] (3.131)

Noting that \( \lambda_j (u) (i = m + 1, \ldots, n) \) are weakly linearly degenerate, by (2.37) and (2.38), we have
\[ B_{ij}(u, c_j) \equiv 0, \forall j \] (3.132)

By Hadamard’s formula, and noting (2.18) and (3.132), we have
\[ B_{ij}(u) = B_{ij}(u) ] \] (3.133)

And
\[ \beta_j(u) - \beta_j(u) = \left( \beta_j(u) - \beta_j(u) \right) \] (3.134)

Then, using Lemma 3.2, similar to (3.88), it follows from (3.131) that
\[ \int_0^1 \left| p_i(t, x) \right| (x) \, dx \leq C_0 \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) \] (3.135)

Similarly, we have
\[ \int_0^1 \left| p_i(t, x) \right| (x) \, dx \leq C_0 \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) \] (3.136)

Similarly, we have
\[ \int_0^1 \left| p_i(t, x) \right| (x) \, dx \leq C_0 \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) \] (3.137)

Thus, we obtain
\[ V_i = C_{0} \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) \] (3.138)

Similarly, we have
\[ V_i = C_{0} \theta (1 + t) \left( u + W_{u}(D^T) \right) + W_{u}(D^T) \left( D^T \right) + V_{u}(D^T) \left( D^T \right) + U_{u}(D^T) \left( D^T \right) \] (3.139)

We next estimate \( V_i(T) \)

(i) For \( r = 1, \ldots, m \), for any fixed point \( t, x \in D_T \), noting (2.17) and (2.18), similar to (3.98), we have
It suffices to estimate $q$ and which intersects one of the noting (2.19) and (y) at a point (3.158), we have

By Hadamard’s formula, we have

Thus, noting (3.93) and the fact that $L > 0$, we obtain from (3.139) that

(ii) For $m = m + 1; \ldots; n$, for any fixed point $(t, x) \in \mathbb{D}^T$ but $(t, x) \notin \mathbb{D}^T$

similar to (3.116), we have

Thus, it follows from (3.143) and (3.144) that

Then, we next estimate $U_1(T)$.

For $i = m + 1; \ldots; n$, for any given $j$th characteristic $\xi_j$, in $\mathbb{D}^T$ (j = i) as in the proof of (3.90), in order to estimate $U_1(T)$ it suffices to estimate

By integrating (2.39) along $\xi = \xi_j(s, y)$ noting (2.41), similar to (3.84), we have

Since $\hat{h}(u)$ is weakly linearly degenerate and $u = (u_1; \ldots; u_m)T$ are normalized coordinates, by (2.43), we have

Then, using Hadamard’s formula, we have

Hence, noting (3.6), (3.11), (3.13) and the fact that

$L > 0$ and $\frac{\partial k(x, y)}{\partial y} > 0$ we obtain from (3.147) and (3.149) that

By Lemma 3.2, similar to (3.88), it follows from (3.150) that

Similarly, we have

Thus, we obtain

We next estimate $U_2(T)$.

(i) For $r = 1; \ldots; m$, for any fixed point $(t, x) \in \mathbb{D}^T$ noting (2.19) and (2.20), similar to (3.98), we have

By using Lemma 3.2 and noting (3.93) and (3.96), it is easy to see that

Thus, noting (3.93) and the fact that $L > 0$, we obtain from (3.157) that

(ii) For $i = m + 1; \ldots; n$, for any fixed point $(t, x) \in \mathbb{D}^T$ but $(t, x) \notin \mathbb{D}^T$

similar to (3.116), we have

Since $\hat{h}(u)$ is weakly linearly degenerate and $u = (u_1; \ldots; u_m)T$ are normalized coordinates, by (2.43), we have

We now estimate $V_{\infty}$

For $i = m + 1; \ldots; n$, passing through any given point $(t, x) \in \mathbb{D}^T$ we draw the $i$th characteristic $\xi = \xi_i(t, x)$ which intersects one of the boundaries of $\mathbb{D}^T$ at one point. For fixing the idea, suppose that this characteristic intersects $x = (\lambda_i(0) + \delta_i)T$ at a point $(y^T \lambda_i(0) + \delta_i, 0)$.
By integrating (2.13) along this characteristic and noting (2.16)-(2.18), we have
\[ \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \beta_\gamma(u) v_i(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \beta_\gamma(u) v_i(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \beta_\gamma(u) v_i(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \beta_\gamma(u) v_i(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \beta_\gamma(u) v_i(u, x, y) ds \] (3.160)

Noting Lemma 3.1 and Lemma 3.2, and using Hadamard’s formula, it follows from (3.160) that
\[ |v(t, x)| \leq C_0 \{ V(D^T) + W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \}
\[ + \{ W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \} \] (3.161)

