Journal of
Applied & Computational Mathematics
ISSN: 2168-9679

Hassana, J Appl Computat Math 2017, 6:3
DOI: 10.4172/2168-9679.1000362

Research Article OMICS International

Green's Function Solution of Non-Homogenous Regular Sturm-Liouville Problem

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Abstract

In this paper, we propose a new method called exp(−ϕ(ξ)) fractional expansion method to seek traveling wave solutions of the nonlinear fractional Sharma-Tasso-Olver equation. The result reveals that the method together with the new fractional ordinary differential equation is a very influential and effective tool for solving nonlinear fractional partial differential equations in mathematical physics and engineering. The obtained solutions have been articulated by the hyperbolic functions, trigonometric functions and rational functions with arbitrary constants.

Introduction

The series solution of differential equation yields an infinite series which often converges slowly. Thus it is difficult to obtain an insight into over-all behavior of the solution [1,2]. The Green's function approach would allow us to have an integral representation of the solution instead of an infinite series.

To obtain the filed y, caused by distributed source we calculate the effect of each elementary portion of source and add (integral) them all. If $G(r,r_0)$ is the field at the observers point r caused by a unit source at the source point r_0, then the field at r caused by distribution $f(r_0)$ is the integral of $f(r_0)$ $G(r,r_0)$ over the whole range of r_0 occupied by the course. The function G is called Green's function [3-5].

The Green's function is powerful tool of mathematical method which used I solving linear non-homogenous differential equation (ordinary and partial) [6-9].

Preliminaries

Sturm-Liouville problem

Consider a linear second order differential equation

$$A(x)\frac{d^2y}{dx^2}+B(x)\frac{dy}{\partial x}+C(\mathrm{x})\mathrm{y}+\lambda\,\mathrm{D}(\mathrm{x})\mathrm{y}=0 \tag{1}$$

Where λ is a parameter to be determined by the boundary conditions? $A(x)$ is positive continuous function, then by dividing every term by $A(x)$, equation (1) can be written as

$$\frac{d^2y}{dx^2}+B(x)\frac{dy}{\partial x}+C(\mathrm{x})\mathrm{y}+\lambda\,\mathrm{D}(\mathrm{x})\mathrm{y}=0 \tag{2}$$

Where $b(\mathrm{x})=\dfrac{B(x)}{A(x)}$, $c(\mathrm{x})=\dfrac{C(x)}{A(x)}$ and $\mathrm{d}(\mathrm{x})=\dfrac{D(x)}{A(x)}$

Let us define integrating factor $p(x)$ by

$$P(x)=\exp\left\{\int_a^x b(\xi)\,d\xi\right\}$$

Multiplying equation (2) by $p(x)$, we have

$$p(x)\frac{d^2y}{dx^2}+p(x)b(x)\frac{dy}{\partial x}+p(\mathrm{x})\mathrm{c}(x)\mathrm{y}+\lambda p(x)d(\mathrm{x})\mathrm{y}=0 \tag{3}$$

Since $\dfrac{dp(x)}{dx}=\dfrac{d}{dx}\left(e^{\int_a^x b(\zeta)d\zeta}\right)=e^{\int_a^x b(\zeta)d\zeta}\dfrac{d}{dx}\left(\int_a^x b(\zeta)\,\mathrm{d}\zeta\right)=p(x)b(x)$

So $\dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right]=p(x)\dfrac{d^2y}{dx^2}+\dfrac{dp(x)}{dx}\dfrac{dy}{dx}=p(x)\dfrac{d^2y}{dx^2}+p(\mathrm{x})b(x)z$

Thus equation (3) can be written as

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y+\lambda r(x)y=0 \tag{4}$$

Where $q(x)=p(x)c(x)$ and $r(x)=p(x)d(x)$

Equation in form (4) is known as Sturm-Liouville equation. Satisfy the boundary conditions

Regular Sturm-Liouville Problem

In case $p(a)\neq 0$ and $p(b)\neq 0$, $p(x)$, $q(x)$, $r(x)$ are continuous, the Sturm-Liouville equation (4) can be expressed as

$$L[y]=\lambda r(x)y \tag{5}$$

$$\text{Where } L=\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x) \tag{6}$$

If the above equation is associated with the following boundary condition

$$\alpha_1 y(a)+\alpha_2 y'(a)=0$$

$$\beta_1 y(b)+\beta_2 y'(b)=0 \tag{7}$$

Where $\alpha_1+\alpha_2\neq 0$ and $\beta_1+\beta_2\neq 0$

The equation (4) and the boundary condition (7) are called regular Sturm -Liouville problem (RSLP).

