Fluid Mechanics: Open Access
ISSN: 2476-2296
Hassan, Fluid Mech Open Acc 2017, 4:2
DOI: 10.4172/2476-2296.1000152
Perspective
OMICS International

Green's Function for the Heat Equation

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Abstract

The solution of problem of non-homogeneous partial differential equations was discussed using the joined Fourier-Laplace transform methods in finding the Green's function of heat equation in different situations.

Keywords: Heat equation; Green's function; Sturm-Liouville problem; Electrical engineering; Quantum mechanics

Introduction

The Green's function is a powerful tool of mathematics method is used in solving some linear non-homogenous PDEs, ODEs. So Green's functions are derived by the specially development method of separation of variables, which uses the properties of Dirac's function. This method was considerable more efficient than the others well known classical methods.

The series solution of differential equation yields an infinite series which often converges slowly. Thus it is difficult to obtain an insight into over-all behavior of the solution. The Green's function approach would allow us to have an integral representation of the solution instead of an infinite series.

To obtain the filed u, caused by distributed source we calculate the effect of each elementary portion of source and add (integral) them all. If $G(r, r_0)$ is the field at the observers point r caused by a unit source at the source point r_0, then the field at r caused by distribution $f(r_0)$ is the integral of $f(r_0)\, G(r, r_0)$ over the whole range of r_0 occupied by the course. The function G is called Green's function.

Preliminaries

Sturm-Liouville problem

Consider a linear second order differential equation:

$$A(x)\frac{d^2y}{dx^2}+B(x)\frac{dy}{\partial x}+C(x)y+\lambda D(x)y=0 \qquad (1)$$

Where λ is a parameter to be determined by the boundary conditions. $A(x)$ is positive continuous function, then by dividing every term by $A(x)$, eqn (1) can be written as:

$$\frac{d^2y}{dx^2}+b(x)\frac{dy}{\partial x}+c(x)y+\lambda d(x)y=0 \qquad (2)$$

Where $b(x)=\frac{B(x)}{A(x)}$, $\quad c(x)=\frac{C(x)}{A(x)}$ and $d(x)=\frac{D(x)}{A(x)}$

Let us define integrating factor $p(x)$ by:

$$p(x)=\exp\left\{\int_a^x b(\zeta)d\zeta\right\}$$

Multiplying eqn (2) by $p(x)$, we have:

$$p(x)\frac{d^2y}{dx^2}+p(x)b(x)\frac{dy}{\partial x}+p(x)c(x)y+\lambda p(x)d(x)y=0 \qquad (3)$$

Since:

$$\frac{dp(x)}{dx}=\frac{d}{dx}\left(e^{\int_a^x b(\zeta)d\zeta}\right)=e^{\int_a^x b(\zeta)d\zeta}\frac{d}{dx}\left(\int_a^x b(\zeta)d\zeta\right)=p(x)b(x)$$

So

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]=p(x)\frac{d^2y}{dx^2}+\frac{dp(x)}{dx}\frac{dy}{dx}=p(x)\frac{d^2y}{dx^2}+p(x)b(x)\frac{dy}{dx}$$

Thus eqn (3) can be written as:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y+\lambda r(x)y=0 \qquad (4)$$

Where $q(x)=p(x)c(x)$ and $r(x)=p(x)d(x)$.

Equation in form (4) is known as Sturm-Liouville equation. Satisfy the boundary conditions [1-5].

Regular Sturm-Liouville problem: In case $p(a)\neq 0$ and $p(b)\neq 0$, $p(x),q(x),r(x)$ are continuous, the Sturm-Liouville eqn (4) can be expressed as:

$$L[y]=\lambda r(x)y \qquad (5)$$

Where $L\equiv\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x) \qquad (6)$

If the above equation is associated with the following boundary condition:

$$\alpha_1 y(a)+\alpha_2 y(a)=0$$

$$\beta_1 y(b)+\beta_2 y(b)=0 \qquad (7)$$

Where $_1+\alpha_2\neq 0$ and $\beta_1+\beta_2\neq 0$

The eqn (4) and the boundary condition (7) are called regular Sturm-Liouville problem (RSLP).

Singular Sturm-Liouville problem: Consider the equation:

$$L[y]+\lambda r(x)y=0 \quad a<x<b \qquad (8)$$

Where L is defined by eqn (6), $p(x)$ is smooth and $r(x)$is positive, then the Sturm-Liouville problem is called singular if one of the following situations is occurred.

(i) If $p(a)=0$ or $p(b)=0$ or both

(ii)The interval (a,b) is infinite.

