

# Higher Bruhat and Tamari Orders and their Realizations

Dimakis A<sup>1</sup> and Müller-Hoissen F<sup>2\*</sup>

<sup>1</sup>Department of Financial and Management Engineering, University of the Aegean, Greece

<sup>2</sup>Max Planck Institute for Dynamics and Self-Organization, 37077 Göttingen, Germany

### Abstract

We briefly describe recent results about higher Bruhat and Tamari orders, the associated simplex equations and generalizations (polygon equations) of the pentagon equation, and the appearance of these orders in soliton interactions.

**Keywords:** Associahedron; Higher Bruhat order; KP equation; Pentagon equation; Permutahedron; Simplex equation; Soliton; Tamari order; Yang-Baxter equation

### Introduction

In this letter, we sketch some of our recent results [1] at the interface between integrable systems and combinatorics. This includes a revision of the relation between *higher Bruhat orders* and *simplex equations* [2,3], a decomposition of higher Bruhat orders, the resulting Tamari orders (expected to be equivalent to higher Stasheff -Tamari orders), and a new family of equations associated with the latter. We finally recall the occurrence of higher Bruhat and Tamari orders in a “tropical limit” of solitons of the famous Kadomtsev-Petviashvili (KP) equation [4,5].

### Higher Bruhat Orders and Simplex Equations

Let  $[N] = \{1, 2, \dots, N\}$  and  $\binom{[N]}{n}$  be the set of  $n$ -element subsets of  $[N]$ . A linear order (permutation) of  $\binom{[N]}{n}$  is called *admissible* if for any  $K \in \binom{[N]}{n+1}$ , the packet  $P(K) := \{n\text{-element subsets of } K\}$  is contained in it in *lexicographical* ( $\bar{P}(K)$ ) or in *reverse lexicographical* order ( $\tilde{P}(K)$ ). Let  $A(N, n)$  be the set of admissible linear orders of  $\binom{[N]}{n}$ . An equivalence relation is defined on  $A(N, n)$  by setting  $\rho \sim \rho'$  if  $\rho$  and  $\rho'$  only differ by exchange of two neighboring elements, not both contained in some packet. The *higher Bruhat order*  $B(N, n)$  is  $A(N, n)/\sim$  supplied with the partial order determined by inversions  $\bar{P}(K) \mapsto \tilde{P}(K)$  of lexicographically ordered packets of neighboring elements. There is a natural correspondence between the elements of  $A(N, n+1)$  and the maximal chains of  $B(N, n)$ . The Bruhat order  $B(3, 1)$  consists of the two maximal chains

$$(1, 2, 3) \xrightarrow{12} (2, 1, 3) \xrightarrow{13} (2, 3, 1) \xrightarrow{23} (3, 2, 1)$$

$$(1, 2, 3) \xrightarrow{23} (1, 3, 2) \xrightarrow{13} (3, 1, 2) \xrightarrow{12} (3, 2, 1)$$

where  $ij$  stands for  $\{i, j\}$  and here indicates the inversion of the packet of it in the respective linear order. A set-theoretical realization

$$i \mapsto U_i, \quad (i, j, k) \mapsto U_i \times U_j \times U_k, \quad ij \mapsto \mathcal{R}_{ij} : U_i \times U_j \rightarrow U_j \times U_i$$

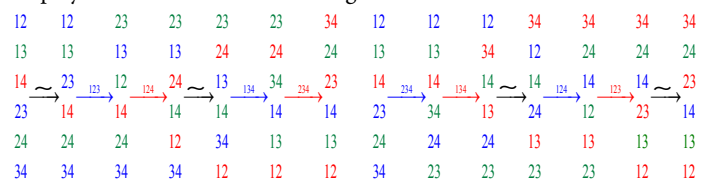
(or a realization using vector spaces and tensor products) leads to the *Yang-Baxter (YB) equation*

$$\mathcal{R}_{23,12} \mathcal{R}_{13,23} \mathcal{R}_{12,13} = \mathcal{R}_{12,23} \mathcal{R}_{13,12} \mathcal{R}_{23,12}$$