(a) Singular Sturm-Liouville problem

Consider the equation

$$\mathrm{L[y]+\lambda r(x)y=0 \quad a<x<b} \tag{8}$$

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Received April 05, 2017; **Accepted** April 25, 2017; **Published** May 11, 2017

Citation: Hassana AA (2017) Green's Function Solution of Non-Homogenous Regular Sturm-Liouville Problem. J Appl Computat Math 6: 362. doi: 10.4172/2168-9679.1000362

Where L is defined by (6), $p(x)$ is smooth and $r(x)$ is positive, then the Sturm- Liouville problem is called singular if one of the following situations is occurred.

(i) If $p(a)=0$ or $p(b)= 0$ or both

(ii)The interval (a,b) is infinite.

(b)Eigenvalue and eigenfunction

The Eigenvalue from equation (4) defining by a Sturm-Liouville operator can be expressed as

$$\lambda = \frac{-1}{r(x)}\left[\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)\right] \tag{9}$$

The non-trival solutions that satisfy the equation and boundary conditions are called eigenfunctions. Therefore the eigenfunction of the Sturm-Liouville problem from complete sets of orthogonal bases for the function space is which the weight function is $r(x)$.

The Dirac delta function

The delta function is defined as

$$\delta(x-\xi) = \begin{cases} 0 & x \neq \xi \\ \infty & x = \xi \end{cases} \tag{10}$$

But such that the integral of $\delta(x-\zeta)$ is normalized to unit

$$\int_{-\infty}^{+\infty} \delta(x-\xi)dx = 1 \tag{11}$$

In fact the first operator where Dirac used the delta function is the integration

$$\int_{-\infty}^{+\infty} f(x)\delta(x-\xi)dx \tag{12}$$

Where $f(x)$ is a continuous function, we have to find the value of the integration (12). Since $\delta(x\zeta)$ is zero for $x\neq\zeta$, the limit of integration may be change to $\zeta-\varepsilon$ and $\zeta+\varepsilon$, where ε is a small positive number, $f(x)$ is continuous at $x-\zeta$, it's values within the interval $(\zeta-\varepsilon,\zeta-\varepsilon)$ will not different much from $f(\zeta)$, approximately that:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-\xi)dx = \int_{\xi-\infty}^{\xi+\infty} f(x)\delta(x-\xi)dx \approx \int_{\xi-\infty}^{\xi+\infty} f(\xi)\delta(x-\xi)dx \tag{13}$$

With the approximation improving as ε approaches zero.

From (11), we have

$$I = \int_{-\infty}^{+\infty} \delta(x-\xi)dx = \int_{\xi-\infty}^{\xi+\infty} \delta(x-\xi)dx = 1 \tag{14}$$

From all values of ε, then by letting $\varepsilon\to0$, we can exactly have

$$\int_{-\infty}^{+\infty} f(x)\delta(x-\xi)dx = f(\zeta) \tag{15}$$

Despite the delta function considered as fundamental role in electrical engineering and quantum mechanics, but no conventional could be found that satisfies (10) and (11), then the delta function sought to be view as the limit of the sequence of strongly peaked function $\delta_n(x)$such that

$$\delta(x) = \lim_{n\to\infty}\delta_n(x) \tag{16}$$

As

$$\left.\begin{aligned} \delta_n(x) &= \frac{n}{\pi\left(1+n^2x^2\right)} \\ \delta_n(x) &= \frac{n}{\sqrt{\pi}}e^{-n^2x^2} \\ \delta_n(x) &= \frac{\sin^2(nx)}{n\pi x^2}etc \end{aligned}\right\} \tag{17}$$

(c) Some important properties of Dirac delta function

Property (1): Symmetry

$$\delta(-x)=\delta(x) \tag{18}$$

Proof

Let $\zeta=-x$, then $dx=-d\zeta$

We can write:

$$\int_{-\infty}^{+\infty} f(x)\delta(-x)dx = \int_{+\infty}^{-\infty} f(\zeta)\delta(\zeta)d\zeta \approx \int_{-\infty}^{+\infty} f(\zeta)\delta(\zeta)d\zeta = f(0) \tag{19}$$