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Received March 01, 2016; **Accepted** March 22, 2017; **Published** April 02, 2017

Citation: Hassan AA (2017) Green's Function for the Heat Equation. Fluid Mech Open Acc 4: 152. doi: 10.4172/2476-2296.1000152

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Eigenvalue and Eigenfunction: The Eigenvalue from eqn (4) defining by a Sturm-Liouville operator can be expressed as:

$$\lambda=\frac{-1}{r(x)}\left[\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)\right] \tag{9}$$

The non-trival solutions that satisfy the equation and boundary conditions are called eigenfunctions. Therefore the eigenfunction of the Sturm-Liouville problem from complete sets of orthogonal bases for the function space is which the weight function is $r(x)$.

The Dirac Delta function

The delta function is defined as:

$$\delta(x-\zeta)=\begin{Bmatrix}0 & x\neq\zeta\\ \infty & x=\zeta\end{Bmatrix} \tag{10}$$

But such that the integral of $\delta\,(x-\xi)$ is normalized to unit;

$$\int_{-\infty}^{+\infty}\delta(x-\zeta)dx=1 \tag{11}$$

In fact the first operator where Dirac used the delta function is the integration:

$$\int_{-\infty}^{+\infty}f(x)\delta(x-\zeta)dx \tag{12}$$

Where $f(x)$ is a continuous function, we have to find the value of the integration eqn (12).

Since $\delta\,(x-\xi)$ is zero for $x\neq\xi$, the limit of integration may be change to $\zeta-$ and $\zeta+\varepsilon$, where ε is a small positive number, $f(x)$ is continuous at $x-\xi$, it's values within the interval $(\zeta-\varepsilon,\zeta-)$ will not different much from $f(\zeta)$, approximately that:

$$\int_{-\infty}^{+\infty}f(x)\delta(x-\zeta)dx=\int_{\zeta-\varepsilon}^{\zeta+\varepsilon}f(x)\delta(x-\zeta)dx\approx f(\zeta)\int_{\zeta-\varepsilon}^{\zeta+\varepsilon}\delta(x-\zeta)dx \tag{13}$$

With the approximation improving as ε approaches zero.

From eqn (11), we have:

$$I=\int_{-\infty}^{+\infty}\delta(x-\zeta)dx=\int_{\zeta-\varepsilon}^{\zeta+\varepsilon}\delta(x-\zeta)dx=1 \tag{14}$$

From all values ofε, then by letting $\varepsilon\to 0$, we can exactly have:

$$\int_{-\infty}^{+\infty}f(x)\delta(x-\zeta)dx=f(\zeta) \tag{15}$$

Despite the delta function considered as fundamental role in electrical engineering and quantum mechanics, but no conventional could be found that satisfies eqns (10) and (11), then the delta function sought to be view as the limit of the sequence of strongly peaked function $\delta_n(x)$ such that:

$$\delta(x)=\lim_{n\to\infty}\delta_n(x) \tag{16}$$

As:

$$\left.\begin{aligned}\delta_n(x)&=\frac{n}{\pi(1+n^2x^2)}\\ \delta_n(x)&=\frac{n}{\sqrt{\pi}}e^{-n^2x^2}\\ \delta_n(x)&=\frac{\sin^2(nx)}{n\pi x^2},etc\end{aligned}\right\} \tag{17}$$

Some important properties of Dirac delta function

Property (1): Symmetry

$$\delta(-x)=\delta(x) \tag{18}$$

Proof: Let $\zeta=-x$, then $dx=-d\zeta$

We can write:

$$\int_{-\infty}^{+\infty}f(x)\delta(-x)dx=-\int_{+\infty}^{-\infty}f(\zeta)\delta(\zeta)d\zeta=\int_{-\infty}^{+\infty}f(\zeta)\delta(\zeta)d\zeta=f(0) \tag{19}$$

But, $\int_{-\infty}^{+\infty}f(x)\delta(x)dx=f(0)$ (20)

Therefore, from eqns (19) and (20), we conclude that $\delta\,(-x)=\delta\,(x)$.

Property (2): Scaling

$$\delta(ax)=\frac{1}{|a|}\delta(x) \tag{21}$$

Proof: Let $\zeta=ax$, then $dx=\frac{1}{a}d\zeta$

If $a>0$, then,

$$\int_{-\infty}^{+\infty}f(x)\delta(ax)dx=\int_{-\infty}^{+\infty}f\left(\frac{\zeta}{a}\right)\delta(\zeta)\frac{1}{a}d\zeta=\frac{1}{a}\int_{-\infty}^{+\infty}f\left(\frac{\zeta}{a}\right)\delta(\zeta)d\zeta=\frac{1}{a}f\left(\frac{0}{a}\right)=\frac{1}{a}f(0)$$