The boldface *position indices* are read off from the above chains. They specify on which pair of sets in the threefold direct product the map  $\mathcal{R}_{ij}$  acts. In terms of  $\hat{\mathcal{R}}_{ij} := \mathcal{R}_{ij} \mathcal{P}$ , where  $\mathcal{P}$  is the transposition, the

YB equation has the more familiar form  $\hat{\mathcal{R}}_{23,12} \hat{\mathcal{R}}_{13,13} \hat{\mathcal{R}}_{12,23} = \hat{\mathcal{R}}_{12,23} \hat{\mathcal{R}}_{13,13} \hat{\mathcal{R}}_{23,12}$ .

For any  $N > 1$ , the higher Bruhat order  $B(N+1, N-1)$  consists of two maximal chains, so that it determines an equation in the same way as in the above example, where  $N=2$ . The associated equation is the *N-simplex equation*. The 3-simplex equation is thus a realization of  $B(4, 2)$ . The two maximal chains of the latter poset are resolved below into elements of  $A(4, 2)$ , and a linear order is now more conveniently displayed as a column. The coloring is referred to later on.



From the above chains we read off the 3-simplex equation

$$\mathcal{R}_{234,123} \mathcal{R}_{134,345} \mathcal{P}_{56} \mathcal{P}_{23} \mathcal{R}_{124,345} \mathcal{R}_{123,123} \mathcal{P}_{34} = \mathcal{P}_{34} \mathcal{R}_{123,456} \mathcal{R}_{124,234} \mathcal{P}_{45} \mathcal{P}_{12} \mathcal{R}_{134,234} \mathcal{R}_{234,456}$$

for maps  $\mathcal{R}_{ijk} : U_i \times U_j \times U_k \rightarrow U_j \times U_k \times U_i, i < j < k$ . An equivalence ( $\sim$ ) is realized by a transposition map  $\mathcal{P}_{ab}$  (acting at positions  $a$  and  $b$ ). In terms of  $\hat{\mathcal{R}}_{ijk} := \mathcal{R}_{ijk} \mathcal{P}_{13} : U_i \times U_j \times U_k \rightarrow U_j \times U_k \times U_i$  the 3-simplex equation takes the form

$$\hat{\mathcal{R}}_{1,123} \hat{\mathcal{R}}_{2,145} \hat{\mathcal{R}}_{3,246} \hat{\mathcal{R}}_{4,356} = \hat{\mathcal{R}}_{4,356} \hat{\mathcal{R}}_{3,246} \hat{\mathcal{R}}_{2,145} \hat{\mathcal{R}}_{1,123}$$

using complementary notation:  $\hat{1} = \{2, 3, 4\} = 234$  etc. This is also known as *tetrahedron equation*, or *Zamolodchikov equation*. It appeared originally as a factorization condition for the S-matrix in a (2+1)-dimensional theory of “straight strings”, and in a related three-dimensional exactly solvable lattice model (see, e.g., [6]).