But, $$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0) \tag{20}$$

Therefore, from (19) and (20), we conclude that $\delta(-x)=\delta(x)$

Property (2): Scaling

$$\delta(ax)=\frac{1}{|a|}\delta(x) \tag{21}$$

Proof

Let $\zeta=ax$, then $dx = \frac{1}{a}d\zeta$

If $a>0$, then

$$\int_{-\infty}^{+\infty} f(x)\delta(ax)dx = \int_{+\infty}^{-\infty} f\left(\frac{\zeta}{a}\right)\delta(\zeta)\frac{1}{a}d\zeta = \frac{1}{a}\int_{-\infty}^{+\infty} f\left(\frac{\zeta}{a}\right)\delta(\zeta)d\zeta = \frac{1}{a}f\left(\frac{0}{a}\right)f(0)$$

Since $$\int_{-\infty}^{+\infty} f(\zeta)\frac{1}{a}\delta(x)dx = \frac{1}{a}\int_{-\infty}^{+\infty} f(x)\delta(x)dx = \left(\frac{1}{a}\right)f(0)$$

Therefore:

$$\delta(ax)=\frac{1}{a}\delta(x)$$

Similarly for $a<0$,

$$\delta(ax)=\frac{1}{-a}\delta(x)$$

then

We can write:

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

Property (3)

$$\delta(x^2-a^2)=\frac{1}{|2a|}\left[\delta(x+a)+\delta(x-a)\right] \tag{22}$$

Proof

The argument of this function goes to zero when $x = a$ and $x=-a$, wherefore

$$\int_{-\infty}^{+\infty} f(\zeta)\delta(x^2-a^2)dx = \int_{-\infty}^{+\infty} f(x)\left[\delta(x+a)+\delta(x-a)\right]dx$$

Only at the zero of the argument of the delta function that is:

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$$\int_{-\infty}^{+\infty} f(\zeta)\delta(x^2-a^2)dx = \int_{-a-e}^{-a+e} f(x)\left[\delta(x^2-a^2)dx + \int_{a-e}^{a+e} f(x)\delta(x^2-a^2)\right]dx \quad (23)$$

Near the two zeros x^2-a^2 can be approximated as:

$$(x^2-a^2)=(x-a)(x+a)=\begin{cases}(-2a)(x+a), & x\to -a\\(+2a)(x-a), & x\to +a\end{cases}$$

In the limit as $\varepsilon\to 0$ the integral (23) becomes:

$$\int_{-\infty}^{+\infty} f(\zeta)\delta(x^2-a^2)dx = \int_{-a-e}^{-a+e} f(x)\delta[(-2a)(x+a)]dx + \int_{a-e}^{a+e} f(x)\delta[(2a)(x-a)]dx$$

$$= \frac{1}{|2a|}\int_{a-e}^{a+e} f(x)\delta[(x)\delta(x+a)] + \frac{1}{|2a|}\int_{a-e}^{a+e} f(x)\delta(x-a)dx$$

$$= \int_{-\infty}^{+\infty} f(x)\frac{1}{|2a|}[\delta[x-a]+\delta(x-a)]dx$$

Therefore: $\delta(x^2-a^2) == \frac{1}{|2a|}[\delta[x-a]+\delta(x-a)]$

Green's Function

The concept of Green's function

In the case of ordinary differential equation we can express this problem as

$$L[y]=f \quad (24)$$

Where L is a linear differential operation $f(x)$ is known function and $y(x)$ is desired solution. We will show that the solution $y(x)$ is given by an integral involving that Green's function $G(x,\xi)$.