Since, $\int_{-\infty}^{+\infty}f(\zeta)\frac{1}{a}\delta(x)dx=\frac{1}{a}\int_{-\infty}^{+\infty}f(x)\delta(x)dx=\frac{1}{a}f(0)$

Therefore: $\delta(ax)=\frac{1}{a}\delta(x)$

Similarly for $a<0$, $\delta(ax)=\frac{1}{-a}\delta(x)$, then,

We can write: $\delta(ax)=\frac{1}{|a|}\delta(x)$

Property (3)

$$\delta(x^2-a^2)=\frac{1}{|2a|}\left[\delta(x+a)+\delta(x-a)\right] \tag{22}$$

Proof: The argument of this function goes to zero when $x=a$ and $x=-a$, wherefore:

$$\int_{-\infty}^{+\infty}f(\zeta)\delta(x^2-a^2)dx=\int_{-\infty}^{+\infty}f(x)\delta[(x+a)(x-a)]dx$$

Only at the zero of the argument of the delta function that is:

$$\int_{-\infty}^{+\infty}f(x)\delta(x^2-a^2)dx=\int_{-a-\varepsilon}^{-a+\varepsilon}f(x)\delta(x^2-a^2)dx+\int_{a-\varepsilon}^{a+\varepsilon}f(x)\delta(x^2-a^2)dx \tag{23}$$

Near the two zeros x^2-a^2 can be approximated as:

$$(x^2-a^2)=(x-a)(x+a)=\begin{cases}(-2a)(x+a), & x\to -a\\ (+2a)(x-a), & x\to +a\end{cases}$$

In the limit as $\varepsilon 0$ the integral eqn (23) becomes:

$$\begin{aligned}\int_{-\infty}^{+\infty}f(x)\delta(x^2-a^2)dx&=\int_{-a-\varepsilon}^{-a+\varepsilon}f(x)\delta[(-2a)(x+a)]dx+\int_{a-\varepsilon}^{a+\varepsilon}f(x)\delta[(2a)(x-a)]dx\\ &=\frac{1}{|2a|}\int_{-a-\varepsilon}^{-a+\varepsilon}f(x)\delta(x+a)dx+\frac{1}{|2a|}\int_{a-\varepsilon}^{a+\varepsilon}f(x)\delta(x-a)dx\\ &=\int_{-\infty}^{+\infty}f(x)\frac{1}{|2a|}\left[\delta(x-a)+\delta(x-a)\right]dx\end{aligned}$$

Therefore: $\delta(x^2-a^2)=\frac{1}{|2a|}\left[\delta(x-a)+\delta(x-a)\right]$

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Green's Function for the Heat

Heat equation over infinite or semi-infinite domains

Consider one dimensional heat equation:

$$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}=-f(x,t) \tag{24}$$

Subject to boundary conditions $|u(x,t)|<\infty$ as $|x|<\infty$ and initial condition. Let $G(x, t, \zeta, \tau)$ be the Green's function for the one-dimensional heat equation, then:

$$\frac{\partial G}{\partial t}-a^2\frac{\partial^2 G}{\partial x^2}=\delta(x-\zeta)\partial(t-\tau),\quad -\infty<x,\quad \zeta<\infty,\quad 0<t,\tau \tag{25}$$

Subject to the boundary condition $|G(x, t, \zeta, \tau)|<\infty$ as $|x|<\infty$, and the initial condition $G(x, t, \zeta, \tau)=0$. Let us find $G(x, t, \zeta, \tau)$.

We begin by taking the Laplace transform of eqn (24) with respect to t, we have:

$$sg(x,s,\zeta,\tau)-g(x,0,\zeta,\tau)-a^2\frac{d^2g}{dx^2}=\delta(x-\zeta)e^{-s\tau}$$

$$\text{So } \frac{d^2g}{dx^2}-\frac{s}{a^2}g=-\delta(x-\zeta)e^{-s\tau} \tag{26}$$

Where $g(x, s, \zeta, \tau)$ the Laplace transform of $G(x, t, \zeta, \tau)$

Now by taking the Fourier transform of eqn (26) with respect to x, so that:

$$(-ik)^2\bar{G}(\mathrm{k,s},\zeta,\tau)-\frac{s}{a^2}\bar{G}(\mathrm{k,s},\zeta,\tau)=-\frac{e^{-ik\zeta-s\tau}}{a^2}$$

$$k^2\bar{G}(\mathrm{k,s},\zeta,\tau)+\frac{s}{a^2}\bar{G}(\mathrm{k,s},\zeta,\tau)=\frac{e^{-ik\zeta-s\tau}}{a^2}$$