On the other hand, for \( i = 1, \ldots, m \), for any fixed point \( (t, x) \in \mathbb{D}^T \) with \( v(t, x) \) can be controlled by \( F(T) \) or \( V(D^T) \). Moreover, for \( i = 1, \ldots, m \), for any fixed point \( (t, x, y) \in \mathbb{D}(T) \) with \( v(t, x) \) can be controlled by \( V(T) \) or \( V(D^T) \) as well. Thus, by using Lemma 3.2 again, we have
\[ V_{\gamma}(T) \leq C_0 \{ \theta + V(D^T) + W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \}
\[ + \{ W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \} \] (3.162)

We finally estimate \( W_{\gamma}(T) \)

For \( i = 1, \ldots, m \), passing through any given point \( (t, x, y) \in \mathbb{D}^T \) similar to (3.160), noticing (2.9), (2.11)-(2.12) and the fact that \( \lambda_i(u) \) is weakly linearly degenerate, we have
\[ u \in (t, x) = e^{-\frac{1}{2} \gamma} \left( \frac{\beta(u) v_i(u, x, y)}{\lambda_i(u)} \right) \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \gamma_i(u) v_j(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \gamma_i(u) v_j(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \gamma_i(u) v_j(u, x, y) ds \]
\[ + \int_{\gamma} e^{\tau - \tau_0} \left( \sum_{i=1}^{m} + \sum_{j=1}^{n} \sum_{k=1}^{m} \right) \gamma_i(u) v_j(u, x, y) ds \] (3.163)

Noting Lemma 3.1 and Lemma 3.2, and using Hadamard’s formula, it follows from (3.163) that
\[ |w(t, x)| \leq C_0 \{ \theta + W(D^T) + W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \}
\[ + \{ W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \} \] (3.164)

Thus, by using the definitions of \( W_{\gamma}(T), W(D^T) \) and \( W(D^T) \) and using Lemma 3.2, we have
\[ W_{\gamma}(T) \leq C_0 \{ \theta + W(D^T) + W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \}
\[ + \{ W_\gamma(T)U_\gamma(T) + U_\gamma(T)W(T) + W_\gamma(T)U_\gamma(T) \} \] (3.165)

We now prove (3.47)-(3.53).

Noting (1.15), evidently we have
\[ W_{\gamma}(0), V_{\gamma}(0), U_{\gamma}(0) \leq C_0 \theta \] (3.166)
\[ W(T) = V(T) = U(T) = W(0) = V(0) = U(0) = 0 \] (3.167)
We assume that there exists a constant $\mu > 0$ such that for any fixed $\theta \in (0, \theta_0]$ on any given domain of existence $\Omega(T) = \{(t, x) | 0 \leq t \leq T, x \geq 0\}$ of the C1 solution $u = u(t; x)$ to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have

$$u = \begin{bmatrix} w \\ v \end{bmatrix}$$

(4.7)

By (4.4), it is easy to see that in a neighborhood of $u_0 = \begin{bmatrix} 0 \\ v_0 \end{bmatrix}$ system (4.1) is strictly hyperbolic and has the following two distinct real eigenvalues:

$$\lambda_j(u) = -\sqrt{\sigma'(w)} < 0 < \lambda_i(u) = \sqrt{\sigma'(w)}$$

(4.8)

The corresponding right eigenvectors are

$$r_j(u) / \sqrt{\sigma'(w)} , r_i(u) / (1 - \sqrt{\sigma'(w)})^T$$

(4.9)

It is easy to see that in a neighborhood of $u_0 = \begin{bmatrix} 0 \\ v_0 \end{bmatrix}$ all characteristics are linearly degenerate, then weakly linearly degenerate, provided that

$$\sigma''(w) \equiv 0, \quad |w| \quad \text{small}$$

(4.10)

The corresponding left eigenvectors can be taken as

$$l_j(u) = (-\sqrt{\sigma'(w)}, 1), l_i(u) = (\sqrt{\sigma'(w)}, 1)$$

(4.11)

Let

$$vi = li(u) u \quad (i = 1, 2)$$

(4.12)

Then, the boundary condition (4.3) can be rewritten as

$$x = 0 : v_1 + v_2 = 2h(t) \triangleq H(t)$$

(4.13)

By Theorem 1.1 we get

Theorem 5.1. Suppose that (4.4) and (4.10) hold. Suppose furthermore that $v_0(x)$, $v_1(x)$ are all C1 functions with respect to their arguments, for which there is a constant $\mu > 0$ such that

$$\theta \triangleq \max_{x \in \Omega(T)} \sup_{i} (1 + x)\theta^x (w_i(x) + |v_i(x)| + |w_i'(x)| + |v_i'(x)|) < +\infty$$

(4.14)

Suppose finally that $h(t) \in C$ satisfies (4.14) and that the conditions of C1 compatibility are satisfied at the point $(0; 0)$. Then there is a sufficiently small $\theta_0 > 0$ such that for any given $\theta \in [0, \theta_0]$, the mixed initial-boundary value problem (4.1)-(4.3) admits a unique global C1 solution $u = u(t; x)$ in the half space $\{(t, x) | t \geq 0, x \geq 0\}$.

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References