Green's function for ordinary differential equation

Here we consider non-homogenous ordinary differential equation

$$L[y]=f \quad (25)$$

Where L is an ordinary linear differential operator that can be represented bySturm-Liouville operator, i.e.

$$L=\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x) \quad (26)$$

And the Sturm-Liouville type is gives by

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y+\lambda r(x)y=-f(x) \quad (27)$$

Where λ is a parameter. Now consider the linear non homogenous ordinary differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=-f(x)a\le x\le b \quad (28)$$

With the boundary condition

$$\left.\begin{aligned}\alpha_1 y(a)+\alpha_2 y'(a)=0\\ \beta_1 y(b)+\beta_2 y'(b)=0\end{aligned}\right\} \quad (29)$$

where the constant are such that $\alpha_1+\alpha_2 0$ and $_1+\beta_2\neq 0$ if $\lambda=0$ then equation (27) and equation (28) are identical in the interval and $r(x)$ are real and positive in that interval.

Now we are seeking to determine the Green's function G for the equation satisfies the following

$$\frac{d}{dx}\left[p(x)\frac{dG}{dx}\right]+q(x)y=-\delta(x-\zeta) \quad (30)$$

With the boundary condition

$$\begin{aligned}\alpha_1 G(a)+\alpha_2 G'(a,\zeta)=0\\ \beta_1 G(b)+\beta_2 G'(b,\zeta)=0\end{aligned} \quad (31)$$

Now consider the region $a\le x<\zeta$.

Let $y_1(x)$ be a nontrivial solution at $x=a$, i.e

$$\alpha_1 y_1(a)+\alpha_2 y'(a)=0 \quad (32)$$

Then $\alpha_1 y_1(a,\zeta)+\alpha_2 G'(a,\zeta)=0$ (33)

The wronskian of y_1 and G must vanish at $x=a$ or

$$y_1(a)G'(a,\zeta)+y'_1(a)G(a,\zeta)=0$$

So $G(x,\zeta)= u_1 y_1(x)$ for $a\le x<\zeta$ (34)

Where u_1 is an arbitrary constant. Similarly if the nontrivial solution $y_2(x)$ satisfies the homogeneous equation and the condition at $x=b$, then

$G(x,\zeta)= u_2 y_2(x)$ for $\zeta\le x<b$ (35)

Now by integrating equation (29) from $\zeta-\varepsilon$ to $\zeta+\varepsilon$ we obtain

$$P(x)\frac{dG(x,\zeta)}{dx}\left[\begin{smallmatrix}\zeta+\varepsilon\\ \zeta+\varepsilon\end{smallmatrix}+\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} q(x)G(x,\zeta)dx=-1\right] \quad (36)$$

Since $G(x,\zeta)$ and $q(x)$ are continuous at $x=\zeta$ then we have

$$\frac{dG\left(\overset{+}{\zeta},\zeta\right)}{dx}=\frac{dG\left(\overset{-}{\zeta},\zeta\right)}{dx}=\frac{1}{p(\zeta)} \quad (37)$$

The continuity condition of G and the Jump discontinuity of G' at $x=\zeta$ from equation (33) , (34) and equation (36) imply

$$\left.\begin{aligned}u_1 y_1(\zeta)-u_2 y_2 &=0\\ u_1 y_1(\zeta)-u_2 y'_2 &=\frac{1}{p(\zeta)}\end{aligned}\right\} \quad (38)$$

we can solve equation (37) for u_1 and u_2 provided the wronskian y_1 and y_2 doesn't varnish at $x=\zeta$ or

$$y_1(\zeta)y'_2(\zeta)y'_1(\zeta)\neq 0 \quad (39)$$

i.e., $w(\zeta)=\begin{vmatrix}y_1(\zeta) & y_1(\zeta)\\ y'_1(\zeta) & y'_2(\zeta)\end{vmatrix}\neq 0$

The system of equation (37) has the solution

$$\left.\begin{aligned}u_1&=\frac{-y'_2(\zeta)}{p(\zeta)w(\zeta)}\\ &and\\ u_2&=\frac{-y'_1(\zeta)}{p(\zeta)w(\zeta)}\end{aligned}\right\} \quad (40)$$

Where $w(\zeta)$ is the wronskian of $y_1(x)$ and $y_2(x)$ at $x=\zeta$

Therefore

$$G(x,\zeta)=\frac{y_1(x<)y_2(x>)}{p(\zeta)w(\zeta)} \quad (41)$$

$$G(x,\zeta)=\left\{\begin{aligned}&\frac{y_1(x)y_2(\zeta)}{p(\zeta)w(\zeta)}, x<\zeta\le b\\ &-\frac{y_1(x<)y_2(x>)}{p(\zeta)w(\zeta)}, a<\zeta\le x\end{aligned}\right\} \quad (42)$$