Where $\bar{G}(\mathrm{k,s},\zeta,\tau)$ is Fourier transform of $g(x, s, \zeta, \tau)$, now let $\frac{s}{a^2}=b^2$

$$(k^2+b^2)\bar{G}(\mathrm{k,s},\zeta,\tau)=\frac{e^{-ik\zeta-s\tau}}{a^2} \tag{27}$$

To find $g(x, s, \zeta, \tau)$, we use the inversion integral:

$$g(x,s,\zeta,\tau)=\frac{e^{-s\tau}}{2\pi a^2}\int_{-\infty}^{\infty}\left(\frac{e^{i(x-\zeta)}}{k^2+b^2}\right)dk \tag{28}$$

Transforming eqn (27) into a closed contour, we evaluate it by the residue theorem and find that:

$$g(x,s,\zeta,\tau)=\frac{e^{-s\tau}}{2\pi a^2}\int_{-\infty}^{\infty}\frac{e^{i(x-\zeta)}}{(k+ib)(k-ib)}dk$$

$$g(x,s,\zeta,\tau)=\frac{e^{-s\tau}}{2\tau a^2}\sum b_i$$

$$at\quad k=\pm ib\quad then\quad \sum b=\frac{1}{2ib}\left(e^{-|x-\zeta|b}-e^{(\quad)}\right)=\frac{1}{2ib}e^{-|x-\zeta|b}$$

$$\therefore g(x,s,\zeta,\tau)=\frac{e^{-s\tau}}{2a^2b}e^{-|x-\zeta|b}=\frac{1}{2ib}e^{-|x-\zeta|b-sr}$$

Now substituting for $b=\frac{\sqrt{s}}{a}$, we have:

$$g(x,s,\zeta,\tau)=\frac{\exp\left(-|x-\zeta|\sqrt{\frac{s}{a}}-s\tau\right)}{2a\sqrt{s}} \tag{29}$$

Taking Laplace transform of eqn (29) we obtain:

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{a\pi a^2(t-\tau)}}\exp\left(\frac{-(x-\zeta)^2}{aa^2(t-\tau)}\right) \tag{30}$$

Example (5.1.1) Consider one dimensional heat equation:

$$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}=-f(x,t) \tag{31}$$

Subject to boundary condition $u(0,t)=0$, $\lim_{x\to 0}|u(x,t)|<\infty$ and initial condition $U(x,0)=0$

Solution: Let us find the Green's function for the following problem:

$$\frac{\partial G}{\partial t}-a^2\frac{\partial^2 G}{\partial x^2}=\delta(x-\zeta)\delta(t-\tau),0<x,\zeta<\infty,0<t,\tau \tag{32}$$

Subject to the boundary conditions $G(0, t, \zeta, \tau)$, $\lim_{x\to\infty}|G(x,t,\zeta,\tau)|<\infty$ and initial condition $G(x, 0, \zeta, \tau)$

From the boundary condition $G(0, t, \zeta, \tau)=0$, we deduce that $G(x, t, \zeta, \tau)$ Green's function by introducing an image source of $-\delta(x-\zeta)$ and resolving eqn (25) with the source $(x-\zeta)\delta(t-\tau)-\delta(x+\zeta)\delta(t-\tau)$. Eqn (30) gives the solution for each delta function and the green's function for eqn (32) can be written:

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\exp\left(\frac{-(x-\zeta)^2}{4a^2(t-\tau)}\right)-\exp\left(\frac{-(x+\zeta)^2}{4a^2(t-\tau)}\right)\right\} \tag{33}$$

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\exp\left(\frac{-x^2+2x\zeta-\zeta^2}{4a^2(t-\tau)}\right)-\exp\left(\frac{-x^2+2x\zeta-\zeta^2}{4a^2(t-\tau)}\right)\right\}$$

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\begin{matrix}\exp\left(\frac{-x^2-\zeta^2}{4a^2(t-\tau)}+\frac{x\zeta}{2a^2(t-\tau)}\right)-\\ \exp\left(\frac{-x^2-\zeta^2}{4a^2(t-\tau)}-\frac{x\zeta}{2a^2(t-\tau)}\right)\end{matrix}\right\}$$

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\begin{matrix}\exp\left(\frac{-x^2+\zeta^2}{4a^2(t-\tau)}\right)\times\\ \left[\exp\left(\frac{x\zeta}{2a^2(t-\tau)}\right)-\exp\left(\frac{-x\zeta}{2a^2(t-\tau)}\right)\right]\end{matrix}\right\}$$

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\exp\left(\frac{-x^2+\zeta^2}{4a^2(t-\tau)}\right)\sinh\left(\frac{x\zeta}{2a^2(t-\tau)}\right)\right\} \tag{34}$$

In a similar manner, if the boundary condition at x=0 changes to $G_x(0, t, \zeta, \tau)=0$, then eqns (31) and (32) become:

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\exp\left(\frac{-(x-\zeta)^2}{4a^2(t-\tau)}\right)+\exp\left(\frac{-(x+\zeta)^2}{4a^2(t-\tau)}\right)\right\}$$

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\exp\left(\frac{-(x-\zeta)^2}{4a^2(t-\tau)}\right)-\exp\left(\frac{-(x+\zeta)^2}{4a^2(t-\tau)}\right)\right\} \tag{35}$$

$$G(x,s,\zeta,\tau)=\frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}}\left\{\exp\left(-\frac{x^2+\zeta^2}{4a^2(t-\tau)}\right)\cosh\left(\frac{x\zeta}{2a^2(t-\tau)}\right)\right\} \tag{36}$$

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Heat Equation within a finite Cartesian Domain

In this section we find Green's function for the heat equation within finite Cartesian domains. These solutions can be written as series involving orthonormal eigenfunction form regular Sturm-Liouville problem [6-10].

Example (5.2.1)

Here we find the Green's function for the one-dimensional heat equation over the interval $0<x<L$.

$$\frac{\partial U}{\partial t}-a^2\frac{\partial^2 U}{\partial x^2}=f(x,t),0<x<L,0<t \tag{37}$$

Where a^2 is the diffusivity constant,

To find the Green's function for this problem, consider the following problem,

$$\frac{\partial G}{\partial t}-a^2\frac{\partial^2 G}{\partial x^2}=\delta(x-\zeta)\delta(t-\tau),0<x,\zeta<L,0<t,\tau \tag{38}$$

With the boundary conditions,

$$\alpha_1 G(0,t,\zeta,\tau)+\beta_1 G_x(0,t,\zeta,t)=0,0<t \tag{39}$$

and

$$\alpha_2 G(L,t,\zeta,\tau)+\beta_2 G_x(L,t,\zeta,t)=0,0<t \tag{40}$$

And the initial condition,

$$G(x,t,\zeta,\tau)=0,\ 0<x<L \tag{41}$$

We begin by taking the Laplace transform of eqn (38) and find that:

$$\begin{aligned}&\frac{s}{a^2}g(x,s,\zeta,\tau)-\frac{s}{a^2}g(x,0,\zeta,\tau)-\frac{\partial^2 g}{\partial x^2}=\frac{\delta(x-\zeta)e^{-st}}{a^2}\\&\frac{\partial^2 g}{\partial x^2}-\frac{s}{a^2}=-\frac{\delta(x-\zeta)e^{-st}}{a^2},0<x<L\end{aligned} \tag{42}$$

With $\alpha_1 g(0,t,\zeta,\tau)+\beta_1 g'(0,t,\zeta,t)=0,$ (43)

And $\alpha_2 g(L,t,\zeta,\tau)+\beta_2 g'(L,t,\zeta,t)=0$ (44)

Applying the technique of the eigenfunction expansions, we have that:

$$g(x,s,\zeta,\tau)=e^{-st}\sum_{n=1}^{\infty}\frac{\phi_n(\zeta)\phi_n(x)}{s+a^2k_n^2} \tag{45}$$

Where $\varphi_n(_x)$ is the nth orthonormal eigenfunction to the regular Sturm-Liouville problem:

$$\varphi(x)+k^2\varphi'(x)=0 \tag{46}$$

Subject to the boundary conditions:

$$\alpha_1\varphi(0)+\beta_1\varphi'(x)=0 \tag{47}$$

And $\alpha_2\varphi(L)+\beta_2\varphi'(L)=0$ (48)

Taking the inverse of eqn (45), we have that:

$$G(x,t,\zeta,\tau)=\left[\sum_{n=1}^{\infty}\frac{\phi_n(\zeta)\phi_n(x)}{s+a^2k_n^2}e^{-kn^2a^2(t-\tau)}\right]H(t-\tau) \tag{49}$$

Heat equation within cylinder

In this section, we turn our attention to cylindrical domains. The techniques used here have much in common with those used in the previous section. However, in place of sines cosines form in Sturm-Liouville problem, we will encounter Bessel functions.

Example (5.3.1)

Find the solution of the following problem by construction the Green's function:

$$\frac{\partial U}{\partial t}-\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)=\frac{f(r,t)}{2\pi r},0<r,\rho<b,0<t,\tau \tag{50}$$

Subject to the boundary conditions $\lim_{x\to 0}|U(r,t)|<\infty$, $U(r,t)=0$, and the initial condition $U(r,0)=0$.