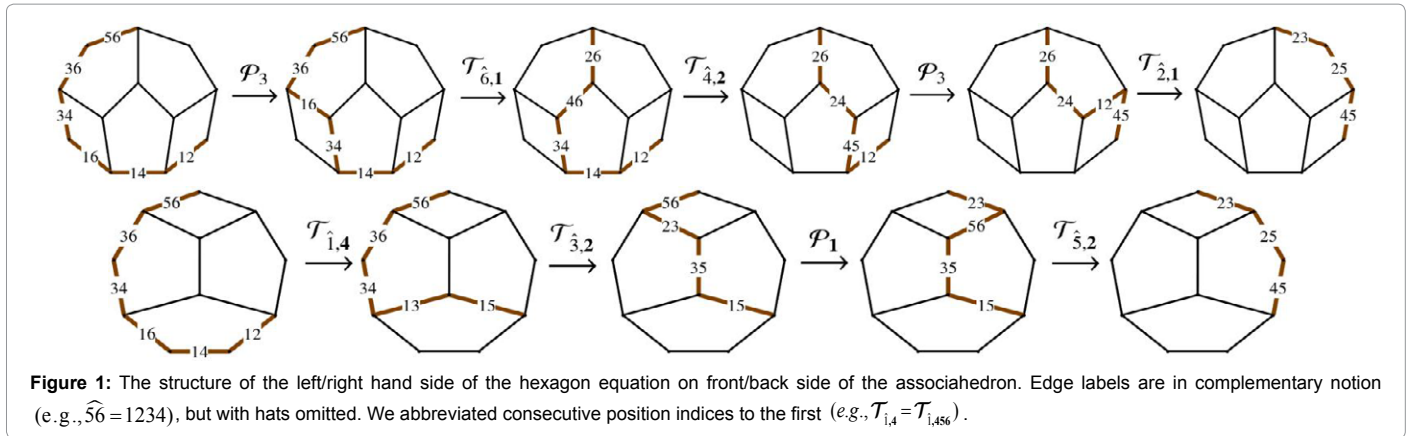
The six linear orders constituting  $A(3, 1)$ , viewed as maximal chains, build  $B(3, 0)$ . It forms a cube, on which the structure of the YB equation is often visualized. There is a counterpart for any simplex equation. For example, the linear orders constituting  $A(4, 2)$  can be viewed as maximal chains of  $B(4, 1)$ , which forms a polyhedron called

\*Corresponding author: Müller-Hoissen F, Max Planck Institute for Dynamics and Self-Organization, 37077 Göttingen, Germany, Tel: 49 551 5176-0; E-mail: [Folkert.Mueller-Hoissen@ds.mpg.de](mailto:Folkert.Mueller-Hoissen@ds.mpg.de)

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permutahedron, and the structure of the 3-simplex equation is nicely visualized on it [1].

$B(N + 2, N - 1)$  contains several copies of  $B(N + 1, N - 1)$ , and thus leads to a system of  $N$ -simplex equations. Let us “localize” them by allowing the maps to depend on additional variables, and such that each of these equations determines a map by taking the variables on the left hand side to those of the right hand side. Then, guided by  $B(N + 2, N)$ , one can deduce that these maps have to satisfy the  $(N + 1)$ -simplex equation [1]. Hence, the latter arises as the consistency condition of a system of localized  $N$ -simplex equations. This is a crucial integrability feature of the simplex equations.

**Decomposition of Bruhat Orders**

The above coloring of elements of the linear orders in the maximal chains of  $B(4, 2)$  indicates projections. The blue one results in

$$(12, 23, 34) \xrightarrow{123} (13, 34) \xrightarrow{134} 14 \quad (12, 23, 34) \xrightarrow{234} (12, 24) \xrightarrow{124} 14$$

and constitutes the Tamari order  $T(4, 2)$ . The red projection of  $B(4, 2)$  is the dual of  $T(4, 2)$ .

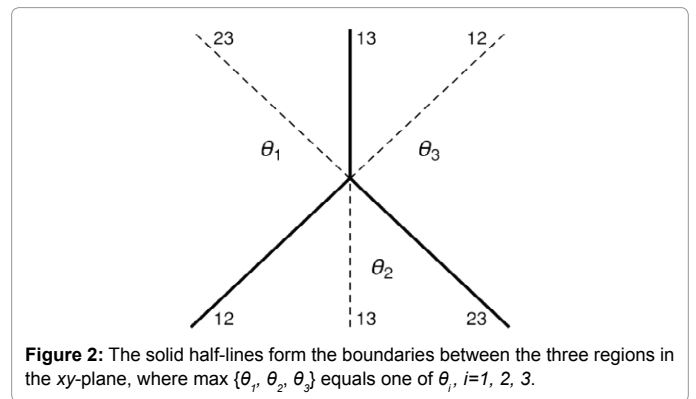
Crucial for the “decomposition” is the splitting  $P(K) = P_o(K) \cup P_e(K)$  of the packet of  $K$  into the half-packets of elements with odd, respectively even position in the lexicographical order. The inversion of lexicographically ordered packets, which governs Bruhat orders, is replaced in a Tamari order by  $\overline{P_o(K)} \mapsto \overline{P_e(K)}$ . For the precise definitions we refer to [1].