Now from (42) the solution (27) can be expressed as

$$y(\mathrm{x})=\int_a^b G(x,\zeta)f(\zeta)d\zeta \tag{43}$$

So $$y(\mathrm{x})=\int_a^x G(x,\zeta)f(\zeta)d\zeta+\int_a^b G(x,\zeta)f(\zeta)d\zeta \tag{44}$$

Some properties of Green's function:

The following properties followed Green's function

Property (i)

$G(x,\zeta)$ is exit because both $p(x)\neq 0$, $w(x)\neq 0$

Proof

From equation (32) and (33) we obtain

$$u_1=\frac{u_1 y_2(x)}{y_1(x)} \tag{45}$$

And from equation (35) at $x=\zeta$ we have

$$u_1 y'_1(x)-u_2 y'_2(x)=-\frac{1}{p(x)} \tag{46}$$

Substituting from (44) into (45) we obtain

$$u_1\left[y_2(x)y_1'(x)-y_1(x)y_2'(x)\right]=-\frac{y_1(x)}{p(x)} \tag{47}$$

Let $w(x)=y_1(x)y'_2(x)-y_2(x)y'_1(x)$

i.e. $$w(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x)\end{vmatrix} \tag{48}$$

$$\left.\begin{array}{l} u_1=-\dfrac{-y_2(x)}{p(x)w(x)} \\ and \\ u_2=-\dfrac{-y_1(x)}{p(x)w(x)}\end{array}\right\} \tag{49}$$

but u_1 and u_2 are arbitrary constant

∴ both $p(x)\neq 0$ and $w(x)\neq 0$

Property (ii)

$G(x,\zeta)$ satisfies the homogenous equation except at $x=\zeta$

Proof

From equation (37) $\delta(x-\zeta)=0$ except at $x=\zeta$

$$p'(x)G'(x,\zeta)+q(x)G(x,\zeta)=0 \tag{50}$$

where $p(x)\neq 0$, $p'(x)$,$q(x)$ are continuous on [a,b]

Property (iii)

$G(x,\zeta)$ is continuous at $x=\zeta$

Proof

$$\lim_{x\to\zeta} G(x,\zeta)=y_2(\zeta)$$

$$\lim_{x\to\zeta} G(x,\zeta)=y_1(\zeta)$$

Therefore $G(x,\zeta)$ is continuous at $x=\zeta$

Property (iv)

The first and second derivatives are continuous for all

$x\neq\zeta$ in $a\leq x$, $\zeta\leq b$

Proof

Differentiation equation (32) with respect to x

$$\frac{d}{dx}\int_a^b G(x,\zeta)f(\zeta)d(\zeta)=\left.\begin{array}{l}\displaystyle\int_a^x \frac{d^2G(x,\zeta)}{dx}f(\zeta)d\zeta+\frac{dG(x,x^-)}{dx}d(\zeta) \\ \displaystyle+\int_x^b \frac{d^2G(x,\zeta)}{dx}f(\zeta)d\zeta-\frac{dG(x,x^+)}{dx}d(\zeta)\end{array}\right\} \tag{51}$$

But $G(x,\zeta)$ is continuous everywhere, there we have

$G(x, x^+)=G(x, x^-)$ so that

$$\frac{d}{dx}\int_a^b G(x,\zeta)f(\zeta)d(\zeta)=\int_a^x \frac{dG(x,\zeta)}{dx}f(\zeta)d\zeta+\int_x^b \frac{dG(x,\zeta)}{dx}f(\zeta)d\zeta$$

Differentiation once more gives

$$\frac{d^2}{dx^2}\int_a^b G(x,\zeta)f(\zeta)d(\zeta)=\left.\begin{array}{l}\displaystyle\int_a^x \frac{d^2G(x,\zeta)}{dx^2}f(\zeta)d\zeta+\frac{dG(x,x^-)}{dx}d\zeta \\ \displaystyle+\int_b^b \frac{d^2G(x,\zeta)}{dx^2}f(\zeta)d\zeta-\frac{dG(x,x^+)}{dx}d\zeta\end{array}\right\} \tag{52}$$