Solution: In this problem, we find Green's function for head equation in cylindrical coordinates:

$$\frac{\partial G}{\partial t}-\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)=\frac{\delta(r-\rho)\delta(t-\tau)}{2\pi r},0<r,\rho<b,0<t,\tau \tag{51}$$

Subject to the boundary conditions $\lim_{x\to 0}|G(r,t,\rho,\tau)|<\infty$, $G(b,t,\rho,\tau)=0$, and the initial condition $G(b,0,\rho,\tau)=0$

Now we begin by taking the Laplace transform to eqn (51) with respect to t:

$$\begin{aligned}&sg(r,s,\rho,\tau)-\frac{a^2}{s}\frac{d}{dr}\left(r\frac{dg}{dr}\right)=\frac{e^{-s\tau}\delta(r-\rho)}{2\pi r}\\&\frac{a^2}{s}\frac{d}{dr}\left(r\frac{dg}{dr}\right)-\frac{s}{a^2}g=-\frac{e^{-s\tau}\delta(r-\rho)}{2\pi a^2 r}\end{aligned} \tag{52}$$

Next we re-express $\frac{\delta(r-\rho)}{r}$ as the Fourier-Bessel expansion:

$$\frac{\delta(r-\rho)}{2\pi r}=\sum_{n=1}^{\infty}C_nJ_0\left(\frac{k_n\bar{r}}{b}\right) \tag{53}$$

Where k_n is the n[th] root of $J_0(k_n)=0$, and

$$C_n\frac{2}{b^2J_1^2(k_n)}\int_0^b\frac{\delta(r-\rho)}{2\pi r}J_0\left(\frac{k_nr}{b}\right)rdr=\frac{J_0\left(\frac{k_nr}{b}\right)}{\pi b^2J_1^2(k_n)} \tag{54}$$

So that $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right)-\frac{s}{a^2}g=-\frac{e^{-s\tau}}{\pi a^2b^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{J_1^2(k_n)} \tag{55}$$

$$\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}-\frac{s}{a^2}g=-\frac{e^{-s\tau}}{\pi a^2b^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{J_1^2}$$

By applying Fourier transform:

$$-k^2\bar{G}+ik^2\bar{G}-\frac{s\bar{G}}{a^2}=-\frac{e^{-s\tau}}{\pi a^2b^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{J_1^2(k_n)}$$

$$G\left(k^2+\frac{s}{a^2}\right)=-\frac{e^{-s\tau}}{\pi a^2b^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{J_1^2(k_n)}$$

$$G(\bar{r},s,\rho,\tau)=\frac{e^{-s\tau}}{\pi a^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{(sb^2+k_n^2a^2)J_1^2(k_n)} \tag{56}$$

Taking the inverse of eqn (56) and applying second shifting theorem,

$$G(r,s,\rho,\tau)=\frac{H(t-\tau)}{\pi b^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{J_1^2(k_n)} \quad (57)$$

If we modify the boundary condition at $r=b$,

$G_x(b,t,\rho,\tau)+hG(b,t,\rho,\tau)=0$ (58)

Where $h0$, our analysis now leads to:

$$G(r,s,\rho,\tau)=\frac{H(t-\tau)}{\pi b^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_n\rho}{b}\right)J_0\left(\frac{k_n\bar{r}}{b}\right)}{J_0^2(k_n)J_1^2(k_n)}\exp\left(\frac{-a^2k_n^2(t-\tau)}{b^2}\right) \quad (59)$$

Where k_n are the positive roots of $kJ_1(k)+hbJ_0(k)=0$.

Problem (3.3.2)

Find the solution for the heat equation in cylindrical by construction the Green's function:

$$\frac{\partial U}{\partial t}-\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)=\frac{f(r,t)}{2\pi r},0<t \quad (60)$$

Subject to the boundary conditions $U(\alpha,t)=U(\beta,t)=0$, and the initial condition $U(r,0)=0$

Let $G(b,t,\rho,\tau)$ Green's function, then:

$$\frac{\partial G}{\partial t}-\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)=\frac{\delta(r-\rho)\delta(t-\tau)}{2\pi r},0<r,\rho<\beta,0<t,\tau \quad (61)$$

Subject to the boundary condition that $G(\alpha,t,\rho,\tau)= G(\beta,t,\rho,\tau)=0$, and the initial condition $G(r,0,\rho,\tau)=0$.