**Polygon Equations**

The Tamari orders  $T(N, N - 2)$  consist of two maximal chains. We define the  $N$ -gon equation as a realization of it, now with maps  $\mathcal{T}_K$  [1]. For  $N = 5$ , this is the ubiquitous *pentagon equation*. As for the YB equation, its structure can be visualized on a cube, which now carries  $T(5, 2)$ , however. The hexagon equation realizes  $T(6, 4)$  and can be visualized on the *associahedron* (Stasheff polytope), formed by the Tamari lattice  $T(6, 3)$ , see Figure 1.

The pentagon and hexagon equations appeared in realizations of Pachner moves of triangulations of a 3-, respectively 4-manifold [7,8]. The heptagon equation involves maps  $\mathcal{T}_{ijklmp} : U_{ijklm} \times U_{ijkmp} \times U_{ikmp} \rightarrow U_{jklmp} \times U_{gilmp} \times U_{ijklp}$ ,  $i < j < k < l < m < p$ . In terms of  $\widehat{\mathcal{T}}_\kappa = \mathcal{T}_\kappa P_{13}$ , it reads

$$\widehat{\mathcal{T}}_{1,123} \widehat{\mathcal{T}}_{3,145} \widehat{\mathcal{T}}_{5,246} \widehat{\mathcal{T}}_{7,356} = \widehat{\mathcal{T}}_{6,356} \widehat{\mathcal{T}}_{4,245} \widehat{\mathcal{T}}_{2,123}$$



A related equation appeared in [9]. Apart from this, polygon equations are rather new terrain. Neighboring polygon equations are related by the same kind of “integrability” that connects neighboring simplex equations, and this is what makes them special and promising. The important role of the pentagon equation (some keywords are “multiplicative unitary”, “Drinfeld associator”, “quantum dilogarithm”, see [1] for references) enhances the expectations for its higher analogs. The aforementioned decomposition of Bruhat orders leads to relations between (solutions of) simplex and polygon equations [1].

**Bruhat and Tamari Orders in Soliton Interactions**

Let  $\theta_i := p_i x + p_i^2 y + p_i^3 t + c_i$ , with real constants  $p_i, c_i, i = 1, 2, \dots, m$ . At fixed  $t, \theta_i = \theta_j$  determines the boundary line  $ij$  between the region where  $\theta_i > \theta_j$  and that where  $\theta_j > \theta_i$ . For  $m = 3$ , there are three such lines, as shown in Figure 2. Proceeding from bottom to top ( $y$ -direction), we observe the Bruhat order  $B(3, 2)$ . Instead, the function  $\max\{\theta_1, \dots, \theta_m\}$  splits the  $xy$ -plane into regions, where one of the  $\theta_i$  dominate, separated by a rooted and generically binary tree. For  $m = 3$ , this is shown in Figure 2 (the dashed half-lines are then “invisible”). It realizes the Tamari order  $T(3, 2)$  and actually appears in nature as a “Miles resonance” on a fluid surface, mathematically described by a special soliton solution of the famous KP equation. Indeed, in a “tropical limit”, where the above maximum function shows up, a subclass of KP soliton solutions realizes all the Tamari lattices  $T(N, N - 3)$  in terms of rooted binary trees [4,5]. Time evolution is then given by right rotation in a tree, which translates to the rightward application of the associativity law in Tamari’s original presentation of the lattices (in terms of bracketings of a monomial of fixed length [10]).

## Further Remarks

Scattering of solitons leads to Yang-Baxter maps, i.e., set-theoretical solutions of the YB equation (see e.g., [11]). Is there also a way to generate solutions of the simplex and polygon equations from (matrix) KP soliton interactions? For different aspects of KP soliton combinatorics, see, e.g., [12]. More generally, one can also ask for the combinatorics underlying soliton interactions in case of soliton equations other than KP. Surely, much more is there to be discovered.

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