The second and fourth terms on the right side will not cancel in this case to the contrary

$$\frac{dG(x,x^-)}{dx}-\frac{dG(x,x^+)}{dx}=\frac{1}{p(\zeta)} \tag{53}$$

We note that the term $\frac{dG(x,x^-)}{dx}$ denotes a differentiation of $G(x,\zeta)$with respect to x using the $x>\zeta$, at $\zeta\to x$. Thus

$$\frac{dG(x,x^-)}{dx}=\lim_{\substack{\zeta\to X\\ \zeta\to\infty}}\frac{y'_1(x)y_1(\zeta)}{p(\zeta)w(\zeta)}=\frac{y'_1(x)y_1(\zeta)}{p(x)w(\zeta)}$$

For: $dG(x, x^+)$ we use the $x<\zeta$ then

$$\frac{dG(x,x^+)}{dx}=\lim_{\substack{\zeta\to X\\ \zeta\to\infty}}\frac{y'_1(x)y_2(\zeta)}{p(\zeta)w(\zeta)}=\frac{y'_1(x)y_2(\zeta)}{p(x)w(\zeta)}.$$

Property (v)

$G(x,\zeta)$ is symmetric in x and ζ

Poof

Let $$G(x,\zeta)=\begin{cases}\dfrac{-y'_1(x)y_2(\zeta)}{p(\zeta)w(\zeta)} & \\ \dfrac{-y'_2(x)y_1(\zeta)}{p(\zeta)w(\zeta)} & a\leq x<\zeta\end{cases}$$

$$\lim_{x+\xi}G(x^+,\zeta)=\frac{y'_2(x)y_1(\zeta)}{p(\zeta)w(\zeta)}$$

and $$\lim_{x+\xi}G(x^+,\zeta)=\frac{y'_2(x)y_1(\zeta)}{p(\zeta)w(\zeta)}$$

Problem(1)

Find the solution of the following problem by construction the Green's function

$$y''+k^2y=-f(x),\ 0<x<L \tag{54}$$

Subject to the boundary conditions

$$y(0)+y(L)=0 \tag{55}$$

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With $k\neq 0$

Solution

Let $G(x,\zeta)$ be the Green's function of the problem, then

$$G''+k^2G=-\delta(x\zeta) \quad (56)$$

With $G(0,\zeta)+G(L,\zeta)=0$ (57)

The general solution to the homogenous equation is given by

$$y(x)=c_1\cos kx+c_2\sin kx \quad (58)$$

Now applying to the above solution

$y(0)=0$

$y(0)=c_1\cos kx+c_2(0)=0\Rightarrow c_1=0$

$$\therefore y_1(x)=c_2\sin kx \quad (59)$$

and $y(L)=0\Rightarrow 0=c_1\cos L+c_2\sin kL$

$$\therefore c_1=-\frac{c_2\sin kL}{\cos L}$$

$$y_2(x)=\frac{c_2}{\cos k\mathrm{L}}(\cos kx\sin kL-\sin kx\cos kL)=\frac{C_2}{CoskL}\mathrm{sink}(\mathrm{L}-x) \quad (60)$$

The Wronskian is given by

$$w(x)=\begin{vmatrix} c_2\sin kx & \frac{C_2}{CoskL}(\mathrm{L}-x) \\ c_2k\cos kx & \frac{C_2k}{CoskL}\mathrm{cosk}(\mathrm{L}-x) \end{vmatrix}=\frac{C_2}{CoskL}\mathrm{sinkL} \quad (61)$$

The Green's function is given by

$$G(x,\zeta)=\begin{cases} \dfrac{y'_1(x)y_2(\zeta)}{p(\zeta)w(\zeta)} & 0\le x<\zeta \\ \dfrac{y'_2(x)y_1(\zeta)}{p(\zeta)w(\zeta)} & \zeta<x\le L \end{cases}$$

$$G(x,\zeta)=\begin{cases} -\dfrac{\sin kx\sin k(L-\zeta)}{k\sin kL} & 0\le x<\zeta \\ -\dfrac{\sin k(L-x)\sin k\zeta}{k\sin kL} & \zeta<x\le L \end{cases} \quad (62)$$