We begin by taking Laplace transform of eqn (61), we obtain:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right)-\frac{s}{a^2}g=-\frac{e^{-s\tau}\delta(r-\rho)}{2\pi a^2 r} \quad (62)$$

Now, we express $\frac{\delta(r-\rho)}{r}$ as a Fourier series by considering the regular Sturm-Liouville problem:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+k^2\phi=0,\ \phi(\alpha)=\phi(\alpha)=0 \quad (63)$$

The eigenfunction that satisfy eqn (63) are:

$$\phi_n(x)=y_0(k_{n\alpha})J_0(k_{nr})-J_0(k_{n\alpha})y_0(k_{nr}) \quad (64)$$

Provided k_n is the nth zero of $y_0(k\alpha)J_0(k_n\beta)-J_0(k\beta)y_0(k_n\alpha)=0$, therefore, the expansion for the delta function in term of $\varphi_n(r)$ is:

$$\frac{\delta(r-\rho)}{r}=\sum_{n=1}^{\infty}C_n\phi_n(r) \quad (65)$$

$$\text{Where } C_n=\frac{\int_\alpha^\beta \delta(r-\rho)\phi_n(r)dr}{\int_\alpha^\beta \phi_n^2(r)dr} \quad (66)$$

Using the orthogonality condition that:

$$\int_\alpha^\beta J_0^2(k_nr)rdr=\frac{1}{2}r^2\left[J_0^n(k_nr)\right]_\alpha^\beta \quad (67)$$

$$\int_\alpha^\beta J_0(k_nr)y_0(k_nr)rdr=\frac{1}{2}r^2\left[J_0(k_nr)y_0(k_nr)+J_1(k_nr)y_1(k_nr)\right]_\alpha^\beta \quad (68)$$

$$\text{And } \int_\alpha^\beta y_0^2(k_nr)rdr=\frac{1}{2}r^2\left[y_0^2(k_nr)+y_1^2(k_nr)\right]_\alpha^\beta \quad (69)$$

with the wronskian relationship:

$$J_0(z)y_1(z)-J_1(z)y_0(z)=\frac{-2}{\pi z} \quad (70)$$

We find that:

$$\int_\alpha^\beta \phi_n^2(r)rdr=\frac{1}{2\pi^2k^2}\left[\frac{J_0^2(k_n\alpha)}{J_0^2(k_n\beta)}-1\right] \quad (71)$$

$$\text{Therefore } g(r,s,\rho,\tau)=\frac{4}{\pi}e^{-s\tau}\sum_{n=1}^{\infty}\frac{k_n^2J_0^2(k_{n\beta})\phi_n(\rho)\phi_n(\tau)}{\left[J_0^2(k_{n\alpha})-J_0^2(k_{n\beta})\right](s+a^2k_n^2)} \quad (72)$$

Inverting the Laplace transform, we obtain:

$$G(r,s,\rho,\tau)=\frac{4}{\pi}H(t-\tau)\sum_{n=1}^{\infty}\frac{k_n^2J_0^2(k_{n\beta})\phi_n(\rho)\phi_n(\tau)}{J_0^2(k_{n\alpha})-J_0^2(k_{n\beta})}\exp\left[-a^2k_n^2(t-\tau)\right] \quad (73)$$

In similar manner, we can solve the general problem eqn (61) with Robin boundary conditions:

$a_1G_r(\alpha,t,\rho,\tau)-a_2G(\alpha,t,\rho,\tau)=0$ (74)

And $b_1G_r(\beta,t,\rho,\tau)-b_2G(\beta,t,\rho,\tau)=0$ (75)

Where a_1, a_2, b_1, $b_2\geq 0$. The solution for this problem is:

$$G(r,s,\rho,\tau)=\frac{4}{\pi}H(t-\tau)\sum_{n=1}^{\infty}\frac{k_n^2}{F(k_n)}\left[b_1k_nJ_1(k_{n\beta})-b_2J_0(k_{n\beta})\right]\times \phi_n(\rho)\phi_n(r)\exp\left[-a^2k_n^2(t-\tau)\right] \quad (76)$$

$$\text{Where } \phi_n(r)=\begin{vmatrix} J_0(k_{nr})\left[a_1k_ny_1(k_{n\beta})+a_2y_0(k_{n\beta})\right] \\ -y_0(k_{nr})\left[a_1k_nJ_1(k_{n\alpha})+a_2J_0(k_{n\alpha})\right]\end{vmatrix} \quad (77)$$

$$\text{And } F(k_n)=\begin{vmatrix}(b_1^2k_n^2+b_2^2)\left[a_1k_nJ_1(k_{n\alpha})+a_2J_0(k_{n\alpha})\right] \\ -y_0(k_{nr})\left[a_1k_ny_1(k_{n\beta})+a_2y_0(k_{n\beta})\right]\end{vmatrix} \quad (78)$$

Heat equation within a sphere

Let us find Green's function for the radically symmetric heat equation within a sphere of radius b. mathematically, we must solve:

$$\frac{\partial G}{\partial t}-\frac{a^2}{r}\frac{\partial^2}{\partial r^2}(r\partial G)=\frac{\delta(r-\rho)\delta(t-\tau)}{4\pi r^2},0<r,\rho<b,0<t,\tau \quad (79)$$

With the boundary condition $\lim_{x\to 0}|G(r,t,\rho,\tau)|<\infty$, $G(b,t,\rho,\tau)=0$, and the initial condition $G(r,0,\rho,\tau)=0$.

Conclusion

We begin by introducing the dependent variable $u(r,t,\rho,\tau)=rG(r,t,\rho,\tau)$, so that condition eqn (69) becomes:

$$\frac{\partial u}{\partial t}-\frac{a^2}{r}\frac{\partial^2 u}{\partial r^2}=\frac{\delta(r-\rho)\delta(t-\tau)}{4\pi r^2},0<r,\rho<b,0<t,\tau \quad (80)$$

With the boundary conditions $u(0,t,\rho,\tau)=u(b,t,\rho,\tau)=0$, and the initial condition $u(r,0,\rho,\tau)=0$. Taking the Laplace transform of eqn (80), we obtain:

$$sU-a^2\frac{d^2U}{dr^2}=\frac{e^{-s\tau}}{4\pi r}\delta(r-\rho)\Rightarrow\frac{d^2U}{dr^2}-\frac{s}{a^2}U=\frac{e^{-s\tau}}{4\pi a^2 r}\delta(r-\rho) \quad (81)$$

Let us expand $\frac{\delta(r-\rho)}{r}$ in the Fourier sine series:

$$\frac{\delta(r-\rho)}{r}=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi r}{b}\right) \tag{82}$$

$$\text{Where } C_n=\frac{2}{b}\int_0^b \delta(r-\rho)\sin\left(\frac{n\pi r}{b}\right)dr=\frac{2}{b\rho}\sin\left(\frac{n\pi\rho}{b}\right) \tag{83}$$

Therefore we write eqn (81) as:

$$\frac{d^2U}{dr^2}-\frac{s}{a^2}U=\frac{e^{-s\tau}}{4\pi a^2 r}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\rho}{b}\right)\sin\left(\frac{n\pi r}{b}\right) \tag{84}$$

$$\text{and } \left(\frac{in\pi}{b}-\frac{s}{a^2}\right)U=\frac{e^{-s\tau}}{4\pi a^2\rho}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\rho}{b}\right)\sin\left(\frac{n\pi\overline{r}}{b}\right)$$

$$\left(\frac{\frac{a^2n^2\pi^2}{b^2}+s}{a^2}\right)U=-\frac{e^{-s\tau}}{4\pi a^2\rho}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\rho}{b}\right)\sin\left(\frac{n\pi\overline{r}}{b}\right)$$

$$U_{(r,t,\rho,\tau)}=-\frac{e^{-s\tau}}{4\pi\rho}\sum_{n=1}^{\infty}\frac{1}{s+\frac{a^2n^2\pi^2}{b^2}}\sin\left(\frac{n\pi\rho}{b}\right)\sin\left(\frac{n\pi\overline{r}}{b}\right) \tag{85}$$

Because this particular solution also satisfies the boundary condition, we do not require any homogeneous solution so that the sum of the particular and homogeneous solution satisfies the boundary conditions.

Taking the inverse of eqn (85), using the second shifting theorem and substituting for $U(r,t,\rho,\tau)$, we finally obtain:

$$G_{(r,t,\rho,\tau)}=\frac{H(t-\tau)}{2\pi br}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\rho}{b}\right)\sin\left(\frac{n\pi r}{b}\right)\exp\left[\frac{-a^2n^2\pi^2(t-\tau)}{b^2}\right] \tag{86}$$

For the case of a sphere $\alpha<r<\beta$, the Green's function can be bound by introducing the new independent variable $x=r-\alpha$ into the partial differential equation:

$$\frac{\partial G}{\partial t}-\frac{a^2}{r}\frac{\partial^2(r\partial G)}{\partial r^2}=\frac{\delta(r-\rho)\delta(t-\tau)}{4\pi r^2},\alpha<r,\rho<b,0<t,\tau \tag{87}$$

With the boundary conditions $G(\alpha,t,\rho,\tau)=G(\beta,t,\rho,\tau)=0$, and the initial condition $G(r,0,\rho,\tau)=0$. Therefore the Green's function in this particular case is:

$$G_{(r,t,\rho,\tau)}=\frac{H(t-\tau)}{2\pi_{(\beta-\alpha)}r\rho}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi(\rho-\alpha)}{(\beta-\alpha)}\right)\sin\left(\frac{n\pi(r-\alpha)}{(\beta-\alpha)}\right)\times \exp\left[\frac{-a^2n^2\pi^2(t-\tau)}{(\beta-\alpha)^2}\right] \tag{88}$$

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