The solution is given by

$$y(\mathrm{x})=\int_a^L G(x,\zeta)f(\zeta)d\zeta$$

$$y(\mathrm{x})=\int_0^X \frac{\sin kx\sin l(L-\zeta)}{k\sin kL}f(\zeta)d\zeta-\int_x^l \frac{\sin k(L-x)\sin k\zeta}{k\sin kL}f(\zeta)d\zeta \quad (63)$$

Problem (2)

Solve the problem by construction the Green's function

$$y''-\frac{1}{a}=-f(x), 0<x<l \quad (64)$$

with the boundary condition

$y(0)+y(L)=0$ where $a\neq 0$ (65)

Solution

Let $G(x,\zeta)$ be the Green's function of the problem, then

$$G''-\frac{1}{a}=-\delta(x-\zeta) \quad (66)$$

With the boundary condition $G(0,\zeta)+G(L,\zeta)=0$

The general solution to the homogenous equation is given by

$$Y(x)=\frac{1}{2a}+c_1x^2+c_2 \quad (67)$$

Applying to the above solution

then, and $y(0)=0$ then $c_2=0$, and

$$\therefore y_1(x)=\frac{x^2}{2a}+c_1x \quad (68)$$

Applying the boundary condition, then $y(L)=0$ $c_2=\left(\frac{L^2}{2a}+c_1L\right)$, and

$$y_2(x)=-\frac{c_2}{2a}(x^2-L^2)+c_1(x-L) \quad (69)$$

The Wronskian is given by

$$w(x)=L\left(\frac{x}{a}+c_1\right)\left(\frac{L}{2a}+c_1\right) \quad (70)$$

The Green's function is given by

$$G(x,\zeta)=\begin{cases} \dfrac{y'_1(x)y_2(\zeta)}{p(\zeta)w(\zeta)} & 0\le x<\zeta \\ \dfrac{y'_2(x)y_1(\zeta)}{p(\zeta)w(\zeta)} \end{cases}$$

$$G(x,\zeta)=\begin{cases} \dfrac{\left(\frac{x^2}{2a}+c_1x\right)\left[\frac{1}{2a}(\zeta^2-L^2)+c_1(\zeta-L)\right]}{L\left(\frac{x}{2a}+c_1\right)\left(\frac{L}{2a}+c_1\right)} & 0\le x<\zeta \\ \dfrac{\left(\frac{x^2}{2a}+c_1x\right)\left[\frac{1}{2a}(\zeta^2-L^2)+c_1(\zeta-L)\right]}{L\left(\frac{x}{2a}+c_1\right)\left(\frac{L}{2a}+c_1\right)} & \zeta<x\le L \end{cases}$$

Where we chose $c_1=1$, then

$$G(x,\zeta)=\begin{cases} \dfrac{x\left(\frac{x}{2a}+1\right)(\zeta-L)\left[\frac{1}{2a}(\zeta+L)+c_1(\zeta-L)\right]}{L\left(\frac{x}{2a}+c_1\right)\left(\frac{L}{2a}+c_1\right)} & 0\le x<\zeta \\ \dfrac{\left(\frac{x^2}{2a}+c_1x\right)\left[\frac{1}{2a}(\zeta^2-L^2)+c_1(\zeta-L)\right]}{L\left(\frac{x}{2a}+c_1\right)\left(\frac{L}{2a}+c_1\right)} & \zeta<x\le L \end{cases} \quad (71)$$

The solution is given by

$$y(x)=\int_a^L G(x,\zeta)f(\zeta)d\zeta$$

$$y(x)=\int_a^x \frac{x\left(\frac{x}{2a}+1\right)(\zeta-L)\left[\frac{1}{2a}(\zeta+L)+1\right]}{L\left(\frac{x}{2a}+1\right)\left(\frac{L}{2a}+1\right)}f(\zeta)d\zeta+\int_x^L \frac{\frac{1}{2a}(x-L)\left[\zeta\left(\frac{1}{2a}+1\right)+\frac{1}{2a}(x+L)\right]}{L\left(\frac{x}{a}+1\right)\left(\frac{L}{2a}+1\right)}f(\zeta)d\zeta \quad (72)$$

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Citation: Hassana AA (2017) Green's Function Solution of Non-Homogenous Regular Sturm-Liouville Problem. J Appl Computat Math 6: 362. doi: 10.4172/2168-9679.1000362